Some new results
for the smoothness of topological equivalence
in uniformly asymptotically stable systems

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Abstract. In this article we revisit a method of topological linearization for nonautonomous and uniformly asymptotically stable ordinary differential equations developed by Kenneth J. Palmer and Faxing Lin. In particular, sufficient conditions are obtained ensuring the smoothness of the above mentioned topological linearization.

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1   Introduction

The smoothness of the topological equivalence or topological conjugacy is a classical topic on autonomous dynamical systems and we refer the reader to [16] for an overview on the latest advances. Nevertheless, the nonautonomous case is considerably less developed than the autonomous one; and the first results go back to the last decade. In this note, we will continue this study in the nonautonomous case.

More specifically, we obtain sufficient conditions ensuring the differentiability of the topological equivalence for certain families of nonautonomous systems

$$\dot{x} = F_1(t, x) \quad \text{for any } t \in J, \tag{1.1}$$

and

$$\dot{y} = F_2(t, y) \quad \text{for any } t \in J, \tag{1.2}$$

where $J \subseteq \mathbb{R}$ is an upperly unbounded interval while the functions $F_i \colon J \times \mathbb{R}^n \to \mathbb{R}^n$ are such that the existence and uniqueness of the solutions on J is ensured. In addition, the solutions of (1.1) and (1.2) passing through x_0 and y_0 at $t = \tau \in J$ will be denoted respectively by $t \mapsto x(t, \tau, x_0)$ and $t \mapsto y(t, \tau, y_0)$.

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The above systems are *J*-topologically equivalent when there exists a family of homeomorphisms parametrized by *J* mapping solutions of a system into solutions of the other one and viceversa; this property is described formally as follows:

Definition 1.1 ([21]). The systems (1.1) and (1.2) are *J*-topologically equivalent if there exists a function $H \colon J \times \mathbb{R}^n \to \mathbb{R}^n$ such that:

i) For any fixed $\tau \in J$, $x_0 \mapsto H_\tau(x_0) := H(\tau, x_0)$ is a homeomorphism of $\mathbb{R}^n$, whose inverse is denoted by $y_0 \mapsto G_\tau(y_0) := G(\tau, y_0)$.

ii) If $t \mapsto x(t, \tau, x_0)$ is a solution of (1.1) then $t \mapsto H(t, x(t, \tau, x_0))$ is a solution of (1.2). Similarly, if $t \mapsto y(t, \tau, y_0)$ is a solution of (1.2), then $t \mapsto G(t, y(t, \tau, y_0))$ is a solution of (1.1). That is, for any $t, \tau \in J$ it follows:

$$\begin{cases} H(t, x(t, \tau, x_0)) = y(t, \tau, H(\tau, x_0)) \\ G(t, y(t, \tau, y_0)) = x(t, \tau, G(\tau, y_0)). \end{cases} \tag{1.3}$$

iii) For any fixed $\tau \in J$, it is verified that the norms

$$\|H(\tau, x_0)\| \to +\infty \quad \text{and} \quad \|G(\tau, y_0)\| \to +\infty \quad \text{as} \quad \|x_0\|, \|y_0\| \to +\infty.$$

Although there is not a universally accepted definition of topological equivalence, the statements i) and ii) hold consistently in the specialized literature, while the asymptotic property iii) can be replaced by other types of conditions; see *e.g.* [18, p. 12], [19, p. 357] and [21] for details.

In the nonautonomous framework, the problem of search sufficient conditions ensuring the differentiability properties of a topological equivalence is relatively recent. In addition, there are diverse approaches to construct the homeomorphisms stated in Definition 1.1. Having this in mind, in Section 2 we describe the two main strategies: the use of the Green's function and the crossing time function.

The rest of the article is organized as follows: Sections 3 and 4 focus on deducting the differentiability for the $\mathbb{R}$-topological equivalence between uniformly asymptotically stable systems (1.1)–(1.2), obtained by F. Lin [14] and K. J. Palmer [18] by using a crossing time approach. Section 5 addresses the higher order differentiability. Section 6 provides an additional result and compares it with the current literature.

Last but not least, we point out that, to the best of our knowledge, there are no smoothness results for the crossing time approach in the nonautonomous context, which is the main novelty and contribution of this article.

2   The topological equivalence problem

The *topological equivalence problem* can be understood as the research of sufficient conditions on the vector fields F_i with $i = 1, 2$ such that (1.1) and (1.2) are *J*-topologically equivalent. In this article, we will distinguish some approaches carried out to cope with this problem: the **Green's function approach** and the **crossing time approach**. Nevertheless, it is important to emphasize that this distinction is not exhaustive.

From now on, the symbol $\| \cdot \|$ denotes either the euclidean vector norm or its induced matrix norm. The particular context of its appearance will indicate what its meaning is. In addition, $u = o(v)$ is the classical Landau's little-o notation.

2.1 The Green's function approach

The topological equivalence problem is studied for the particular case of systems (1.1)–(1.2), described by the linear system

$$\dot{x} = A(t)x \quad \text{for any } t \in J, \tag{2.1}$$

and a family of quasilinear perturbations, such as:

$$\dot{y} = A(t)y + f(t,y) \quad \text{for any } t \in J, \tag{2.2}$$

which is to say $F_1(t,x) = A(t)x$ and $F_2(t,y) = A(t)y + f(t,y)$ with $J = \mathbb{R}$ or $J = [0, +\infty)$.

A pivotal assumption of this approach is that (2.1) has a *dichotomy* property on J, which is defined as follows:

Definition 2.1. The system (2.1) has a dichotomy on J if there exist a projector $t \mapsto P(t) \in M_n(\mathbb{R})$, positive constants K, α and two functions $h, \mu \colon J \to [1, +\infty)$ continuous, increasing and verifying $\mu = o(h^\alpha)$ such that any fundamental matrix $t \mapsto \Phi(t)$ of (2.1) verifies:

$$P(t)\Phi(t,s) = \Phi(t,s)P(s) \quad \text{for any } t, s \in J$$

and

$$\begin{cases} \|\Phi(t,s)P(s)\| \leq K\mu(|s|) \left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s \text{ with } t, s \in J, \\ \|\Phi(t,s)Q(s)\| \leq K\mu(|s|) \left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t \text{ with } t, s \in J, \end{cases}$$

where $\Phi(t,s) := \Phi(t)\Phi^{-1}(s)$ and $Q(t) = I - P(t)$.

Note that any nontrivial solution $t \mapsto x(t, \tau, x_0) = \Phi(t, \tau)x_0$ of (2.1) can be splitted as

$$x(t, \tau, x_0) = \underbrace{\Phi(t, \tau)P(\tau)x_0}_{=x^+(t,\tau,x_0)} + \underbrace{\Phi(t, \tau)Q(\tau)x_0}_{=x^-(t,\tau,x_0)},$$

where $t \mapsto x^+(t, \tau, x_0) := x^+(t)$ and $t \mapsto x^-(t, \tau, x_0) := x^-(t)$ verify

$$\|x^+(t)\| \leq K\|P(\tau)x_0\|\mu(|\tau|) \left(\frac{h(t)}{h(\tau)}\right)^{-\alpha} \quad \text{and} \quad \left(\frac{h(t)}{h(\tau)}\right)^\alpha \frac{\|Q(\tau)x_0\|}{K\mu(|t|)} \leq \|x^-(t)\|,$$

for any $t \geq \tau$.

The above mentioned properties of μ, h and α allow to deduce that $x^+(t)$ is a forward contraction and $x^-(t)$ is a forward expansion. This splitting and its *dichotomic* asymptotic behavior motivate the use of the name dichotomy.

There exist several kinds of dichotomies describing the contractions and expansions at a specific rate; we refer the reader to the Table 1 from [23] and references therein for a detailed description.

The Green's function associated to the above mentioned dichotomy property is

$$\mathcal{G}(t,s) = \begin{cases} \Phi(t,s)P(s) & \text{if } t \geq s, \\ -\Phi(t,s)Q(s) & \text{if } t < s, \end{cases}$$

and allows an explicit construction of the homeomorphisms H_t and their inverses G_t mentioned on Definition 1.1.

The first homeomorphism was established by K. J. Palmer [17], which was constructed under the following assumptions: (2.1) has an exponential dichotomy on $\mathbb{R}$, namely $h(t) = e^t$ and $\mu(t) = 1$, the function f is uniformly bounded on $\mathbb{R} \times \mathbb{R}^n$ and $x \mapsto f(t, x)$ is uniformly Lipschitz with respect to t.

The first improvement of Palmer´s result was done by J. Shi and K. Xiong in [21], who demonstrated that the maps $\xi \mapsto H_t(\xi)$ and $\xi \mapsto G_t(\xi)$ are uniformly continuous with respect to t.

There exist a vast corpus of literature devoted to the topological equivalence problem by following this approach. In general, the problem is addressed by considering dichotomies more general than the exponential one; and, at the same time, imposing more restrictive assumptions on the perturbation f. In this context, we highlight the work of L. Barreira and C. Valls [2], which assumes that (2.1) has a nonuniform exponential dichotomy on $\mathbb{R}$, that is, $h(t) = e^t$ and $\mu(t) = e^{\varepsilon|t|}$. We refer the reader again to the Table 1 from [23] and [19] for more results.

2.2   The crossing time approach

In the work [18] of K. J. Palmer, the topological equivalence problem is considered for systems (1.1)–(1.2), where the maps $F_1 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $F_2 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following properties:

(P1) The origin is an equilibrium for any $t \in \mathbb{R}$, that is

$$F_1(t, 0) = F_2(t, 0) = 0 \quad \text{for any } t \in \mathbb{R},$$

(P2) There exists $L > 0$ such that, given any $t \in \mathbb{R}$ and $x, \tilde{x} \in \mathbb{R}^n$,

$$\|F_1(t, x) - F_1(t, \tilde{x})\| \leq L\|x - \tilde{x}\| \quad \text{and} \quad \|F_2(t, x) - F_2(t, \tilde{x})\| \leq L\|x - \tilde{x}\|.$$

(P3) There exists a continuous function $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and positive constants C_1, C_2 and β such that

$$C_1\|x\|^{\beta} \leq V(t, x) \leq C_2\|x\|^{\beta} \quad \text{for any } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

(P4) There exists $\eta > 0$ such that any solution $t \mapsto \phi(t)$ either of (1.1) or (1.2) verifies

$$DV_-(t, \phi(t)) := \liminf_{h \to 0^+} \frac{V(t, \phi(t)) - V(t - h, \phi(t - h))}{h} \leq -\eta\|\phi(t)\|^{\beta}.$$

A consequence of **(P3)** and **(P4)** is that V is a Lyapunov function for the systems (1.1) and (1.2). Then, classical results of Lyapunov's stability [13, Theorem 4.9], [20, Chapter 1] imply that the origin is a globally uniformly asymptotically stable equilibrium of (1.1) and (1.2), which also implies the existence and uniqueness of the *crossing times* $T := T(\tau, x_0)$ and $S := S(\tau, y_0)$, namely, the unique times such that

$$V(T, x(T, \tau, x_0)) = V(S, y(S, \tau, y_0)) = 1. \tag{2.3}$$

We now state the following result obtained by K. J. Palmer in [18, Lemma]:

Proposition 2.2. *If the systems* (1.1) *and* (1.2) *verify* **(P1)–(P4)***, then,* (1.1) *and* (1.2) *are* $\mathbb{R}$*-topologically equivalent with H and G defined by:*

$$H(\tau, x_0) = \begin{cases} y(\tau, T(\tau, x_0), x(T(\tau, x_0), \tau, x_0)) & x_0 \neq 0, \\ 0 & x_0 = 0, \end{cases} \tag{2.4}$$

and

$$G(\tau, y_0) = \begin{cases} x(\tau, S(\tau, y_0), y(S(\tau, y_0), \tau, y_0)) & y_0 \neq 0, \\ 0 & y_0 = 0. \end{cases} \tag{2.5}$$

From now on, for each $\tau \in \mathbb{R}$, the maps H_τ and G_τ will be called as the **Palmer's homeomorphism**.

A strong assumption of Palmer's result is that (1.1) and (1.2) must have the same Lypaunov's function; a particular example of this result is studied by F. Lin [14], which considers the linear diagonal system

$$\dot{x} = -\frac{\delta}{2}x, \tag{2.6}$$

and also the quasilinear system

$$\dot{y} = C(t)y + B(t)y + g(t, y), \tag{2.7}$$

such that $x, y \in \mathbb{R}^n$, $\delta > 0$ while the functions $C \colon \mathbb{R} \to M_n(\mathbb{R})$, $B \colon \mathbb{R} \to M_n(\mathbb{R})$ and $g \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and also verify:

(L1) The function $t \mapsto C(t) = \{c_{ij}(t)\}_{i,j=1}^n$ is bounded in $\mathbb{R}$ and $C(t)$ is a diagonal matrix with $c_{ii}(t) \leq -\delta$ for any $t \in \mathbb{R}$,

(L2) For any $t \in \mathbb{R}$, it follows that $\|B(t)\| \leq \frac{\delta}{4}$,

(L3) For any $t \in \mathbb{R}$ and any couple $y, \tilde{y} \in \mathbb{R}^n$, it is satisfied that

$$\|g(t, y) - g(t, \tilde{y})\| \leq \frac{\delta}{4}\|y - \tilde{y}\| \quad \text{and} \quad g(t, 0) = 0.$$

A careful reading of [14, Proposition 7, p. 41] allows us to deduce that a consequence of **(L1)–(L3)** is that the systems (2.6) and (2.7) have the same Lyapunov function and, consequently, the origin is a uniformly asymptotically stable equilibrium, emulating the properties **(P3)** and **(P4)** considered by Palmer.

The following result is obtained by F. Lin in [14, Lemma 1]:

Proposition 2.3. *If the systems* (2.6)–(2.7) *verify* **(L1)–(L3)***, then,* (2.6)–(2.7) *are* $\mathbb{R}$*-topologically equivalent with H and G defined by:*

$$H(\tau, x_0) = \begin{cases} y\left(\tau, T(\tau, x_0), e^{-\frac{\delta}{2}(T(\tau, x_0) - \tau)}x_0\right) & x_0 \neq 0, \\ 0 & x_0 = 0, \end{cases} \tag{2.8}$$

and

$$G(\tau, y_0) = \begin{cases} y(S(\tau, y_0), \tau, y_0)e^{-\frac{\delta}{2}(\tau - S(\tau, y_0))} & y_0 \neq 0, \\ 0 & y_0 = 0, \end{cases} \tag{2.9}$$

where $T := T(\tau, x_0)$ *and* $S := S(\tau, y_0)$ *are the unique times such that the euclidean norm of its solutions verify*

$$\|x(T(\tau, x_0), \tau, x_0)\|^2 = \|y(S(\tau, y_0), \tau, y_0)\|^2 = 1. \tag{2.10}$$

From now on, for each $\tau \in \mathbb{R}$, the maps H_τ and G_τ will be called as the **Lin's homeomorphism**.

Remark 2.4. If $x_0 \neq 0$ and $y_0 \neq 0$, the identity $x(t, t_0, x_0) = e^{-\frac{\delta}{2}(t-t_0)} x_0$ implies that $H(\tau, x_0)$ and $G(\tau, y_0)$ have the alternative characterizations:

$$H(\tau, x_0) = y(\tau, T(\tau, x_0), x(T(\tau, x_0), \tau, x_0)),$$

and

$$G(\tau, y_0) = x(\tau, S(\tau, y_0), y(S(\tau, y_0), \tau, y_0)),$$

which coincide with (2.4)–(2.5) and also implies the identities

$$y(T(\tau, x_0), \tau, H(\tau, x_0)) = x(T(\tau, x_0), \tau, x_0),$$

and

$$x(S(\tau, y_0), \tau, G(\tau, y_0)) = y(S(\tau, y_0), \tau, y_0).$$

It is important to emphasize that the literature devoted to the crossing time based homeomorphisms is considerably less developed in comparison with the Green's function approach. In fact, while the topological linearization via the Green's function has become an interesting topic in itself, the linearization via crossing time has been used as a technical step inside more general results. For example, in [18] the crossing time is used to relate topological equivalence with exponential dichotomy; furthermore, in [14] is a tool employed to obtain a topological equivalence result for a more general family of systems that can be reduced to (2.6)–(2.7).

2.3 The smoothness of the topological equivalence and the main novelty of this work

While the topological equivalence problem goes back to the 70's, the study of the differentiability properties of the homeomorphisms H_t and G_t of Definition 1.1 started in the 2010's decade and, obviously, is considerably less studied.

The first results on the smoothness of the maps H_t and G_t were based on the Green's function approach and were obtained for the contractive case in [4–7], while the contractive/expansive case is treated later in [11] under strong assumptions on the quasilinear perturbation.

It is important to stress that less restrictive smoothness results have recently been obtained in the contractive/expansive case by Cuong *et al.* and Dragičević *et al.* both cases are inspired by the ideas developed by Sternberg's, and considering resonance conditions described in terms of the spectra associated to the uniform exponential dichotomy [8] and the nonuniform exponential dichotomy [9, 10]. In this context, we also highlight the noticeable contributions of Backes & Dragičević in [1], Barreira & Valls in [3] and Lu *et al.* in [15].

Surprisingly, and to the best of our knowledge; there are no studies about the smoothness properties of homeomorphisms H_t and G_t when considering the crossing time approach and this work can be seen as a contribution on this subject.

3 Smoothness of Lin's homeomorphism

Throughout this section, we will assume that the conditions **(L1)–(L3)** of the Proposition 2.3 are verified and, in consequence, the systems (2.6) and (2.7) are $\mathbb{R}$-topologically equivalent

with maps H and G described respectively by (2.8) and (2.9). Moreover, as a convenient shorthand, we will refer to $\mathbb{R}_0^n$ rather than $\mathbb{R}^n \setminus \{0\}$ ($n \geq 1$) in all that follows.

Firstly, we will study the smoothness properties of the crossing time function S stated in (2.9). In order to do that, it will be useful to introduce the map $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\mathcal{F}(s,y) = C(s)y + B(s)y + g(s,y), \tag{3.1}$$

and, if $y \mapsto \mathcal{F}(s,y)$ is derivable on $\mathbb{R}^n$, its jacobian matrix for any fixed $s \in \mathbb{R}$ will be denoted by $D_y \mathcal{F}(s,y)$.

Lemma 3.1. *If the system* (2.7) *satisfies* **(L1)–(L3)** *and the maps* $(t,y) \mapsto g(t,y)$, $(t,y) \mapsto C(t)y$ *and* $(t,y) \mapsto B(t)y$ *belong to* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ *then the crossing time* $S : \mathbb{R} \times \mathbb{R}_0^n \to \mathbb{R}$ *is continuously differentiable on its domain of definition.*

Moreover, for any fixed $\tau \in \mathbb{R}$, *the partial derivative of* S *with respect to* ξ *is given explicitly by*

$$D_\xi S(\tau,\xi) = -\frac{D_\xi y(S(\tau,\xi),\tau,\xi) y(S(\tau,\xi),\tau,\xi)}{\mathcal{F}(\tau, y(S(\tau,\xi),\tau,\xi)) \cdot y(S(\tau,\xi),\tau,\xi)}. \tag{3.2}$$

Proof. As a first step, let us define the auxiliary map $\psi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by:

$$\psi(\tau,\xi,t) = \|y(t,\tau,\xi)\|^2 - 1.$$

The above assumptions imply that $\mathcal{F} \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Therefore, the differentiability of the solutions of (2.7) with respect to the initial conditions [22, Theorem 6.1, p. 89] states that $(t,\tau,\xi) \mapsto y(t,\tau,\xi) \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $D_\xi y(t,\tau,\xi)$ is solution of the linear variational equation

$$Y' = D_y \mathcal{F}(t, y(t,\tau,\xi)) Y \quad \text{with} \quad Y(\tau) = I,$$

which leads to

$$D_\xi \psi(\tau,\xi,t) = 2 D_\xi y(t,\tau,\xi) y(t,\tau,\xi). \tag{3.3}$$

Moreover, by (2.7) it is straightforward to verify that

$$D_t \psi(\tau,\xi,t) = 2 \mathcal{F}(t, y(t,\tau,\xi)) \cdot y(t,\tau,\xi), \tag{3.4}$$

where $\mathcal{F}$ is defined in (3.1). By gathering the above derivatives and recalling the assumptions, we have that $\psi \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

As a second step, let us consider the Banach spaces $X = (\mathbb{R} \times \mathbb{R}^n, |\cdot|_X)$ and $Y = Z = (\mathbb{R}, |\cdot|)$, where $|(t,x)|_X = \|x\| + |t|$.

Given the open set $\mathcal{O} = \mathbb{R} \times \mathbb{R}_0^n \times \mathbb{R} \subseteq X \times Y$, we define F as the restriction of ψ into the set $\mathcal{O}$, namely, $F : \mathcal{O} \to \mathbb{R}$ is defined by

$$F(\tau,\xi,t) = \psi(\tau,\xi,t) = \|y(t,\tau,\xi)\|^2 - 1,$$

which belongs to $\mathscr{C}^1(\mathcal{O}, \mathbb{R})$. Moreover, by (2.10), it follows that

$$F(\tau_0,\xi_0, S(\tau_0,\xi_0)) = 0 \quad \text{for any } \xi_0 \neq 0 \text{ and } \tau_0 \in \mathbb{R}. \tag{3.5}$$

The next step applies a Lin's estimation, namely, (2.10) and the proof of [14, Proposition 7, p. 41–42], which allow us to deduce that

$$D_t F(\tau_0,\xi_0, S(\tau_0,\xi_0)) \leq -\delta \underbrace{\|y(S(\tau_0,\xi_0),\tau_0,\xi_0)\|^2}_{=1} = -\delta < 0. \tag{3.6}$$

By using (3.5) combined with the implicit function theorem [22, Theorem 5.7, p. 82] applied to F, we can prove the existence of $\varphi \in \mathscr{C}^1(U, W)$, where U is a neighborhood of (τ_0, ξ_0) while W is one of $S(\tau_0, \xi_0)$, such that $\varphi(\tau_0, \xi_0) = S(\tau_0, \xi_0)$ with $U \times W \subseteq \mathcal{O}$ and

$$F(\tau, \xi, \varphi(\tau, \xi)) = 0 \quad \text{for any } (\tau, \xi) \in U,$$

which is equivalent to

$$\|y(\varphi(\tau, \xi), \tau, \xi)\|^2 = 1 = \|y(S(\tau, \xi), \tau, \xi)\|^2 \quad \text{for any} \quad (\tau, \xi) \in U,$$

then, the uniqueness of S implies $S(\tau, \xi) = \varphi(\tau, \xi)$ on U and $S \in \mathscr{C}^1(U, W)$.

In particular, we have that S is continuously differentiable on each $(\tau_0, \xi_0) \in \mathbb{R} \times \mathbb{R}_0^n$ and it follows that $S \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$. In addition, the partial derivative can be explicitly computed as

$$D_\xi S(\tau, \xi) = -\left[D_t F(\tau, \xi, S(\tau, \xi))\right]^{-1} D_\xi F(\tau, \xi, S(\tau, \xi)).$$

As a final step, the identity $F = \psi$ on $\mathcal{O}$ combined with (3.1) imply that the above partial derivatives coincides with those described by (3.3)–(3.4), then the above identity becomes (3.2) and the result follows. $\qquad \square$

Corollary 3.2. *The crossing time T corresponding to the solutions of* (2.6) *verifies* $T \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$ *and the partial derivative of T with respect to ξ satisfies*

$$D_\xi T(\tau, \xi) = \frac{2}{\delta} \frac{\xi}{\|\xi\|^2} \tag{3.7}$$

for any $\xi \neq 0$ and any fixed $\tau \in \mathbb{R}$.

Proof. Let us define $C_0, B_0 : \mathbb{R} \to M_n(\mathbb{R})$ and $g_0 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$C_0(t) = -\delta I, \quad B_0(t) = \frac{\delta}{4} I \quad \text{and} \quad g_0(t, x) = \frac{\delta}{4} x.$$

For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we have

$$C_0(t)x + B_0(t)x + g_0(t, x) = -\frac{\delta}{2} x,$$

and the matrix functions C_0, B_0 verify **(L1)** and **(L2)**, while g_0 verifies **(L3)** of the Proposition 2.3. Moreover, we have that $(t, x) \mapsto C_0(t)x, B_0(t)x$ and $(t, x) \mapsto g_0(t, x)$ are maps of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Then, Lemma 3.1 implies that the crossing time T is a function of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

In order to verify (3.7), let us remember that

$$x(t, \tau, \xi) = e^{-\frac{\delta}{2}(t-\tau)} \xi,$$

and we can obtain an explicit description for the crossing time:

$$\|x(T(\tau, \xi), \tau, \xi)\|^2 = 1 \iff T(\tau, \xi) = \frac{2}{\delta} \ln(\|\xi\|) + \tau \tag{3.8}$$

and (3.7) follows by calculating the derivative of $(\tau, \xi) \mapsto \frac{2}{\delta} \ln(\|\xi\|) + \tau$ with respect to ξ. $\quad \square$

The results of continuous differentiability for the crossing time functions will be useful to achieve the following result:

Theorem 3.3. *If the systems* (2.6)–(2.7) *satisfy* **(L1)–(L3)** *and the maps* $(t, y) \mapsto g(t, y)$, $(t, y) \mapsto C(t)y$ *and* $(t, y) \mapsto B(t)y$ *belong to* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, *the Lin's homeomorphism* H_τ *is a diffeomorphism of class* $\mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$ *for any fixed* $\tau \in \mathbb{R}$.

Moreover, the derivative of $H_\tau := H(\tau, \cdot)$ *with respect to* ξ *is given by*

$$D_\xi H(\tau, \xi) = \frac{2}{\delta} \left\{ D_T y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right) \frac{\xi_j}{\|\xi\|^2} \right\}_{i,j=1}^n \tag{3.9}$$
$$+ D_\xi y \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right) D_\xi \left[\frac{\xi}{\|\xi\|} \right],$$

while the derivative of $G(\tau, \cdot) := H_\tau^{-1}$ *with respect to* ξ *is*

$$D_\xi G(\tau, \xi) = e^{\frac{\delta}{2}(S(\tau, \xi) - \tau)} \left[\frac{\delta}{2} \mathcal{G}(\tau, \xi) + \mathcal{V}(\tau, \xi) + D_\xi y(S(\tau, \xi), \tau, \xi) \right] \tag{3.10}$$

where $\mathcal{G}$ *and* $\mathcal{V}$ *are n-th order matrices with*

$$\mathcal{G}_{i,j}(\tau, \xi) := \frac{\partial S(\tau, \xi)}{\partial \xi_j} y_i(S(\tau, \xi), \tau, \xi) \quad \text{and} \quad \mathcal{V}_{i,j}(\tau, \xi) := \mathcal{F}_i(S, y(S, \tau, \xi)) \frac{\partial S(\tau, \xi)}{\partial \xi_j}$$

for any $i, j \in \{1, \dots, n\}$ *and* $\mathcal{F}_i$ *is the i-th coordinate of* $\mathcal{F}$ *defined in* (3.1).

Proof. By Lemma 3.1 and Corollary 3.2, the crossing time functions S and T are of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$, whose proofs used that the map $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$ is of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

The above facts combined with (2.8) imply that, for any fixed $\tau \in \mathbb{R}$, the map $0 \neq \xi \mapsto H_\tau(\xi) = y(\tau, T(\tau, \xi), e^{-\frac{\delta}{2}(T(\tau, \xi) - \tau)}\xi)$ can be seen as a composition of functions of class $\mathscr{C}^1$, which leads to $H_\tau \in \mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

Similarly, by using (2.9), the map $0 \neq \xi \mapsto G_\tau(\xi) = y(\tau, S(\tau, \xi), \xi)e^{-\frac{\delta}{2}(\tau - S(\tau, \xi))}$ is a composition and product of functions of class $\mathscr{C}^1$. In consequence, we have $G_\tau \in \mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$ and H_τ is a diffeomorphism of class $\mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

In order to verify (3.9), the explicit characterization of the crossing T given by (3.8) allows to obtain a simpler expression

$$H(\tau, \xi) = y \left(\tau, T(\tau, \xi), e^{-\frac{\delta}{2}(T(\tau, \xi) - \tau)}\xi \right) = y \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right).$$

Let $H_i(\tau, \xi)$ be the *i*-th coordinate of the map $H(\tau, \cdot)$. By using $\frac{\partial \tau}{\partial \xi_j} = 0$ combined with the chain rule and (3.7), we can deduce that the partial derivatives are

$$\frac{\partial H_i(\tau, \xi)}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \left\{ y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right) \right\}$$
$$= \frac{\partial y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right)}{\partial \tau} \underbrace{\frac{\partial \tau}{\partial \xi_j}}_{=0} + \frac{\partial y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right)}{\partial T(\tau, \xi)} \frac{\partial T(\tau, \xi)}{\partial \xi_j}$$
$$+ \sum_{k=1}^n \frac{\partial y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right)}{\partial \xi_k} \frac{\partial}{\partial \xi_j} \left\{ \frac{\xi_k}{\|\xi\|} \right\}$$
$$= \frac{2}{\delta} \frac{\partial y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right)}{\partial T} \frac{\xi_j}{\|\xi\|^2} + \sum_{k=1}^n \frac{\partial y_i \left(\tau, T(\tau, \xi), \frac{\xi}{\|\xi\|} \right)}{\partial \xi_k} \frac{\partial}{\partial \xi_j} \left\{ \frac{\xi_k}{\|\xi\|} \right\},$$

which corresponds to the (i, j)-coefficient of (3.9).

In order to verify the identity (3.10), if $G_i(\tau, \xi)$ is the i-th coordinate of $G(\tau, \cdot)$, notice that

$$
\frac{\partial G_i(\tau, \xi)}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \left[y_i(S(\tau, \xi), \tau, \xi) e^{-\frac{\delta}{2}(\tau - S(\tau, \xi))} \right]
$$

$$
= \frac{\delta}{2} e^{\frac{\delta}{2}(S(\tau, \xi) - \tau)} \frac{\partial S(\tau, \xi)}{\partial \xi_j} y_i(S(\tau, \xi), \tau, \xi)
$$

$$
+ e^{\frac{\delta}{2}(S(\tau, \xi) - \tau)} \left(\underbrace{\frac{\partial y_i(S(\tau, \xi), \tau, \xi)}{\partial S}}_{=\mathcal{F}_i(S, y(S, \tau, \xi))} \frac{\partial S(\tau, \xi)}{\partial \xi_j} + \frac{\partial y_i(S(\tau, \xi), \tau, \xi)}{\partial \xi_j} \right)
$$

$$
= e^{\frac{\delta}{2}(S(\tau, \xi) - \tau)} \left\{ \frac{\delta}{2} \mathcal{G}_{i,j}(\tau, \xi) + \mathcal{V}_{i,j}(\tau, \xi) + \frac{\partial y_i(S(\tau, \xi), \tau, \xi)}{\partial \xi_j} \right\}
$$

and similarly, we can verify that it corresponds to the (i, j)-coefficient of (3.10), and the Theorem follows. $\qquad \square$

Corollary 3.4. *Under the assumptions of Theorem 3.3, the Lin's homeomorphism $G_\tau : \mathbb{R}_0^n \to \mathbb{R}_0^n$ is a preserving orientation diffeomorphism for $n \geq 2$.*

Proof. Let $\tau \in \mathbb{R}$ fixed. Firstly, as H_τ is a bijective map with inverse G_τ, it follows that

$$
H_\tau(G_\tau(\xi)) = \xi \quad \text{for any } \xi \in \mathbb{R}_0^n.
$$

Then, by the Theorem 3.3, we have that H_τ is a diffeomorphism of $\mathbb{R}_0^n$ on itself and

$$
D_G H_\tau(G_\tau(\xi)) D_\xi G_\tau(\xi) = I \implies \det[D_G H_\tau(G_\tau(\xi))] \det[D_\xi G_\tau(\xi)] = 1,
$$

where $\det[D_\xi G_\tau(\xi)] \neq 0$ for any $\xi \in \mathbb{R}_0^n$.

Therefore, we only have to verify that $\det[D_\xi G_\tau(\xi)] > 0$ for any $\xi \in \mathbb{R}_0^n$. In order to prove this property, we construct the function $\Gamma : \mathbb{R}_0^n \to \mathbb{R} \setminus \{0\}$ defined by $\Gamma(\xi) = \det[D_\xi G_\tau(\xi)]$. Note that Γ can be seen as a composition of continuous maps described by:

$$
\mathbb{R}_0^n \xrightarrow{D_\xi G_\tau} GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R} \setminus \{0\}
$$

$$
\xi \longmapsto D_\xi G_\tau(\xi) \longmapsto \det[D_\xi G_\tau(\xi)] = \Gamma(\xi).
$$

By the continuity of Γ on the connected set $\mathbb{R}_0^n$ for $n \geq 2$, we have that $\Gamma(\mathbb{R}_0^n)$ is connected, then we have that

$$
\text{either} \quad \Gamma(\mathbb{R}_0^n) \subseteq \left]-\infty, 0\right[\quad \text{or} \quad \Gamma(\mathbb{R}_0^n) \subseteq \left]0, +\infty\right[. \tag{3.11}
$$

Hence, in order to prove that $\det D_\xi G(\tau, \xi) > 0$ for any $\xi \in \mathbb{R}_0^n$, we have to show that $\Gamma(\mathbb{R}_0^n) \subseteq \left]0, +\infty\right[$. By the above paragraph, we only need to show that $\det D_\xi G_\tau(\xi) > 0$ for some specific $\xi \in \mathbb{R}_0^n$. Indeed, we will verify this property for $\xi = (1, 0, \dots, 0) \in \mathbb{R}_0^n$.

Now, as $\|\xi\| = 1$, we have that

$$
\|y(\tau, \tau, \xi)\|^2 = \|\xi\|^2 = 1 = \|y(S(\tau, \xi), \tau, \xi)\|^2,
$$

which implies that $S(\tau, \xi) = \tau$ by the uniqueness of S. In addition, by [22, Theorem 6.1, p. 189], we have that $t \mapsto D_\xi y(t, \tau, \xi)$ is solution of the linear variational equation

$$
\frac{dY}{dt} = D_y \mathcal{F}(t, y(t, \tau, \xi)) Y \quad \text{with} \quad Y(\tau) = I,
$$

and (3.2) combined with $S(\tau, \xi) = \tau$ imply that

$$D_\xi S(\tau, \xi) = -\frac{D_\xi y(S(\tau, \xi), \tau, \xi) \, y(S(\tau, \xi), \tau, \xi)}{\mathcal{F}(\tau, y(S(\tau, \xi), \tau, \xi)) \cdot y(S(\tau, \xi), \tau, \xi)} = -\frac{\xi}{\mathcal{F}(\tau, \xi) \cdot \xi}$$

then we have

$$\frac{\partial S(\tau, \xi)}{\partial \xi_j} = -\frac{\xi_j}{\mathcal{F}(\tau, \xi) \cdot \xi} \quad \text{for any } 1 \le j \le n. \tag{3.12}$$

On the other hand, we have that $\frac{\partial y_i(\tau, \tau, \xi)}{\partial \xi_j} = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Then, (3.10), (3.12) and $S(\tau, \xi) = \tau$ imply that the i, j-coordinate of $D_\xi G_\tau(\xi)$ is

$$\frac{\partial G_i(\tau, \xi)}{\partial \xi_j} = \frac{\delta}{2} e^{\frac{\delta}{2}(S(\tau, \xi) - \tau)} \frac{\partial S(\tau, \xi)}{\partial \xi_j} y_i(S(\tau, \xi), \tau, \xi)$$

$$+ e^{\frac{\delta}{2}(S(\tau, \xi) - \tau)} \left\{ \mathcal{F}_i(S(\tau, \xi), y(S(\tau, \xi), \tau, \xi)) \frac{\partial S(\tau, \xi)}{\partial \xi_j} + \frac{\partial y_i(S(\tau, \xi), \tau, \xi)}{\partial \xi_j} \right\}$$

$$= \frac{\partial S(\tau, \xi)}{\partial \xi_j} \left\{ \frac{\delta}{2} y_i(S(\tau, \xi), \tau, \xi) + \mathcal{F}_i(S(\tau, \xi), y(S(\tau, \xi), \tau, \xi)) \right\} + \frac{\partial y_i(S(\tau, \xi), \tau, \xi)}{\partial \xi_j}$$

$$= -\frac{\xi_j}{\mathcal{F}(\tau, \xi) \cdot \xi} \left\{ \frac{\delta}{2} \xi_i + \mathcal{F}_i(\tau, \xi) \right\} + \delta_{ij}.$$

By considering $\xi = (1, 0, 0 \dots, 0) = (\xi_1, \xi_2, \dots, \xi_n)$, we can deduce that

$$\frac{\partial G_i(\tau, \xi)}{\partial \xi_j} = \begin{cases} -\dfrac{\mathcal{F}_i(\tau, \xi)}{\mathcal{F}_1(\tau, \xi)} & j = 1 \ne i, \\ 0 & j \ne i, j \ne 1, \\ -\dfrac{\delta}{2\mathcal{F}_1(\tau, \xi)} & i = j = 1 \\ 1 & i = j, i \ne 1, \end{cases}$$

and the derivative $D_\xi G_\tau(\xi)$ is described by the block matrix

$$D_\xi G_\tau(\xi) = \left[\begin{array}{c|c} -\frac{\delta}{2\mathcal{F}_1(\tau, \xi)} & 0_{1 \times n} \\ \hline \mathcal{D} & I_{n-1} \end{array} \right] \quad \text{where} \quad \mathcal{D} = \left[\begin{array}{c} -\frac{\mathcal{F}_2(\tau, \xi)}{\mathcal{F}_1(\tau, \xi)} \\ -\frac{\mathcal{F}_3(\tau, \xi)}{\mathcal{F}_1(\tau, \xi)} \\ \vdots \\ -\frac{\mathcal{F}_n(\tau, \xi)}{\mathcal{F}_1(\tau, \xi)} \end{array} \right],$$

and $I_{n-1} \in M_{n-1}(\mathbb{R})$ is the identity matrix. That is, $D_\xi G_\tau(\xi)$ is a lower triangular matrix where its diagonal terms are

$$\frac{\partial G_i(\tau, \xi)}{\partial \xi_i} = \begin{cases} -\dfrac{\delta}{2\mathcal{F}_1(\tau, \xi)} & i = 1, \\ 1 & i \ne 1, \end{cases}$$

and we can explicitely see that $\det[D_\xi G_\tau(\xi)] = -\frac{\delta}{2\mathcal{F}_1(\tau, \xi)}$.

Let us recall that, in the proof of the Lemma 3.1, we constructed the function $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(\tau, \xi, t) = \|y(t, \tau, \xi)\|^2 - 1 \quad \text{for any } (\tau, \xi, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},$$

which satisfies (3.6). This property combined with $S(\tau, \xi) = \tau$ lead to

$$
\begin{aligned}
\mathcal{F}_1(\tau, \xi) = \sum_{i=1}^{n} \mathcal{F}_i(\tau, \xi)\xi_i &= \mathcal{F}(\tau, \xi) \cdot \xi \\
&= \mathcal{F}(S(\tau, \xi), y(S(\tau, \xi), \tau, \xi)) \cdot y(S(\tau, \xi), \tau, \xi) \\
&= \frac{1}{2}D_t F(\tau, \xi, S(\tau, \xi)) \leq -\frac{\delta}{2} < 0,
\end{aligned}
$$

then we have that $\mathcal{F}_1(\tau, \xi) < 0$ and consequently $\det[D_\xi G_\tau(\xi)] = -\frac{\delta}{2\mathcal{F}_1(\tau,\xi)} > 0$.

We have verified the existence of $\xi \in \mathbb{R}_0^n$ such that $\det[D_\xi G_\tau(\xi)] = \Gamma(\xi) > 0$. The connectedness of $\Gamma(\mathbb{R}_0^n)$ and (3.11) imply that $\Gamma(\mathbb{R}_0^n) \subseteq]0, +\infty[$, or equivalently, $\det[D_\xi G_\tau(\tilde{\xi})] > 0$ for every $\tilde{\xi} \in \mathbb{R}_0^n$ and we have proved that G_τ is a preserving orientation diffeomorphism of $\mathbb{R}_0^n$ on itself for any $n \geq 2$. $\square$

The connectedness of $\mathbb{R}_0^n$ with $n \geq 2$ played a key role in the above proof. Nevertheless, we will make minor adaptations to cope with the case $n = 1$.

Corollary 3.5. *Under the assumptions of Theorem 3.3, the Lin's homeomorphism $G_\tau : \mathbb{R}_0^n \to \mathbb{R}_0^n$ is also a preserving orientation diffeomorphism for $n = 1$.*

Proof. Let be $\Gamma : \mathbb{R}_0 \to \mathbb{R}_0$ as in the previous proof, where $\mathbb{R}_0$ is the disconnected set

$$\mathbb{R}_0 := \mathbb{R} \setminus \{0\} =]-\infty, 0[\cup]0, +\infty[.$$

However, we will emulate the proof of the Corollary (3.4) in the connected components of $\mathbb{R}_0$, namely, $\mathcal{C}^+ :=]0, +\infty[$ and $\mathcal{C}^- :=]-\infty, 0[$.

Let us consider $\xi^+ := 1 \in \mathcal{C}^+$ and $\xi^- := -1 \in \mathcal{C}^-$. Then we have $|\xi^-| = |\xi^+| = 1$, which leads to

$$S(\tau, \xi^-) = \tau = S(\tau, \xi^+).$$

By replying the proof of the Corollary 3.4, we have that

$$\frac{\partial S(\tau, \xi^+)}{\partial \xi} = \frac{-1}{\mathcal{F}(\tau, 1)} \quad \text{and} \quad \det[D_\xi G_\tau(\xi^+)] = D_\xi G_\tau(\xi^+) = -\frac{\delta}{2\mathcal{F}(\tau, 1)},$$

and we verify, similarly as in the previous result, that $\det D_\xi[G_\tau(\xi^+)] = \Gamma(\xi^+) > 0$, which implies $\Gamma(\mathcal{C}^+) \in]0, +\infty[$ and it follows that $\det[D_\xi G_\tau(\xi)] = \Gamma(\xi) > 0$ for any $\xi \in \mathcal{C}^+$.

By proceeding analogously, we also can verify that

$$\frac{\partial S(\tau, \xi^-)}{\partial \xi} = \frac{-1}{\mathcal{F}(\tau, -1)} \quad \text{and} \quad D_\xi G_\tau(\xi^-) = \frac{\delta}{2\mathcal{F}(\tau, -1)}.$$

Now, let us recall that $\mathcal{F}(t, \xi) = C(t)\xi + B(t)\xi + g(t, \xi)$ where $C, B : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the properties **(L1)–(L3)** stated in Section 2 with $n = 1$, then, we have

$$C(\tau)\xi^- = -C(\tau) \geq \delta, \quad B(\tau)\xi^- = -B(\tau) \geq -\frac{\delta}{4}, \quad \text{and} \quad g(\tau, \xi^-) \geq -\frac{\delta}{4},$$

which implies that $\mathcal{F}(\tau, \xi^-) = -C(\tau) - B(\tau) + g(\tau, \xi^-) \geq \frac{\delta}{4} > 0$ and we have that

$$\Gamma(\xi^-) = \det[D_{\xi} G_{\tau}(\xi^-)] = D_{\xi} G_{\tau}(\xi^-) = \frac{\delta}{2\mathcal{F}(\tau, \xi^-)} > 0.$$

By the connectedness of $\Gamma(\mathcal{C}^-)$, we have that $\Gamma(\mathcal{C}^-) \subseteq \,]0, +\infty[$ and $\det[D_{\xi} G_{\tau}(\xi)] = \Gamma(\xi) > 0$ for any $\xi \in \mathcal{C}^-$ and the result holds. $\qquad\square$

4 Smoothness of Palmer's homeomorphism

This section studies the differentiability properties of the homeomorphisms H_t and G_t defined by (2.4) and (2.5) in the Proposition 2.2.

Lemma 4.1. *If the systems* (1.1) *and* (1.2) *verify* **(P1)–(P4)** *while the functions* $F_1, F_2 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *are of class* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ *and the Lyapunov function* V *is of class* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, *the crossing times* T *and* S *of* (2.3) *are of class* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

Moreover, the derivative of $T := T(\tau, \cdot)$ *with respect to* $\xi \in \mathbb{R}_0^n$ *verifies*

$$D_{\xi} T(\tau, \xi) = -\frac{D_{\xi} x(T, \tau, \xi)\, D_{\xi} V(T, x(T, \tau, \xi))}{D_T V(T, x(T, \tau, \xi)) + D_{\xi} V(T, x(T, \tau, \xi)) \cdot F_1(T, x(T, \tau, \xi))}. \tag{4.1}$$

while, the derivative of $S := S(\tau, \cdot)$ *with respect to* $\xi \in \mathbb{R}_0^n$ *is given by*

$$D_{\xi} S(\tau, \xi) = -\frac{D_{\xi} y(S, \tau, \xi)\, D_{\xi} V(S, y(S, \tau, \xi))}{D_S V(S, y(S, \tau, \xi)) + D_{\xi} V(S, y(S, \tau, \xi)) \cdot F_2(S, y(S, \tau, \xi))}. \tag{4.2}$$

Proof. We will work with the same Banach spaces X, Y and Z and the same open set $\mathcal{O} \subseteq X \times Y$ of the proof of the Lemma 3.1.

Moreover, we will construct a $\mathscr{C}^1(\mathcal{O}, Z)$-map ϕ verifying $\phi(\tau, \xi, T(\tau, \xi)) = 0$ in order to apply the implicit function theorem for Banach spaces [22, Theorem 5.7, p. 82].

Firstly, let us define the auxiliary map $\nu : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\nu(\tau, \xi, t) = V(t, x(t, \tau, \xi)) - 1.$$

As $F_1 \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, the differentiability of the solutions of (1.1) with respect to the initial conditions states that $(t, \tau, \xi) \mapsto x(t, \tau, \xi) \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $t \mapsto D_{\xi} x(t, \tau, \xi)$ is solution of the linear variational equation

$$X' = D_x F_1(t, x(t, \tau, \xi))X \quad \text{with} \quad X(\tau) = I.$$

By hypothesis, $(t, x) \mapsto V(t, x)$ is of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, which leads to

$$D_{\xi} \nu(\tau, \xi, t) \;=\; D_x V(t, x(t, \tau, \xi)) D_{\xi} x(t, \tau, \xi). \tag{4.3}$$

Moreover, by (1.1) it is straightforward to verify that

$$D_t \nu(\tau, \xi, t) = D_t V(t, x(t, \tau, \xi)) + D_x V(t, x(t, \tau, \xi)) F_1(t, x(t, \tau, \xi)), \tag{4.4}$$

and, by recalling the above assumptions, we can see that $\nu \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

Now, let us construct the restriction on ν into $\mathcal{O}$, namely, $\phi : \mathcal{O} \to \mathbb{R}$ defined by

$$\phi(\tau, \xi, t) = \nu(\tau, \xi, t) = V(t, x(t, \tau, \xi)) - 1,$$

which clearly belongs to $\mathscr{C}^1(\mathcal{O}, \mathbb{R})$. In addition, (2.3) implies that

$$\phi(\tau_0, \xi_0, T(\tau_0, \xi_0)) = 0 \quad \text{for any } \xi_0 \neq 0 \text{ and } \tau_0 \in \mathbb{R}. \tag{4.5}$$

By the assumptions **(P3)–(P4)** of the Proposition 2.2 we can deduce that

$$\begin{aligned}
D_t\phi(\tau, \xi, t) = D_t\nu(\tau, \xi, t) &= D_t\{V(t, x(t, \tau, \xi)) - 1\} \\
&= D_t V(t, x(t, \tau, \xi)) \leq -\eta \|x(t, \tau, \xi)\|^\beta,
\end{aligned}$$

for all $(\tau, \xi, t) \in \mathcal{O}$. Moreover, by **(P1)**, **(P2)** and [14, Prop 2, p. 40], we have the inequality $\|\xi\|e^{-L|t-\tau|} \leq \|x(t, \tau, \xi)\|$, obtaining the sharper estimation:

$$D_t\phi(\tau_0, \xi_0, T(\tau_0, \xi_0)) \leq -\eta e^{-L\beta|T(\tau_0,\xi_0)-\tau_0|}\|\xi_0\|^\beta < 0, \tag{4.6}$$

then, the implicit function theorem and (4.5) establish the existence of $\mathscr{C}^1(U, W)$-map φ, where U is a neighborhood of (τ_0, ξ_0) while W is one of $T(\tau_0, \xi_0)$, which verifies

$$\varphi(\tau_0, \xi_0) = T(\tau_0, \xi_0) \quad \text{with} \quad U \times W \subseteq \mathcal{O} \text{ and } \phi(\tau, \xi, \varphi(\tau, \xi)) = 0 \quad \text{for any } (\tau, \xi) \in U,$$

which also can be written as

$$V(\varphi(\tau, \xi), x(\varphi(\tau, \xi), \tau, \xi)) = 1 = V(T(\tau, \xi), x(T(\tau, \xi), \tau, \xi)) \quad \text{on } U,$$

which leads to $T(\tau, \xi) = \varphi(\tau, \xi)$ for any $(\tau, \xi) \in U$ by the uniqueness of T, which also implies that $T \in \mathscr{C}^1(U, W)$.

In particular, we have the continuous differentiability of T on each arbitrary $(\tau_0, \xi_0) \in \mathbb{R} \times \mathbb{R}_0^n$, this implies that $T \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$. The partial derivative is obtained explicitly as

$$D_\xi T(\tau, \xi) = -\left[D_t\phi(\tau, \xi, T(\tau, \xi))\right]^{-1} D_\xi\,\phi(\tau, \xi, T(\tau, \xi)).$$

By using that $\nu\mid_{\mathcal{O}} = \phi$, the above partial derivatives coincides with (4.3)–(4.4) and the above identity becomes (4.1).

Finally, in order to show (4.2) and $S \in \mathscr{C}^1$, we can make an identical proof to the previous one by using the maps F_2 and $t \mapsto y(t, \tau, \xi)$ instead of F_1 and $t \mapsto x(t, \tau, \xi)$, respectively, and the result follows. □

The above result of continuous differentiability for the crossing times will be useful to achieve the continuous differentiability of the Palmer's homeomorphisms H_τ and G_τ.

Theorem 4.2. *Under the assumptions of the Lemma 4.1, the Palmer's homeomorphism H_τ described in (2.4) is a diffeomorphism of class $\mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$ for any fixed $\tau \in \mathbb{R}$.*

Moreover, if we use the notation $S := S(\tau, \xi)$ and $T := T(\tau, \xi)$, the derivative of $H_\tau := H(\tau, \cdot)$, with respect to ξ is given by

$$D_\xi H(\tau, \xi) = \mathcal{V}(\tau, \xi) + D_\xi y(\tau, T, x(T, \tau, \xi))[\mathcal{A}(\tau, \xi) + D_\xi x(T, \tau, \xi)]. \tag{4.7}$$

where the (i, j)-coordinates of $\mathcal{V}(\tau, \xi)$ and $\mathcal{A}(\tau, \xi)$ are given by

$$\mathcal{V}_{i,j}(\tau, \xi) = D_T y_i(\tau, T, x(T, \tau, \xi)) D_{\xi_j} T(\tau, \xi), \quad \mathcal{A}_{i,j}(\tau, \xi) = D_T x_i(T, \tau, \xi) D_{\xi_j} T(\tau, \xi),$$

while the derivative of $G(\tau, \cdot) := H_\tau^{-1}$ with respect to ξ is

$$D_\xi G(\tau, \xi) = \mathcal{W}(\tau, \xi) + D_\xi x(\tau, S, x(S, \tau, \xi))[\mathcal{B}(\tau, \xi) + D_\xi y(S, \tau, \xi)] \tag{4.8}$$

where the (i, j)-coordinates of $\mathcal{W}(\tau, \xi)$ and $\mathcal{B}(\tau, \xi)$ are given by

$$\mathcal{W}_{i,j}(\tau, \xi) = D_S x_i(\tau, S, x(S, \tau, \xi)) D_{\xi_j} S(\tau, \xi), \quad \mathcal{B}_{i,j}(\tau, \xi) = D_S y_i(S, \tau, \xi) D_{\xi_j} S(\tau, \xi).$$

Proof. Note that $S, T \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$ by Lemma 4.1, whose proof used that the maps $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$ and $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$ are of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

The above paragraph together with (2.4) imply that the map

$$0 \neq \xi \mapsto H_\tau(\xi) = y(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi)),$$

is a composition of $\mathscr{C}^1$ maps, and we have that $H_\tau \in \mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$ for any fixed $\tau \in \mathbb{R}$.

To verify (4.7), let $H_i(\tau, \xi)$ be the i-th coordinate of the map $H(\tau, \cdot)$. By using (2.4) and the chain's rule, we can deduce that the partial derivatives are

$$\frac{\partial H_i(\tau, \xi)}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \left\{ y_i\left(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi) \right) \right\}$$

$$= \frac{\partial y_i(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi))}{\partial T} \frac{\partial T(\tau, \xi)}{\partial \xi_j}$$

$$+ \sum_{k=1}^{n} \frac{\partial y_i\left(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi) \right)}{\partial x_k} \frac{\partial x_k(T(\tau, \xi), \tau, \xi)}{\partial T} \frac{\partial T(\tau, \xi)}{\partial \xi_j}$$

$$+ \sum_{k=1}^{n} \frac{\partial y_i\left(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi) \right)}{\partial x_k} \frac{\partial x_k(T(\tau, \xi), \tau, \xi)}{\partial \xi_j}$$

$$= \mathcal{V}_{i,j}(\tau, \xi) + \sum_{k=1}^{n} \frac{\partial y_i\left(\tau, T, x(T, \tau, \xi) \right)}{\partial x_k} \left\{ \mathcal{A}_{k,j}(\tau, \xi) + \frac{\partial x_k(T, \tau, \xi)}{\partial \xi_j} \right\},$$

where the last equation is due by using $T = T(\tau, \xi)$. Then, we can verify that it corresponds to the (i, j)-coefficient of (4.7).

In a similar way, by (2.5), the map $0 \neq \xi \mapsto G_\tau(\xi) = x(\tau, S(\tau, \xi), y(S(\tau, \xi), \tau, \xi))$ is a composition of continuously differentiable functions. In consequence, G_τ is also a continuously differentiable map of $\mathbb{R}_0^n$ on itself. Therefore, H_τ is a diffeomorphism of class $\mathscr{C}^1(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

The proof of the identity (4.8) is similar to (4.7) by using $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$ and $(\tau, \xi) \mapsto S(\tau, \xi)$ instead of $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$ and $(\tau, \xi) \mapsto T(\tau, \xi)$, respectively. Therefore, we can prove that it corresponds to the (i, j)-coefficient of the identity (4.8), and the Theorem follows. $\qquad\square$

5 The smoothness of class $\mathscr{C}^k$ for the Palmer's and Lin's homeomorphism

Throughout this section, we will see that; provided some additional properties; the Palmer's and Lin's homeomorphisms via crossing times are diffeomorphisms of class $\mathscr{C}^k$ for any fixed $\tau \in \mathbb{R}$ and $k \geq 2$.

We will start by studying the Palmer's homeomorphism:

Lemma 5.1. *If the systems* (1.1) *and* (1.2) *verify* **(P1)–(P4)**, F_1 *and* F_2 *are in* $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ *while the Lyapunov function V verifies* $V \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, *then, the crossing times S and T of* (2.3) *verify* $T, S \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

Proof. We will consider the Banach spaces X, Y and Z together with the open set $\mathcal{O} \subseteq X \times Y$ and the functions ν and $\phi := \nu|_{\mathcal{O}}$ of the proof of Lemma 4.1.

By [22, Cor. 6.1, p. 92], $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$ is of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. In addition, ν is a composition of $\mathscr{C}^k$-maps, which implies that $\nu \in \mathscr{C}^k(X \times Y, Z)$, and we conclude that $\phi \in \mathscr{C}^k(\mathcal{O}, Z)$.

By (4.5), we have that $\phi(\tau_0, \xi_0, T(\tau_0, \xi_0)) = 0$ while $D_t\phi(\tau_0, \xi_0, T(\tau_0, \xi_0)) \neq 0$ by (4.6). Hence, the implicit function theorem for maps in $\mathscr{C}^k$ [22, Cor. 5.1, p. 84] implies the existence of $\varphi : U_0 \to V_0$ of class $\mathscr{C}^k$, where U_0 and V_0 are neighborhoods of (τ_0, ξ_0) and $T(\tau_0, \xi_0)$, respectively, and ϕ verifies

$$\phi(\tau, \xi, \varphi(\tau, \xi)) = 0 \quad \text{for any } (\tau, \xi) \in U_0.$$

Moreover, the definition of T and the above property of φ establish that

$$V(\varphi(\tau, \xi), x(\varphi(\tau, \xi), \tau, \xi)) = 1 = V(T(\tau, \xi), x(T(\tau, \xi), \tau, \xi)),$$

and the uniqueness of T implies that $\varphi(\tau, \xi) = T(\tau, \xi)$ for any $(\tau, \xi) \in U_0$. Then, T has continuous k-th derivatives on any $(\tau_0, \xi_0) \in \mathbb{R} \times \mathbb{R}_0^n$, which implies that $T \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

By using $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$, F_2 and S instead of $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$, F_1 and T respectively, we have that $S \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$. $\qquad\square$

Theorem 5.2. *If the systems* (1.1) *and* (1.2) *verify* **(P1)–(P4)**, F_1, F_2 *are in* $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ *and the Lyapunov function also verifies that* $V \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, *then the Palmer's homeomorphism* H_τ *is a diffeomorphism of class* $\mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$ *for any fixed* $\tau \in \mathbb{R}$.

Proof. The crossing time functions S and T are of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$ by Lemma 5.1, whose proof used that $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$ and $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$ are of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

A byproduct of the above facts combined with (2.4) is that, for any fixed $\tau \in \mathbb{R}$, the map $0 \neq \xi \mapsto H_\tau(\xi) = y(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi))$ can be seen as a composition of function of class $\mathscr{C}^k$, which leads to $H_\tau := H(\tau, \cdot) \in \mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

Similarly, we have that $0 \neq \xi \mapsto G_\tau(\xi) = x(\tau, S(\tau, \xi), y(S(\tau, \xi), \tau, \xi))$ is a composition of $\mathscr{C}^k$-functions, leading to $G_\tau := G(\tau, \cdot) \in \mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$. This implies that each H_τ is a diffeomorphism of class $\mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$. $\qquad\square$

As a byproduct of the above result, we can study the smoothness properties for the Lin's homeomorphism:

Corollary 5.3. *Under the assumptions of Lemma 3.1, if the maps* $(t, \xi) \mapsto C(t)\xi$, $(t, \xi) \mapsto B(t)\xi$ *and* g *are of class* $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, *the crossing times* T *and* S *of* (2.10) *are of class* $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

Moreover, for all $\tau \in \mathbb{R}$ *fixed, the Lin's homeomorphism* H_τ *is a diffeomorphism of class* $\mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

Proof. Let us define the functions $F_1, F_2 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_1(t, \xi) = C(t)\xi + B(t)\xi + g(t, \xi), \quad \text{and} \quad F_2(t, \xi) = -\frac{\delta}{2}I\xi \quad \text{for any } (t, \xi) \in \mathbb{R} \times \mathbb{R}^n,$$

and we will verify the properties **(P1)–(P4)**.

Firstly, note that $F_1(t, 0) = F_2(t, 0) = 0$ for any $t \in \mathbb{R}$. In fact, one identity is trivial while the other one is by **(L3)**, then, **(P1)** follows.

To verify **(P2)**, note that $t \mapsto C(t)$ is a bounded matrix function by **(L1)**, then, we can define $M = \sup_{t \in \mathbb{R}} \|C(t)\|$ and $L := M + \frac{\delta}{2}$. Now, by **(L1)–(L3)**, we can deduce that for any $\xi_1, \xi_2 \in \mathbb{R}^n$ and $t \in \mathbb{R}$

$$\|F_1(t, \xi_1) - F_1(t, \xi_2)\| \leq L\|\xi_1 - \xi_2\| \quad \text{and} \quad \|F_2(t, \xi_1) - F_2(t, \xi_2)\| \leq L\|\xi_1 - \xi_2\|$$

and we have verified **(P2)**.

We will see that **(P3)** and **(P4)** are verified if we consider $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(t, \xi) := \|\xi\|^2 = \sum_{i=1}^{n} \xi_i^2.$$

In fact, if $C_1 = C_2 = 1$ and $\beta = 2$, we have that

$$C_1 \|\xi\|^\beta \leq \|\xi\|^2 = V(t, \xi) \leq C_2 \|\xi\|^\beta$$

for any $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, this proves **(P3)**.

The last step consists in verify the existence of $\eta > 0$ such that

$$DV(t, \gamma(t)) \leq -\eta \|\gamma(t)\|^\beta,$$

for any solution $t \mapsto \gamma(t)$ either of (2.6) or (2.7). But, let us define $\eta = \delta > 0$. By the proof of [14, Proposition 7, p. 41], any solution $t \mapsto y(t, \tau, \xi)$ of (2.7) satisfies

$$DV(t, y(t, \tau, \xi)) = \frac{d}{dt}\left(\|y(t, \tau, \xi)\|^2\right) \leq -\delta \|y(t, \tau, \xi)\|^2,$$

and any solution $t \mapsto x(t, \tau, \xi)$ can be written as a solution of (2.7) with the maps

$$C_0(t) = -\delta I, \quad B_0(t) = \frac{\delta}{4} I \quad \text{and} \quad g_0(t, \xi) = \frac{\delta}{4} I\xi \quad \text{for any } (t, \xi) \in \mathbb{R} \times \mathbb{R}^n,$$

then

$$DV(t, x(t, \tau, \xi)) \leq -\delta \|x(t, \tau, \xi)\|^2,$$

and we proved **(P4)**.

Now, the function F_1 is a sum of functions of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ which implies that $F_1 \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

On the order hand, we have that $F_2 = -\frac{\delta}{2} I\xi$ belongs to $\mathscr{C}^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ which leads to $F_2 \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

In addition, we have that V is a quadratic polinomial map of n variables, then $V \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and we have that T and S are in $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n_0, \mathbb{R})$ by Lemma 5.1.

Finally, the Lin's homeomorphism can be seen as a Palmer homeomorphism's between the systems (2.6) and (2.7) by Remark 2.4, then the Lin's homeomorphism is a diffeomorphism of class $\mathscr{C}^k(\mathbb{R}^n_0, \mathbb{R}^n_0)$ by the Theorem 5.2. $\square$

6 A generalization of Theorem 3.3

Under the assumption that the properties **(L1)** and **(L2)** are verified, let us consider the diagonal dominant linear system:

$$\dot{y} = [C(t) + B(t)]y. \tag{6.1}$$

By the variation of parameters method, we have that any solution $t \mapsto x(t, t_0, \xi)$ of (6.1) passing by ξ at $t = t_0$ verifies

$$x(t, t_0, \xi) = \Phi_C(t, t_0)\xi + \int_{t_0}^t \Phi_C(t, s)B(s)x(s, t_0, \xi)\, ds,$$

where $t \mapsto \Phi_C(t, s)$ is a transition matrix of $z' = C(t)z$.

On the other hand, we have that $x(t, t_0, \xi) = \Phi_{C+B}(t, t_0)\xi$ where $\Phi_{B+C}(t, t_0)$ is a transition matrix of (6.1). In addition, by considering

$$\Phi_C(t, s) = \mathrm{Diag}\left\{ e^{\int_s^t c_{ii}(\tau)\, d\tau} \right\}_{i=1}^n,$$

and **(L1)**, we can deduce that $\|\Phi_C(t, s)\| \leq e^{-\delta(t-s)}$ for any $t \geq s$. The estimate of Φ_C combined with **(L2)** imply that

$$e^{\delta t}\|\Phi_{B+C}(t, t_0)\xi\| = e^{\delta t}\|x(t, t_0, \xi)\| \leq e^{\delta t_0}\|\xi\| + \int_{t_0}^t \frac{\delta}{4} e^{\delta s}\|x(s, t_0, \xi)\|\, ds$$

$$= e^{\delta t_0}\|\xi\| + \int_{t_0}^t \frac{\delta}{4} e^{\delta s}\|\Phi_{C+B}(s, t_0)\xi\|\, ds$$

then, by using the classical Gronwall's inequality, it is straightforward to infer that

$$\|\Phi_{B+C}(t, t_0)\| \leq e^{-\frac{\delta}{2}(t-t_0)} \quad \text{for any } t \geq t_0, \tag{6.2}$$

namely, the linear system (6.1) is $\mathbb{R}$-uniformly exponentially stable.

Similarly, as it was stated in the subsection 2.2, the properties **(L1)**–**(L3)** imply that any solution $t \mapsto y(t, t_0, y_0)$ of (2.7) verifies

$$\|y(t, t_0, y_0)\| \leq \|y_0\| e^{-\frac{\delta}{2}(t-t_0)} \quad \text{for any } t \geq t_0.$$

A nice consequence of the Lin's homeomorphism is that the linear system (6.1) is topologically equivalent to its quasilinear perturbation (2.7) by the proof of [14, Lemma 2].

A direct byproduct of our previous results is the smoothness of the above mentioned topological equivalence

Theorem 6.1. *If* **(L1)**–**(L3)** *are satisfied and the maps* $(t, y) \mapsto g(t, y)$, $(t, y) \mapsto C(t)y$ *and* $(t, y) \mapsto B(t)y$ *belong to* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, *then the linear system* (6.1) *and its quasilinear perturbation* (2.7) *are* $\mathbb{R}$-*topologically equivalent via a function* $P \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *which, for any fixed t, is a preserving orientation diffeomorphism of class* $\mathscr{C}^1$ *on* $\mathbb{R}_0^n$.

Proof. By Theorem 3.3 and Corollary 3.4, the quasilinear system (2.7) and the diagonal autonomous system (2.6) are $\mathbb{R}$-topologically equivalent via the function $H \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ described by (2.8), which is a preserving orientation diffeomorphism of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$ for any fixed t.

On the other hand, as pointed out by Lin in [14], the function $(t, x) \mapsto g(t, x) \equiv 0$ satisfies **(L3)** and belongs to $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, then, the linear system (6.1), and the diagonal autonomous system (2.6), are $\mathbb{R}$-topologically equivalent via a function $Q \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which for any fixed t, is a preserving orientation diffeomorphism of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$.

Finally, as the $\mathbb{R}$-topological equivalence is an equivalence relation; it follows that the linear system (6.1) and its quasilineal perturbation (2.7), are $\mathbb{R}$-topologically equivalent via a function $P = H \circ Q \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which, for any fixed t, is a composition of two preserving orientation diffeomorphisms of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$. $\qquad\square$

It is of interest to point out that the Theorem 6.1 has a similar structure that a result obtained in [6] by following a Green's function approach. Both results are about the smoothness of a $\mathbb{R}$-topological equivalence between a uniformly exponentially stable linear system and a quasilinear perturbation. Now, it is interesting for us to describe the advantages and drawbacks of our result compared with the one obtained in [6].

A difference at first glance is that the Theorem 6.1 does not assumes that the quasilinear perturbation g is bounded on $\mathbb{R} \times \mathbb{R}^n$. For example, a linear perturbation $g(t, x) = H(t)x$ with $\|H(t)\| \leq \frac{\delta}{4}$ is covered by our result when $t \mapsto H(t)$ is continuously differentiable. On the other hand, in [6] the global boundedness of the perturbation is an essential assumption to the construction of the homeomorphism; nevertheless, Theorem 6.1 assumes that $g(t, 0) = 0$ for any $t \in \mathbb{R}$, which is not necessary in [6].

A second difference is that Theorem 6.1 allows an easier generalization to derivatives of higher order, which is not the case in [6]. From this perspective, our approach has a clear advantage.

Finally, the result of [6] only assumes that the linear part is a uniformly asymptotically stable linear system, while in Theorem 6.1 this assumption is restricted for the special case of linear systems with diagonal dominance, making our result a more restrictive one.

Acknowledgements

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