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Existence and uniqueness of Carathéodory and Filippov solutions for discontinuous systems of differential equations

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Abstract. We use essential limits inferior and superior of the nonlinear part of a discontinuous ODE to introduce some novel transversality conditions which imply that Filippov solutions are Carathéodory solutions. We also prove some uniqueness criteria based on different Lipschitz conditions on different parts of the domain separated from one another by boundaries which satisfy certain transversality conditions.

Keywords: discontinuous differential equations, Carathéodory solutions, Filippov solutions, differential inclusions.

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1 Introduction

Consider the initial value problem

$$x' = f(t,x), \qquad t \in I = [t_0, t_0 + L], \quad x(t_0) = x_0, \tag{1.1}$$

where $t_0, L \in \mathbb{R}$, $L > 0$, $x_0 \in \mathbb{R}^n$ ($n \in \mathbb{N}$) and $f = (f_1, f_2, \dots, f_n) : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ need not be continuous. In this paper, we prove new existence and uniqueness results on Carathéodory and Filippov solutions to (1.1).

Assume that for a.a. $t \in I$ the mapping $f(t, \cdot)$ is locally essentially bounded. A Filippov solution of (1.1) is defined as an absolutely continuous function $x : I \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$ and

$$x'(t) \in \bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\mathrm{co}}\, f\left(t, B_\varepsilon(x(t)) \setminus N\right) \quad \text{for a.a. } t \in I, \tag{1.2}$$

where m is the Lebesgue measure, $\overline{\mathrm{co}}$ means closed convex hull and $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$. Here and henceforth, we denote by $\|x\|$ the usual norm of a vector $x \in \mathbb{R}^n$. Observe that, in the scalar case ($n = 1$), we have $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

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Filippov solutions satisfy

$$x'(t) \in \prod_{j=1}^{n} \left[\operatorname*{ess\,lim\,inf}_{y \to x(t)} f_j(t,y), \operatorname*{ess\,lim\,sup}_{y \to x(t)} f_j(t,y) \right] \quad \text{for a.a. } t \in I, \tag{1.3}$$

where ess lim inf and ess lim sup stand for the essential limit inferior and superior, respectively. Namely, for each $j \in \{1, 2, \dots, n\}$,

$$\operatorname*{ess\,lim\,inf}_{y \to x} f_j(t,y) = \lim_{\varepsilon \to 0^+} \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f_j(t,y) = \sup_{\varepsilon>0} \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f_j(t,y).$$

The essential limit superior is defined analogously.

With this information on Filippov solutions we shall deduce some sufficient conditions for the existence of Carathédory solutions in terms of essential limits.

This paper is organized as follows. In Section 2 we prove (1.3). In Section 3 we introduce novel transversality conditions on $f(t,x)$ in terms of essential limits which ensure that, first, Filippov solutions of (1.1) exist and, second, every Filippov solution is a Carathéodory solution. In Section 4, we deduce new uniqueness results for both Carathéodory and Filippov solutions of (1.1).

2 Preliminaries

This section is mainly devoted to proving that Filippov solutions of (1.1) satisfy (1.3). We thank the anonymous reviewer of a previous version of this paper for having brought to our attention reference [9], where Filippov himself uses (1.3) omiting its proof.

Proposition 2.1. *Let $f = (f_1, f_2, \dots, f_n) : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an arbitrary function.*

For all $t \in I$ and every $x \in \mathbb{R}^n$ such that $f(t,\cdot)$ is essentially bounded on a neighborhood of x, we have

$$\bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\left(t, B_\varepsilon(x) \setminus N\right) \subset \prod_{j=1}^{n} \left[\operatorname*{ess\,lim\,inf}_{y \to x} f_j(t,y), \operatorname*{ess\,lim\,sup}_{y \to x} f_j(t,y) \right]. \tag{2.1}$$

Moreover, in the scalar case ($n = 1$) we have

$$\bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\left(t, B_\varepsilon(x) \setminus N\right) = \left[\operatorname*{ess\,lim\,inf}_{y \to x} f(t,y), \operatorname*{ess\,lim\,sup}_{y \to x} f(t,y) \right]. \tag{2.2}$$

Proof. For each $j \in \{1, 2, \dots, n\}$ and each sufficiently small $\varepsilon > 0$ there exist $c_*(j)$, $c^*(j) \in \mathbb{R}$, essential lower and upper bounds of the set

$$A_\varepsilon(j) = \{f_j(t,y) \,:\, 0 < \|x-y\| < \varepsilon\},$$

i.e., there exists a null measure set N_ε such that

$$c_*(j) \le f_j(t,y) \le c^*(j) \quad \text{provided that } 0 < \|x-y\| < \varepsilon,\, y \notin N_\varepsilon.$$

We may (and we do) assume that $x \in N_\varepsilon$ and that N_ε does not depend on j. Hence

$$\overline{\text{co}}\, f\left(t, B_\varepsilon(x) \setminus N_\varepsilon\right) \subset \prod_{j=1}^{n} [c_*(j), c^*(j)],$$

which implies that
$$\bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} [c_*(j), c^*(j)].$$
Since $c_*(j)$ and $c^*(j)$ were arbitrary essential lower and upper bounds of $A_\varepsilon(j)$, we deduce that
$$\bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} \left[\operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f_j(t,y), \operatorname*{ess\,sup}_{0<\|x-y\|<\varepsilon} f_j(t,y) \right].$$
Finally, since ε was fixed arbitrarily, we conclude that
$$\bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} \left[\text{ess} \liminf_{y\to x} f_j(t,y), \text{ess} \limsup_{y \to x} f_j(t,y) \right]. \tag{2.3}$$

Next we prove (2.2) in the scalar case. Notice that if
$$\xi \in \left[\text{ess} \liminf_{y\to x} f(t,y), \text{ess} \limsup_{y\to x} f(t,y) \right]$$
then for each $\varepsilon > 0$ and each $N \subset \mathbb{R}$, $m(N) = 0$, we have
$$\xi \geq \text{ess} \liminf_{y\to x} f(t,y) \geq \operatorname*{ess\,inf}_{B_\varepsilon(x)} f(t,y) \geq \inf_{B_\varepsilon(x)\setminus N} f(t,y),$$
and, analogously,
$$\xi \leq \sup_{B_\varepsilon(x)\setminus N} f(t,y).$$
Hence, for each $\varepsilon > 0$ and each null measure set $N \subset \mathbb{R}$, we have
$$\xi \in \left[\inf_{B_\varepsilon(x)\setminus N} f(t,y), \sup_{B_\varepsilon(x)\setminus N} f(t,y) \right] = \overline{\text{co}} f\,(t, B_\varepsilon(x) \setminus N)\,,$$
as desired. □

In applications we shall often have some more assumptions on $f(t, x)$, which yield a clearer version of (2.2). An interesting particular case is considered in our next lemma.

Lemma 2.2. *Assume that $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies that for a.a. $t \in I$ there is a null measure set $N(t)$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous.*

Then, for a.a. $t \in I$ and every $x \in \mathbb{R}^n$ such that $f(t,\cdot)$ is essentially bounded on a neighborhood of x, we have
$$\text{ess} \liminf_{y\to x} f(t,y) = \liminf_{y\to x,\, y\notin N(t)} f(t,y), \tag{2.4}$$
and
$$\text{ess} \limsup_{y\to x} f(t,y) = \limsup_{y\to x,\, y\notin N(t)} f(t,y). \tag{2.5}$$

Proof. Let us prove (2.4). The proof of (2.5) is analogous and we omit it.

Les us fix $t \in I$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous and $N(t)$ is null. Observe that (2.4) is obviously true in case $x \in \mathbb{R}^n \setminus N(t)$ because the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous at x, so we assume that $x \in N(t)$.

By definition, we have

$$\operatorname*{ess\,lim\,inf}_{y\to x} f(t,y) = \lim_{\varepsilon\to 0^+} \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f(t,y),$$

so it suffices to check that for all sufficiently small $\varepsilon > 0$ we have

$$\eta := \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f(t,y) = \inf_{0<\|x-y\|<\varepsilon,\, y\notin N(t)} f(t,y) =: \iota. \tag{2.6}$$

Take any $\varepsilon > 0$ such that $f(t,\cdot)$ is essentially bounded on $B_\varepsilon(x)$. Clearly, ι is an essential lower bound for the set $\{f(t,y) : 0 < \|x-y\| < \varepsilon\}$, hence $\iota \le \eta$.

Now assume, reasoning by contradiction, that $\iota < \eta$. The definition of ι guarantees that we can find $y_0 \in \mathbb{R}^n \setminus N(t)$, $0 < \|x-y_0\| < \varepsilon$, such that

$$\iota \le f(t,y_0) < \eta.$$

Since the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous at y_0, there is a neighborhood of y_0 relative to $\mathbb{R}^n \setminus N(t)$ of points y satisfying $0 < \|x-y\| < \varepsilon$ for which we have $f(t,y) < \eta$, so η cannot be an essential lower bound for the set $\{f(t,y) : 0 < \|x-y\| < \varepsilon\}$, a contradiction. The proof of (2.6) is complete. □

3 Existence of Carathéodory solutions

In this section we use a deep existence result due to Filippov [10, Theorem 8, page 85] along with Proposition 2.1 and Lemma 2.2, to establish a new existence result of Carathéodory solutions for (1.1). We recall that a Carathéodory solution of (1.1) is an absolutely continuous function $x : I \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$ and $x'(t) = f(t,x(t))$ for a.a. $t \in I$.

For the convenience of the reader, we gather the main ingredients we need from [10, Theorem 8, page 85] in the following proposition.

Proposition 3.1. *Assume that $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the following conditions.*

(i) The function $f(t,x)$ is measurable;

(ii) There exists $\psi \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}^n$ we have

$$|f(t,x)| \le \psi(t).$$

Then, problem (1.1) has at least one Filippov solution.

We shall also employ the following result, which follows from [3, Lemma 5.8.13].

Lemma 3.2. *Let a, $b \in \mathbb{R}$, $a < b$. If $\varphi : [a,b] \longrightarrow \mathbb{R}$ is almost everywhere differentiable on $[a,b]$, then for each null measure set $A \subset \mathbb{R}$ there exists a null measure set $B \subset \varphi^{-1}(A)$ such that*

$$\varphi'(t) = 0 \quad \text{for all } t \in \varphi^{-1}(A) \setminus B.$$

We are now in a position to prove a result on the existence of Carathéodory solutions for (1.1).

Theorem 3.3. *In the conditions of Proposition 3.1, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k, b_k] \subset I$, such that for a.a. $t \in I$ the following conditions hold:*

(a) *There exists a null measure set $N(t) \subset \mathbb{R}^n$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous;*

(b) *For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, $\tau_k(t,x) \in A_k$, and*

$$\nabla \tau_k(t,x) \cdot (1,z) \neq 0 \quad \text{for all } z \in \prod_{j=1}^{n} \left[\liminf_{y \to x,\, y \notin N(t)} f_j(t,y), \limsup_{y \to x,\, y \notin N(t)} f_j(t,y) \right]. \tag{3.1}$$

Then, problem (1.1) has at least one Carathéodory solution, which is also a Filippov solution.

Proof. By virtue of Proposition 2.1, problem (1.1) has at least one Filippov solution $x : I \longrightarrow \mathbb{R}^n$, which, according to (1.2), (2.1) and (2.4)–(2.5), satisfies

$$x'(t) \in \prod_{j=1}^{n} \left[\liminf_{y \to x(t),\, y \notin N(t)} f_j(t,y), \limsup_{y \to x(t),\, y \notin N(t)} f_j(t,y) \right] \quad \text{for a.a. } t \in I.$$

We shall prove that x is a Carathéodory solution of (1.1).

Let $E \subset I$ be a null measure set such that, first, conditions (a) and (b) hold for all $t \in I \setminus E$, and, second,

$$x'(t) \in \prod_{j=1}^{n} \left[\liminf_{y \to x(t),\, y \notin N(t)} f_j(t,y), \limsup_{y \to x(t),\, y \notin N(t)} f_j(t,y) \right] \quad \text{for all } t \in I \setminus E.$$

Observe that for each $t \in I \setminus E$ such that $x(t) \notin N(t)$ condition (a) ensures that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous at $x(t)$ and therefore

$$x'(t) \in \prod_{j=1}^{n} \left[\liminf_{y \to x(t),\, y \notin N(t)} f_j(t,y), \limsup_{y \to x(t),\, y \notin N(t)} f_j(t,y) \right] = \{f(t,x(t))\}.$$

Hence, it suffices to prove that the set $J = \{t \in I \setminus E \,:\, x(t) \in N(t)\}$ is null.

We deduce from condition (b) that

$$J \subset \bigcup_{k \in \mathcal{C}} \{t \in [a_k, b_k] \setminus E \,:\, \tau_k(t, x(t)) \in A_k\},$$

so the proof is reduced to showing that each $J_k = \{t \in [a_k, b_k] \setminus E \,:\, \tau_k(t,x(t)) \in A_k\}$ is a null measure set. For an arbitrarily fixed $k \in \mathcal{C}$, we define $\varphi(t) = \tau_k(t, x(t))$ for all $t \in [a_k, b_k]$, so that $J_k \subset \varphi^{-1}(A_k)$ and it suffices to prove that $\varphi^{-1}(A_k)$ is null. Since $m(A_k) = 0$, Lemma 3.2 guarantees the existence of a set $B \subset \varphi^{-1}(A_k)$, with $m(B) = 0$, such that for every $t \in \varphi^{-1}(A_k) \setminus B$ we have $\varphi'(t) = 0$, i.e.

$$\frac{d}{dt} \tau_k(t, x(t)) = 0. \tag{3.2}$$

Let us prove that $\varphi^{-1}(A_k) \subset B \cup E$, thus showing that $\varphi^{-1}(A_k)$ is null. Reasoning by contradiction, we assume that there is some $t \in \varphi^{-1}(A_k)$ such that $t \notin B \cup E$, and then we can use the chain rule in (3.2) to deduce that

$$\nabla \tau_k(t, x(t)) \cdot (1, x'(t)) = 0,$$

a contradiction with condition (3.1). □

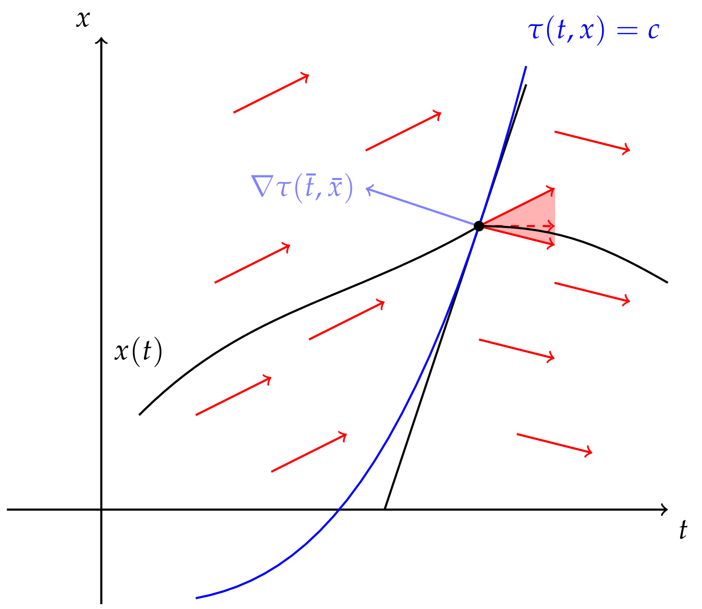

Figure 3.1: Visualization of the transversality condition (3.1) at $(\bar{t},\bar{x})=(\bar{t},x(\bar{t}))$.

Remark 3.4. Observe that, under the assumptions of Theorem 3.3, every Filippov solution of (1.1) is in fact a Carathéodory solution.

Note that, in general, Carathéodory solutions need not be Filippov solutions. Indeed, the constant function $x(t)\equiv 0$ is a Carathéodory solution of the initial value problem

$$x'=f(x)=\begin{cases}1, & \text{if } x\neq 0,\\ 0, & \text{if } x=0,\end{cases}\qquad t\in[0,1],\quad x(0)=0, \tag{3.3}$$

but it is not a Filippov solution. Readers can find in [15] a good account on the relations between Carathéodory and Filippov solutions.

The transversality condition (3.1) is based on an original idea by Bressan and Shen [6], later improved and applied by the authors in [12, 13]. All those papers assumed that

$$f(t,x)=F(t,g_1(\tau_1(t,x),x),g_2(\tau_2(t,x),x),\dots,g_N(\tau_N(t,x),x))\quad \text{for some } N\in\mathbb{N},$$

for functions F and g_k under suitable conditions, a technical drawback which we avoid in this paper.

Figure 3.1 can help readers to have a clearer intuition of what (3.1) means, at least in the very specific setting of one dimension and just one discontinuity curve $\tau(t,x)=c$. Note that vectors $(1,z)$ in condition (3.1) are represented as the red triangle in the figure, and condition (3.1) means that the red triangle cannot contain tangent vectors to $\tau(t,x)=c$ at $(\bar{t},\bar{x})$.

In addition, in [12, 13], we first look for Krasovskij solutions and then we use a transversality condition to prove that they are Carathéodory solutions. In this paper, we use Filippov solutions instead of Krasovskij's, thus getting a milder transversality condition in terms of essential limits. Both transversality conditions are compared in our next example.

Example 3.5. Consider the initial value problem (3.3). Theorem 3.3 ensures that (3.3) has at least one Carathéodory solution. Indeed, the function f is continuous on $\mathbb{R}\setminus\{0\}$ and so conditions (a) and (b) in Theorem 3.3 hold with $\mathcal{C} = \{1\}$, $N = A_1 = \{0\}$ and $\tau_1(t,x) = x$, since

$$\left[\liminf_{y\to 0} f(y), \limsup_{y\to 0} f(y)\right] = \{1\}.$$

The main results in [12, 13] do not apply because they are based on the larger Krasovskij envelope

$$\mathcal{K}f(x) := \left[\min\left\{f(x), \liminf_{y\to x} f(y)\right\}, \max\left\{f(x), \limsup_{y\to x} f(y)\right\}\right],$$

so the transversality condition in [12, 13], namely,

$$\nabla\tau_1(t,x)\cdot(1,z) = z \neq 0 \quad \text{for all } z \in \mathcal{K}f(0) = [0,1],$$

fails (at $z = 0$).

We also stress that the information provided by Proposition 2.1 and Lemma 2.2 concerning the Filippov envelope is useful in order to reduce the regularity required to the function f. Note that in the previous mentioned papers [12, 13], it was basically assumed that for a.a. $t \in I$ there exists a null measure set $N(t) \subset \mathbb{R}^n$ such that $f(t,\cdot)$ is continuous on $\mathbb{R}^n \setminus N(t)$, instead of the weaker assumption (a) in Theorem 3.3. We highlight that Theorem 3.3 can be even applied to functions f which are discontinuous at every point of its domain, as shown by the following example.

Example 3.6. Any planar system of the form

$$\begin{cases} x' = f_1(t,x,y), & x(0) = 0, \\ y' = f_2(t,x,y), & y(0) = 0, \end{cases}$$

where f_1 is continuous and bounded and f_2 is measurable, bounded and its restriction to $[0,L] \times (\mathbb{R}\setminus A) \times \mathbb{R}$ is continuous with A a null measure set, has at least one absolutely continuous solution defined in the interval $[0,L]$ provided that $f_1(t,x,y) \neq 0$ for all (t,x,y) such that $x \in A$.

Indeed, it suffices to apply Theorem 3.3 with $\mathcal{C} = \{1\}$, $\tau_1(t,x,y) = x$, $A_1 = A$ and $N(t) = A \times \mathbb{R}$. Note that the transversality condition (3.1) can be written in this case as

$$z_1 \neq 0 \quad \text{for all } (z_1,z_2) \in \prod_{j=1}^{2}\left[\liminf_{(u,v)\to(x,y),\, u\notin A} f_j(t,u,v), \limsup_{(u,v)\to(x,y),\, u\notin A} f_j(t,u,v)\right],$$

for each $(x,y) \in \mathbb{R}^2$ such that $x \in A$. Since f_1 is continuous,

$$\left[\liminf_{(u,v)\to(x,y),\, u\notin A} f_1(t,u,v), \limsup_{(u,v)\to(x,y),\, u\notin A} f_1(t,u,v)\right] = \{f_1(t,x,y)\},$$

and thus the conclusion follows from the fact that f_1 does not vanish at the points (t,x,y) with $x \in A$.

For instance, we can choose $f_1(t,x,y) = \cos^2(x\,y) + e^{t-x^2-y^2}$ and $f_2(t,x,y) = \varphi(x)e^{\sin(t+y)}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by

$$\varphi(x) = \chi_{\mathbb{Q}}(x) - \chi_{\mathbb{R}\setminus\mathbb{Q}}(x),$$

where χ_B denotes the characteristic function of the set $B \subset \mathbb{R}$. It is worth mentioning that φ is discontinuous at every point which, to the best of the authors' knowledge, falls outside the scope of earlier existence results. Observe, however that its restriction to the set $\mathbb{R}\setminus\mathbb{Q}$ is continuous and therefore Theorem 3.3 applies.

Our existence result applies for discontinuous ODE-systems associated with two-phase flows, that is, initial value problem (1.1) with a nonlinearity f which is discontinuous over a single hypersurface $\Sigma(t)$ defined as

$$\Sigma(t) := \{x \in \mathbb{R}^n \,:\, \tau(t,x) = 0\},$$

where $\tau : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable mapping. More precisely, let us consider an initial value problem of type

$$x' = f(t,x) = \begin{cases} f^+(t,x) & \text{if } x \in \Sigma^+(t), \\ f^-(t,x) & \text{if } x \in \Sigma^-(t), \end{cases} \qquad t \in I, \quad x(t_0) = x_0, \tag{3.4}$$

where

$$\Sigma^+(t) = \{x \in \mathbb{R}^n \,:\, \tau(t,x) > 0\} \text{ and } \Sigma^-(t) = \{x \in \mathbb{R}^n \,:\, \tau(t,x) < 0\},$$

and $f^\pm : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are L^1–bounded Carathéodory mappings. Note that the definition of f on $\Sigma(t)$ is not relevant in order to apply Theorem 3.3, so we may assume that either $f(t,x) = f^+(t,x)$ or $f(t,x) = f^-(t,x)$ on $\Sigma(t)$.

As a straightforward consequence of Theorem 3.3, we obtain the following existence result for (3.4).

Corollary 3.7. *Assume that for a.a. $t \in I$ and for each $x \in \Sigma(t)$ we have*

$$\nabla\tau(t,x) \cdot (1,z) \neq 0 \quad \textit{for all } z \in \overline{\text{co}}\left\{f^-(t,x), f^+(t,x)\right\}. \tag{3.5}$$

Then problem (3.4) has at least one Carathéodory solution.

Note that we need less regularity on $f^\pm$ than some related results (cf. Step 1 of [4, Theorem 1], where they are required to be locally Lipschitz continuous in x instead of merely continuous in x).

Let us now focus on the scalar case of (1.1), i.e., $n = 1$. By the implicit function theorem, if τ is regular enough, the discontinuity curve $\tau(t,x) = c$ can be seen, at least locally, as the graph of a time-dependent curve $x = \gamma(t)$ provided that $\frac{\partial \tau}{\partial x}(t,x) \neq 0$. Note that the transversality condition (3.1) implies that $\nabla\tau(t,x) \neq (0,0)$ over the discontinuity points of f which satisfy $\tau(t,x) = c$, where c belongs to a suitable null measure set.

With this in mind, we have the following alternative version of Theorem 3.3.

Corollary 3.8. *In the conditions of Proposition 3.1 and in the case $n = 1$, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\gamma_k : [a_k, b_k] \subset I \longrightarrow \mathbb{R}$ such that for a.a. $t \in I$ the following conditions hold:*

(a) the restriction of $f(t,\cdot)$ to the set $\mathbb{R} \setminus N(t)$ is continuous, where

$$N(t) = \bigcup_{\{k \in \mathcal{C} \,:\, t \in [a_k,b_k]\}} \bigcup_{c \in A_k} \{\gamma_k(t) + c\};$$

(b) for each $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, and each $c \in A_k$, we have either

$$\gamma_k'(t) < \liminf_{y \to \gamma_k(t),\, y \notin N(t)} f(t, y + c) \tag{3.6}$$

or

$$\gamma_k'(t) > \limsup_{y \to \gamma_k(t),\, y \notin N(t)} f(t, y + c). \tag{3.7}$$

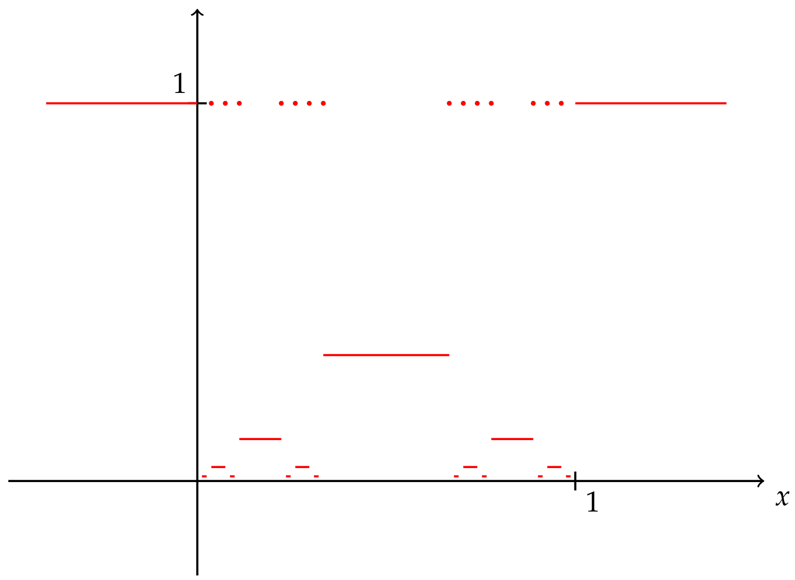

Figure 3.2: Approximate plot of $x \mapsto f(0,x)$, discontinuous at every point of Cantor's ternary set.

Then, problem (1.1) *has at least one Carathéodory solution.*

Proof. It suffices to apply Theorem 3.3 with $\tau_k(t,x) = x - \gamma_k(t)$. □

Example 3.9. Let C denote Cantor's ternary set. We have

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ if $n \neq m$.

Define $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ as $f(t,x) = b_n - a_n$ provided that $x + t \in (a_n, b_n)$ for some $n \in \mathbb{N}$, and $f(t,x) = 1$ otherwise. See Figure 3.2 for a plot of $x \mapsto f(0,x)$.

Corollary 3.8 guarantees that the corresponding initial value problem (1.1) (with $t_0 = x_0 = 0$ and $L = 1$) has at least one Carathéodory solution. To prove it, just define $\mathcal{C} = \{1\}$, $A_1 = C$, and $\gamma_1(t) = -t$ for $t \in [0,1]$. In this case, $N(t) = -t + C$ for all $t \in [0,1]$ and the restriction of $f(t,\cdot)$ to $\mathbb{R} \setminus (-t + C)$ is continuous; moreover, condition (3.6) holds for all $t \in [0,1]$.

Observe that the set $-t + C$ is not countable for any t, so discontinuities of f cannot be covered by countable unions of curves, as required, for instance, in Corollary 3.7 or Corollary 3.8 in [7]. Remarkably, solutions are increasing on $[0,1]$, so they cross every line $x + t = c$, $c \in C$.

The previous result does not cover some situations in which the set of Filippov solutions is a subset of that of Carathéodory solutions for (1.1). Indeed, one may easily verify that every Filippov solution of the initial value problem

$$x' = f(t,x) = \begin{cases} 0, & \text{if } x > t, \\ 1, & \text{if } x = t, \\ 2, & \text{if } x < t, \end{cases} \qquad t \in [0,1], \quad x(0) = 0, \tag{3.8}$$

is in fact a Carathéodory solution. Nevertheless, f is discontinuous over the line $x = \gamma(t) := t$, $t \in [0,1]$, which does not satisfy neither (3.6) nor (3.7), since

$$\gamma'(t) = 1 \in [0,2] = \left[\liminf_{y \to t} f(t,y), \limsup_{y \to t} f(t,y)\right].$$

Note that γ is a solution of the initial value problem (3.8).

In the following result, we admit that f be discontinuous over the graphs of a countable family of solutions of the differential equation $x' = f(t,x)$.

Proposition 3.10. *In the conditions of Proposition 3.1 and in the case $n = 1$, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, $j \in \mathcal{D} \subset \mathbb{N}$, and differentiable mappings $\gamma_k : [a_k, b_k] \subset I \longrightarrow \mathbb{R}$ and $\psi_j : [\tilde{a}_j, \tilde{b}_j] \subset I \longrightarrow \mathbb{R}$ such that for a.a. $t \in I$ the following conditions hold:*

(a) the restriction of $f(t,\cdot)$ to the set $\mathbb{R} \setminus N(t)$ is continuous, where

$$N(t) = N_1(t) \cup N_2(t), \qquad N_1(t) = \bigcup_{k \in \mathcal{C}} \bigcup_{c_k \in A_k} \{\gamma_k(t) + c_k\} \quad \text{and} \quad N_2(t) = \bigcup_{j \in \mathcal{D}} \{\psi_j(t)\};$$

(b) for each $k \in \mathcal{C}$ and each $c_k \in A_k$, the function γ_k satisfies that for a.a. $t \in [a_k, b_k]$ either (3.6) or (3.7) holds;

(c) for each $j \in \mathcal{D}$, $\psi_j'(t) = f(t, \psi_j(t))$ for a.a. $t \in [\tilde{a}_j, \tilde{b}_j]$.

Then, problem (1.1) has at least one Carathéodory solution.

Proof. It follows from Proposition 3.1 that problem (1.1) has at least one Filippov solution x. Let us prove that x is a Carathéodory solution.

It can be shown (just as in the proof of Theorem 3.3) that the set

$$J^\gamma = \bigcup_{k \in \mathcal{C}} \{t \in [a_k, b_k] : x(t) - \gamma_k(t) \in A_k\}$$

has Lebesgue measure zero. Suppose that there exists $j \in \mathcal{D}$ such that $m(J_j^\psi) > 0$, with

$$J_j^\psi = \{t \in [\tilde{a}_j, \tilde{b}_j] : x(t) = \psi_j(t)\} \quad \text{and} \quad J^\psi = \bigcup_{j \in \mathcal{D}} J_j^\psi.$$

Then $x'(t) = \psi_j'(t)$ for a.a. $t \in J_j^\psi$. By the definition of ψ_j, we have that $\psi_j'(t) = f(t, \psi_j(t))$ for a.a. $t \in J_j^\psi$ and thus $x'(t) = f(t, x(t))$ for a.a. $t \in J_j^\psi$. Hence, $x'(t) = f(t, x(t))$ for a.a. $t \in J^\psi$.

Finally, since the restriction of $f(t,\cdot)$ to $\mathbb{R} \setminus N(t)$ is continuous at $x(t)$ for a.a. $t \in I \setminus (J^\gamma \cup J^\psi)$, we conclude that x is a Carathéodory solution of (1.1). □

We shall say that both the functions γ_k satisfying (3.6) or (3.7) and the functions of type ψ_j are *admissible discontinuity curves*(cf. [8]).

Note that the previous result is sharp in the following sense: if there exists a differentiable function $\gamma : [t_0, t_1] \subset I \to \mathbb{R}$ which is not an admissible discontinuity curve and such that $\gamma(t_0) = x_0$ and for each $t \in [t_0, t_1]$, $f(t,\cdot)$ is discontinuous at $\gamma(t)$, then γ can be extended to a Filippov solution of (1.1) which is not a solution in the sense of Carathéodory.

4 Uniqueness of Carathéodory and Filippov solutions

In this section we show that a similar transversality condition to that employed in Theorem 3.3 can be used to deduce uniqueness of Filippov or Carathéodory solutions of (1.1). Note that recently Fjordholm [11] gave necessary and sufficient conditions for the uniqueness of Filippov solutions in the scalar autonomous case of (1.1), complementing the results for Carathéodory solutions due to Binding [2]. Our results concern non-autonomous discontinuous systems.

First, we use some transversality conditions to prove uniqueness of Filippov solutions of (1.1). Surprisingly enough, it does not guarantee uniqueness of Carathéodory solutions.

Basically, we assume that $f(t,x)$ satisfies a Lipschitz condition with respect to x in every gap delimited in $I \times \mathbb{R}^n$ by a set of hypersurfaces (not necessarily a finite set) where f may be discontinuous and where some suitable transversality conditions hold.

We need some notation for our first main result on uniqueness. Let $A \subset \mathbb{R}$ be a non-empty set. We will say that $x_0 \in A$ is a *left-isolated point*(*right-isolated point*) of A if there is $\delta > 0$ such that $(x_0 - \delta, x_0) \cap A = \emptyset$ ($(x_0, x_0 + \delta) \cap A = \emptyset$). In the sequel, we denote by $\mathcal{I}^-(A)$ the set of left-isolated points of A and by $\mathcal{I}^+(A)$ that of right-isolated points. Obviously, if $x_0 \in \mathcal{I}^-(A) \cap \mathcal{I}^+(A)$, it is an isolated point of the subset A.

Theorem 4.1. *In the conditions of Proposition 3.1, assume also that there exist continuously differentiable mappings $\tau_k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ ($k = 1, 2, \ldots, m$, $m \in \mathbb{N}$) and countable closed sets $A_k \subset \mathbb{R}$ satisfying that $A_k = \mathcal{I}^-(A_k) \cup \mathcal{I}^+(A_k)$ such that the following conditions hold:*

(i) For each $k \in \{1, 2, \ldots, m\}$ and each $(t,x) \in \tau_k^{-1}(A_k)$, there exists $\varepsilon > 0$ such that for a.a. $s \in (t, t+\varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have

$$\nabla \tau_k(s,\zeta) \cdot (1,z) > 0 \quad \textit{for all } z \in \prod_{j=1}^{n} \left[\operatorname{ess\,lim\,inf}_{y \to \zeta} f_j(s,y), \operatorname{ess\,lim\,sup}_{y \to \zeta} f_j(s,y) \right] \tag{4.1}$$

if $\tau_k(t,x) \in \mathcal{I}^+(A_k) \setminus \mathcal{I}^-(A_k)$;

$$\nabla \tau_k(s,\zeta) \cdot (1,z) < 0 \quad \textit{for all } z \in \prod_{j=1}^{n} \left[\operatorname{ess\,lim\,inf}_{y \to \zeta} f_j(s,y), \operatorname{ess\,lim\,sup}_{y \to \zeta} f_j(s,y) \right] \tag{4.2}$$

if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \setminus \mathcal{I}^+(A_k)$; and either (4.1) or (4.2) if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \cap \mathcal{I}^+(A_k)$.

(ii) For each connected component, $\mathcal{O}$, of the set $I \times \mathbb{R}^n \setminus \bigcup_{k=1}^m \tau_k^{-1}(A_k) = \bigcap_{k=1}^m \tau_k^{-1}(\mathbb{R} \setminus A_k)$, there exists $l \in L^1(I)$ such that for a.a. $t \in I$ and all x, y such that (t,x), $(t,y) \in \mathcal{O}$ we have

$$\|f(t,x) - f(t,y)\| \le l(t) \|x - y\|. \tag{4.3}$$

Then, problem (1.1) has exactly one Filippov solution.

Proof. We can assume, without loss of generality, that $A_k = \mathcal{I}^+(A_k)$ for every $k = 1, 2, \ldots, m$. Indeed, if for some k we have $\mathcal{I}^-(A_k) \neq \emptyset$, then we replace the set A_k by two sets, namely, $A_{k,1} = \mathcal{I}^+(A_k)$ (which satisfies (4.1) with τ_k) and $A_{k,2} = -\mathcal{I}^-(A_k)$ (which satisfies $A_{k,2} = \mathcal{I}^+(A_{k,2})$) and we define a new function $\tilde{\tau}_k = -\tau_k$. Now, condition (4.2) for $\mathcal{I}^-(A_k)$ and τ_k implies condition (4.1) for $A_{k,2}$ and $\tilde{\tau}_k$.

By virtue of Proposition 3.1, problem (1.1) has at least one Filippov solution. Let us prove uniqueness. Let $x(t)$ and $y(t)$ be Filippov solutions of (1.1); we shall prove that $x(t) = y(t)$ for all $t \in I$. Reasoning by contradiction, we assume that there exists some $t_1 \in [t_0, t_0 + L)$ and $\rho > 0$ such that $x(t_1) = y(t_1)$ and $\|x(t) - y(t)\| > 0$ for all $t \in (t_1, t_1 + \rho)$.

Let $z(t)$ be an arbitrary solution of (1.1) such that $z(t_1) = x(t_1)$. Observe that for all $t \in I$, $t \geq t_1$, we have

$$\|z(t) - x(t_1)\| \leq \int_{t_1}^{t} \|z'(r)\|\, dr \leq \int_{t_1}^{t} \psi(r)\, dr,$$

so for any $\varepsilon > 0$ there exists $\mu > 0$ (independent of the solution $z(t)$) such that $z(t) \in B_\varepsilon(x(t_1))$ for all $t \in [t_1, t_1 + \mu]$.

Let $k \in \{1, 2, \ldots, m\}$ be fixed. If $\tau_k(t_1, x(t_1)) \notin A_k$, then there exists an open interval $I_k \subset \mathbb{R} \setminus A_k$ such that $\tau_k(t_1, z(t_1)) = \tau_k(t_1, x(t_1)) \in I_k$ (observe that I_k does not depend on the specific solution $z(t)$). Since τ_k is continuous at $(t_1, x(t_1))$, there exists $r_k > 0$ (independent of the solution $z(t)$) such that $\tau_k(t, z(t)) \in I_k$ for all $t \in [t_1, t_1 + r_k)$, or, equivalently, $(t, z(t)) \in \tau_k^{-1}(I_k)$ for all $t \in [t_1, t_1 + r_k)$.

If $\tau_k(t_1, x(t_1)) =: a \in A_k = \mathcal{I}^+(A_k)$, then there is $\delta > 0$ such that $(a, a + \delta) \cap A_k = \emptyset$ and we define $I_k = (a, a + \delta)$ (which does not depend on the solution $z(t)$). Then, assumption (i) implies that there exists $\varepsilon > 0$ such that for a.a. $s \in (t_1, t_1 + \varepsilon)$ and $\zeta \in B_\varepsilon(x(t_1))$ we have

$$\nabla \tau_k(s, \zeta) \cdot (1, z) > 0 \quad \text{for all } z \in \prod_{j=1}^{n} \left[\operatorname*{ess\,lim\,inf}_{y \to \zeta} f_j(s, y), \operatorname*{ess\,lim\,sup}_{y \to \zeta} f_j(s, y) \right]. \tag{4.4}$$

Take a sufficiently small $r_k \in (0, \varepsilon)$ such that $z(t) \in B_\varepsilon(x(t_1))$ for all $t \in [t_1, t_1 + r_k)$. For a.a. $t \in [t_1, t_1 + r_k)$ we use the chain rule and (4.4) to deduce that

$$\frac{d}{dt} \tau_k(t, z(t)) = \nabla \tau_k(t, z(t)) \cdot (1, z'(t)) > 0,$$

because, as a Filippov solution, $z(t)$ satisfies (1.3). Note that the composition $\tau_k(\cdot, z(\cdot))$ is absolutely continuous, so the previous inequality implies that $\tau_k(t, z(t)) > a$ for all $t \in (t_1, t_1 + r_k)$.

Let $r = \min\{r_1, r_2, \ldots, r_m\}$; we have proven that for every $t \in (t_1, t_1 + r)$ we have

$$(t, z(t)) \in \bigcap_{k=1}^{m} \tau_k^{-1}(I_k),$$

and r does not depend on the specific solution $z(t)$ such that $z(t_1) = x(t_1)$.

We know from [10, Theorem 9] that the set of all Filippov solutions of (1.1) with initial condition $(t_1, x(t_1))$ is a connected subset of $\mathcal{C}([t_1, t_1 + r])$. Hence, for a fixed $t^* \in (t_1, t_1 + r)$ the set

$$S(t^*) = \{(t^*, z(t^*)) \,:\, z(t) \text{ solution}, z(t_1) = x(t_1)\}$$

is a connected subset of $I \times \mathbb{R}^n$. This implies that the set

$$S = \{(t, z(t)) \,:\, z(t) \text{ solution}, z(t_1) = x(t_1) \text{ and } t \in (t_1, t_1 + r)\}$$

is a connected subset of $I \times \mathbb{R}^n$ because it is the union of all the graphs $\{(t, z(t)) : t \in (t_1, t_1 + r)\}$ which are connected and each contains a point in $S(t^*)$. Therefore, S must be inside one of the connected components of $\bigcap_{k=1}^{m} \tau_k^{-1}(I_k)$, which is contained in a connected component of $I \times \mathbb{R}^n \setminus \bigcup_{k=1}^{m} \tau_k^{-1}(A_k)$. Now condition (ii) ensures the existence of some $l \in L^1(I)$ such that

$$\|x(t) - y(t)\| \leq \int_{t_1}^{t} \|f(s, x(s)) - f(s, y(s))\|\, ds \leq \int_{t_1}^{t} l(s) \|x(s) - y(s)\|\, ds, \quad t \in [t_1, t_1 + r],$$

and we deduce from Gronwall's inequality that $\|x - y\| = 0$ on $[t_1, t_1 + r]$, a contradiction. □

Remark 4.2. The mappings τ_k may be defined in $[a_k, b_k] \times \mathbb{R}^n$, with $[a_k, b_k] \subset I$, instead of the whole $I \times \mathbb{R}^n$. Indeed, the set $\{a_1, b_1, a_2, b_2, \ldots, a_m, b_m\}$ defines a partition of the interval $[t_0, t_0 + L]$ and it suffices to apply Theorem 4.1 in each subinterval defined by the partition in order to obtain uniqueness of Filippov solutions to (1.1).

Remark 4.3. The Lipschitz type condition (4.3) can be replaced, for instance, by one-sided Lipschitz, Osgood's or Montel–Tonelli's conditions (see [1]) in such a way that uniqueness of Filippov solutions is proven in a similar manner.

Observe that, in the hypotheses of Theorems 3.3 and 4.1, although existence of Carathéodory solutions for the initial value problem (1.1) is guaranteed, uniqueness cannot be ensured, as shown by the Cauchy problem (3.3). Hence, to obtain uniqueness of Carathéodory solutions, it is necessary to reinforce the assumptions on Theorem 4.1. Obviously, it would be sufficient to ensure that the set of Carathéodory solutions and the set of Filippov solutions coincide (as pointed out in Remark 3.4, Filippov solutions are Carathéodory solutions, so it only remains to ensure the reverse inclusion). In case of autonomous systems, some comparison between them can be found in [15]. In general, this can be directly obtained if one assumes that f is a selection of the Filippov envelope or, even less, if for a.e. t and all x,

$$\operatorname{ess}\liminf_{y \to x} f_j(t, y) \leq f_j(t, x) \leq \operatorname{ess}\limsup_{y \to x} f_j(t, y), \qquad j = 1, \ldots, n, \tag{4.5}$$

which provides uniqueness of Carathéodory solutions as a straightforward consequence of Theorem 4.1.

Corollary 4.4. *In the conditions of Theorem 4.1, if in addition f satisfies (4.5), then problem (1.1) has exactly one Carathéodory solution.*

This simple uniqueness criterion can be useful in practice. In particular, it enables us to establish uniqueness of Carathéodory solutions for discontinuous ODE-systems associated with two-phase flows, namely, initial value problems of type (3.4).

Corollary 4.5. *Let $f^{\pm} : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be L^1-bounded Carathéodory mappings. Assume that $f^{\pm}$ are Lipschitz continuous in x and that for a.a. $t \in I$ and for each $x \in \Sigma(t)$, there exists $\varepsilon > 0$ such that for all $s \in (t, t + \varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have either*

$$\nabla \tau(s, \zeta) \cdot (1, z) > 0 \quad \text{for all } z \in \overline{\text{co}} \left\{ f^-(s, \zeta), f^+(s, \zeta) \right\}$$

or

$$\nabla \tau(s, \zeta) \cdot (1, z) < 0 \quad \text{for all } z \in \overline{\text{co}} \left\{ f^-(s, \zeta), f^+(s, \zeta) \right\}.$$

Then problem (3.4) has a unique Carathéodory solution.

Once existence is guaranteed, uniqueness of both Carathéodory and Filippov solutions can be obtained simultaneously, without assuming (4.5), provided that there exists a unique Krasovskij solution of (1.1). To do so, we shall strengthen the transversality condition (i) in Theorem 4.1.

Theorem 4.6. *In the conditions of Theorems 3.3 and 4.1, assume that hypothesis (i) is replaced by the following one:*

(i) For each $k \in \{1, 2, \ldots, m\}$ and each $(t, x) \in \tau_k^{-1}(A_k)$, there exists $\varepsilon > 0$ such that for a.a. $s \in (t, t + \varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have*

$$\nabla \tau_k(s, \zeta) \cdot (1, z) > 0 \quad \text{for all } z \in \mathcal{K}f(s, \zeta) \tag{4.6}$$

if $\tau_k(t, x) \in \mathcal{I}^+(A_k) \setminus \mathcal{I}^-(A_k)$;

$$\nabla \tau_k(s, \zeta) \cdot (1, z) < 0 \quad \text{for all } z \in \mathcal{K}f(s, \zeta), \tag{4.7}$$

if $\tau_k(t, x) \in \mathcal{I}^-(A_k) \setminus \mathcal{I}^+(A_k)$; and either (4.6) or (4.7) if $\tau_k(t, x) \in \mathcal{I}^-(A_k) \cap \mathcal{I}^+(A_k)$ (where $\mathcal{K}f$ denotes the Krasovskij envelope of f).

Then problem (1.1) has exactly one Krasovskij solution, which is also the unique Carathéodory and Filippov solution of (1.1).

Proof. It follows, in an analogous way to the proof of Theorem 4.1, that the differential inclusion

$$x' \in \mathcal{K}f(t, x), \quad t \in I, \; x(t_0) = x_0, \tag{4.8}$$

has a unique solution.

To conclude, it suffices to observe that any Carathéodory, Filippov or Krasovskij solution of (1.1) is, in particular, an absolutely continuous solution of (4.8). □

Remark 4.7. Note that the Krasovskij envelope of the function f satisfies that

$$\begin{aligned}\mathcal{K}f(s, \zeta) &:= \bigcap_{\varepsilon > 0} \overline{\text{co}}\, f\,(s, B_\varepsilon(\zeta)) \\ &\subset \prod_{j=1}^{n} \left[\min\left\{ f_j(s, \zeta), \liminf_{y \to \zeta} f_j(s, y) \right\}, \max\left\{ f_j(s, \zeta), \limsup_{y \to \zeta} f_j(s, y) \right\} \right].\end{aligned}$$

Example 4.8. Let $\lfloor \cdot \rfloor$ be the floor function. The system

$$\begin{cases} x' = f_1(x, y), & x(0) = 0, \\ y' = f_2(x, y), & y(0) = 0, \end{cases}$$

with $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = \begin{cases} \left(5x + \dfrac{1}{\lfloor 1/(1 - x^2 - y^2) \rfloor}, 5y + \dfrac{1}{\lfloor 1/(1 - x^2 - y^2) \rfloor} \right), & \text{if } x^2 + y^2 < 1, \\ \left(5x\, e^{1 - x^2 - y^2}, 5y\, e^{1 - x^2 - y^2} \right), & \text{otherwise,} \end{cases}$$

has exactly one Carathéodory solution in any interval $[0, L]$ ($L > 0$).

Observe that Theorem 4.6 can be applied with $m = 1$, $\tau_1(x, y) = x^2 + y^2$ and the closed countable set $A_1 = \{1 - 1/(k+1) : k \in \mathbb{N}\} \cup \{1\}$ (clearly, $A_1 = \mathcal{I}^+(A_1)$ and for any open interval I the set $\tau_1^{-1}(I)$ is empty or connected). Indeed, in order to check condition (i*), notice that

$$\frac{1}{\lfloor 1/(1 - x^2 - y^2) \rfloor} = \frac{1}{j} \quad (j \in \mathbb{N}) \quad \text{if and only if} \quad 1 - \frac{1}{j} \le x^2 + y^2 < 1 - \frac{1}{j+1}.$$

Hence, for $j \in \mathbb{N}$, $j \geq 2$, and $x^2 + y^2 = 1 - 1/j$ fixed, we have that

$$\begin{aligned}\mathcal{K}f(x,y) &\subset \prod_{i=1}^{2} \left[\liminf_{(z_1,z_2)\to(x,y)} f_i(z_1,z_2), \limsup_{(z_1,z_2)\to(x,y)} f_i(z_1,z_2) \right] \\ &\subset \left[5x + \frac{1}{j}, 5x + \frac{1}{j-1} \right] \times \left[5y + \frac{1}{j}, 5y + \frac{1}{j-1} \right].\end{aligned}$$

It follows that for $(z_1, z_2) \in \mathcal{K}f(x,y)$,

$$\nabla\tau_1(x,y) \cdot (z_1,z_2) = (2x,2y) \cdot (5x + a_j, 5y + b_j) = 10(x^2+y^2) + 2(xa_j + yb_j)$$

where $a_j, b_j \in [1/j, 1/(j-1)]$. Then

$$\nabla\tau_1(x,y) \cdot (z_1,z_2) \geq 10\left(1 - \frac{1}{j}\right) - 2\left(|x| + |y|\right)\frac{1}{j-1} > 10\left(1 - \frac{1}{j}\right) - 4 \geq 1$$

for all $(z_1, z_2) \in \mathcal{K}f(x,y)$.

Note that f is continuous at any (x,y) such that $x^2 + y^2 = 1$, so $\mathcal{K}f(x,y) = \{f(x,y)\}$ and

$$\nabla\tau_1(x,y) \cdot (f_1(x,y), f_2(x,y)) = (2x,2y) \cdot (5x,5y) = 10.$$

Finally, by continuity, we deduce that condition (i*) holds. Moreover, f is Lipschitz continuous in the sets $\mathcal{O}_- = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $\mathcal{O}_+ = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ and

$$\mathcal{O}_j = \left\{ (x,y) \in \mathbb{R}^2 : 1 - \frac{1}{j} < x^2 + y^2 < 1 - \frac{1}{j+1} \right\} \quad (j \in \mathbb{N},\ j \geq 2).$$

Therefore, Theorem 4.6 is applicable and the conclusion follows.

As a consequence of Theorem 4.6, we can also deduce a discontinuous version of a classical uniqueness criterion due to Norris and Driver [14]. Observe that condition (c) is slightly stronger than condition (i^*) in Theorem 4.6 because we need to use Theorem 3.3.

Corollary 4.9. *Let $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a measurable function satisfying the following hypotheses:*

(a) There exist a constant $K > 0$ and functions $g_k : \mathbb{R} \longrightarrow \mathbb{R}$ and $\tau_k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$, for $k = 1, 2, \dots, m$, such that

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\| + K \sum_{k=1}^{m} |g_k(\tau_k(t,x)) - g_k(\tau_k(t,y))|$$

for all $(t,x), (t,y) \in I \times \mathbb{R}^n$.

(b) Each function $g_k : \mathbb{R} \longrightarrow \mathbb{R}$ is bounded in $\mathbb{R}$ and Lipschitz continuous in each bounded interval contained in $\mathbb{R} \setminus A_k$, where A_k is a countable closed subset of $\mathbb{R}$ such that $A_k = \mathcal{I}^-(A_k) \cup \mathcal{I}^+(A_k)$.

(c) Each function $\tau_k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuously differentiable and for each $(t,x) \in \tau_k^{-1}(A_k)$, there exists $\varepsilon > 0$ such that for a.a. $s \in [t, t+\varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have

$$\nabla\tau_k(s,\zeta) \cdot (1,z) > 0 \quad \text{for all } z \in \mathcal{K}f(s,\zeta) \tag{4.9}$$

if $\tau_k(t,x) \in \mathcal{I}^+(A_k) \setminus \mathcal{I}^-(A_k)$;

$$\nabla\tau_k(s,\zeta) \cdot (1,z) < 0 \quad \text{for all } z \in \mathcal{K}f(s,\zeta), \tag{4.10}$$

if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \setminus \mathcal{I}^+(A_k)$; and either (4.9) or (4.10) if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \cap \mathcal{I}^+(A_k)$ (where $\mathcal{K}f$ denotes the Krasovskij envelope of f).

Then the initial value problem

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{4.11}$$

has a unique local Carathéodory solution.

Proof. Existence follows from Theorem 3.3. Let us prove uniqueness as a consequence of Theorem 4.6.

Consider an arbitrary collection of open bounded intervals $I_k \subset \mathbb{R} \setminus A_k$, $k = 1, 2, \dots, m$, such that

$$\mathcal{O} = \bigcap_{k=1}^{m} \tau_k^{-1}(I_k) \neq \emptyset.$$

For each $k \in \{1, 2, \dots, m\}$, the function g_k is Lipschitz continuous in the interval I_k, so there is $L_k > 0$ such that

$$|g_k(z_1) - g_k(z_2)| \le L_k\, |z_1 - z_2| \quad \text{for all } z_1, z_2 \in I_k.$$

Hence, for all $(t,x), (t,y) \in \mathcal{O}$, we have that $\tau_k(t,x), \tau_k(t,y) \in I_k$ and thus

$$\sum_{k=1}^{m} |g_k(\tau_k(t,x)) - g_k(\tau_k(t,y))| \le \sum_{k=1}^{m} L_k\, |\tau_k(t,x) - \tau_k(t,y)| \le C\, \|x - y\|\,,$$

for some positive constant C, since the functions τ_k are Lipschitz continuous w.r.t. x on any compact set which contains the graphs of the solutions. Therefore, for all $(t,x), (t,y) \in \mathcal{O}$ we have

$$\|f(t,x) - f(t,y)\| \le (K + KC)\, \|x - y\|\,.$$

The conclusion follows now from Theorem 4.6 guaranteeing uniqueness of solutions for (4.11) in the interval $I = [t_0, t_0 + \delta]$. □

In view of the general assumptions on Theorem 3.3 concerning existence of Carathéodory solutions, one may wonder whether a Lipschitz continuous condition outside the set of discontinuities of f implies uniqueness of Carathéodory solutions of (1.1).

Theorem 4.10. *In the conditions of Proposition 3.1, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k, b_k] \subset I$, such that for a.a. $t \in I$ the following conditions hold:*

(a) There exists a null measure set $N(t) \subset \mathbb{R}^n$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is locally Lipschitz continuous, i.e., for each compact set $K \subset \mathbb{R}^n$ there exists $l_K \in L^1(I)$ such that for a.a. $t \in I$ and all $x, y \in K \cap (\mathbb{R}^n \setminus N(t))$ we have

$$\|f(t,x) - f(t,y)\| \le l_K(t) \|x - y\|.$$

(b) For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, $\tau_k(t,x) \in A_k$ and

$$\nabla \tau_k(t,x) \cdot (1, z) \neq 0 \quad \textit{for all } z \in \mathcal{K} f(t,x). \tag{4.12}$$

Then, problem (1.1) has exactly one Carathéodory solution, which is also the unique Filippov solution.

Proof. Existence follows from Theorem 3.3. For uniqueness, first note that if x is a Carathéodory solution, then it is a solution in the sense of Krasovskij and so the transversality condition (4.12) implies that

$$m\left(\{t \in I : x(t) \in N(t)\}\right) = 0.$$

Let $x(t)$ and $y(t)$ be Carathéodory solutions of (1.1); we shall prove that $x(t) = y(t)$ for all $t \in I$. Reasoning by contradiction, we assume that there exists some $t_1 \in [t_0, t_0 + L)$ such that $x(t_1) = y(t_1)$ and $\|x - y\| > 0$ on $(t_1, t_1 + \rho)$ for some $\rho > 0$. Note that $x(t), y(t) \in \mathbb{R}^n \setminus N(t)$ for a.a. $t \in (t_1, t_1 + \rho)$. Hence, there exists $l \in L^1(I)$ such that for a.a. $t \in (t_1, t_1 + \rho)$ we have

$$\|f(t, x(t)) - f(t, y(t))\| \le l(t)\|x(t) - y(t)\|.$$

Then

$$\|x(t) - y(t)\| \le \int_{t_1}^{t} \|f(s, x(s)) - f(s, y(s))\|\, ds \le \int_{t_1}^{t} l(s)\|x(s) - y(s)\|\, ds, \quad t \in [t_1, t_1 + \rho),$$

and we deduce from Gronwall's inequality that $\|x - y\| = 0$ on $[t_1, t_1 + \rho)$, a contradiction. □

Remark 4.11. Note that the transversality condition (4.12) cannot be replaced by (3.1) in Theorem 4.10 in order to ensure uniqueness of Carathéodory solutions for (1.1), as shown once again by Example 3.3. Nevertheless, it is enough to ensure uniqueness of Filippov solutions for (1.1).

Example 4.12. The planar system

$$\begin{cases} x' = f_1(t, x, y), & x(0) = 0, \\ y' = f_2(t, x, y), & y(0) = 0, \end{cases}$$

where $f_1(t, x, y) = \cos^2(x\,y) + e^{t - x^2 - y^2}$ and $f_2(t, x, y) = \left(\chi_{\mathbb{Q}}(x) - \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)\right) e^{\sin(t+y)}$, already considered in Example 3.6, has a unique Carathéodory solution. Indeed, the restriction of the function $f = (f_1, f_2)$ to $I \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$ is clearly locally Lipschitz continuous with respect to (x, y) and, moreover, condition (b) in Theorem 4.10 can be verified as in Example 3.6 since f_1 is continuous.

Unlike assumption (a) in Theorem 3.3, which is a reasonable condition to obtain existence for discontinuous ODEs, condition (a) in Theorem 4.10 imposes strong restrictions on the discontinuities of f (for instance, f cannot have jump discontinuities).

Finally, let us observe that the following simple result, which leans on local directional Lipschitz conditions in the line of [5], can be useful in many situations with discontinuous nonlinearities.

Theorem 4.13. *In the conditions of Proposition 3.1, assume also that for each $t \in [t_0, t_0 + L)$ and each $\xi \in \mathbb{R}^n$ there exist $\varepsilon = \varepsilon(t, \xi) > 0$ and $l = l_{(t,\xi)} \in L^1(I)$ such that for a.a. $s \in (t, t + \varepsilon)$ we have*

$$\|f(s, x) - f(s, y)\| \le l(s)\|x - y\| \quad \text{for all } x, y \in \prod_{j=1}^{n} \left[\xi_j - \int_t^s \psi(r)\, dr, \xi_j + \int_t^s \psi(r)\, dr\right]. \tag{4.13}$$

Then problem (1.1) has at most one Carathéodory solution.

Proof. Let $x(t)$ and $y(t)$ be Carathéodory solutions of (1.1); we shall prove that $x(t) = y(t)$ for all $t \in I$. Reasoning by contradiction, we assume that there exists some $t_1 \in [t_0, t_0 + L)$ such that $x(t_1) = y(t_1)$ and $\|x - y\| > 0$ on $(t_1, t_1 + \rho)$ for some $\rho > 0$.

Take $\varepsilon \in (0, \rho)$ and $l \in L^1(I)$ in the conditions of (4.13) for the point $(t, \xi) = (t_1, x(t_1))$. We have

$$\|x(t) - y(t)\| \le \int_{t_1}^{t} \|f(s, x(s)) - f(s, y(s))\| \, ds \le \int_{t_1}^{t} l(s) \|x(s) - y(s)\| \, ds, \quad t \in [t_1, t_1 + \varepsilon),$$

and then Gronwall's inequality yields $\|x - y\| = 0$ on $[t_1, t_1 + \varepsilon)$, a contradiction. □

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