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Oscillation criteria for perturbed half-linear differential equations

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Abstract. Oscillatory properties of perturbed half-linear differential equations are investigated. We make use of the modified Riccati technique. A certain linear differential equation associated with the modified Riccati equation plays an important part. Improved oscillation criteria for a perturbed half-linear Riemann–Weber differential equation can be obtained.

Keywords: half-linear differential equation, oscillation criteria, Riemann–Weber differential equation, principal solution.

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1 Introduction

In this paper we consider the second order half-linear ordinary differential equation

$$(p(t)\Phi_\alpha(x'))' + q(t)\Phi_\alpha(x) = 0, \quad t \geq t_0, \tag{1.1}$$

where $\Phi_\alpha(x) = |x|^\alpha \operatorname{sgn} x$ with $\alpha > 0$, $p(t)$ and $q(t)$ are real-valued continuous functions on $[t_0, \infty)$, and $p(t) > 0$ for $t \geq t_0$. If $\alpha = 1$, then (1.1) reduces to the linear equation

$$(p(t)x')' + q(t)x = 0, \quad t \geq t_0. \tag{1.2}$$

The half-linear equation (1.1) can be seen as a natural generalization of the linear equation (1.2).

For a solution $x(t)$ of (1.1), the vector function

$$(x(t), y(t)) = (x(t), p(t)\Phi_\alpha(x'(t)))$$

is a solution of the two-dimensional nonlinear system

$$x' = p(t)^{-1/\alpha}\Phi_{1/\alpha}(y), \quad y' = -q(t)\Phi_\alpha(x). \tag{1.3}$$

Conversely, for a solution $(x(t), y(t))$ of (1.3), the first component $x(t)$ is a solution of (1.1). The system of the type (1.3) was considered by Mirzov [11]. Using the result of Mirzov

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[11, Lemma 2.1], we see that all local solutions of (1.1) can be continued to t_0 and ∞, and so all solutions of (1.1) exist on the entire interval $[t_0, \infty)$. Analogues of Sturm's comparison theorem and Sturm's separation theorem remain valid for (1.1) (Mirzov [11, Theorem 1.1]). Hence, if the equation (1.1) has a nonoscillatory solution, then any other nontrivial solution is also nonoscillatory. If the equation (1.1) has an oscillatory solution, then any other nontrivial solution is also oscillatory. Clearly, if $x(t)$ is a solution of (1.1), then so is $-x(t)$. Therefore we can suppose without loss of generality that a nonoscillatory solution of (1.1) is eventually positive.

In the last three decades, many results have been obtained in the theory of oscillatory and asymptotic behavior of solutions of half-linear differential equations. It is known that basic results for the second order linear equations can be generalized to the second order half-linear equations. The important works are summarized in the book of Došlý and Řehák [8]. For the recent results to half-linear equations we refer the reader to, for example, [1–7,9,10,12–16]. The present paper is strongly motivated by oscillatory and nonoscilaltory results in [2–4,6,7,9].

For the equation (1.1), it is sometimes assumed that

$$\int_{t_0}^{\infty} p(s)^{-1/\alpha} ds = \lim_{t\to\infty} \int_{t_0}^{t} p(s)^{-1/\alpha} ds = \infty \tag{1.4}$$

and

$$\begin{cases} \displaystyle\int_{t_0}^{\infty} q(s)ds = \lim_{t\to\infty} \int_{t_0}^{t} q(s)ds \quad \text{is convergent, and} \\ \displaystyle\int_{t}^{\infty} q(s)ds \geq 0, \not\equiv 0 \quad \text{on} \quad [t_0^+, \infty) \quad \text{for any} \quad t_0^+ \geq t_0. \end{cases} \tag{1.5}$$

We point out that for any nonoscillatory solution $x(t)$ of (1.1) with (1.4) and (1.5) the derivative $x'(t)$ does not vanish eventually. More precisely,

Proposition 1.1. *Consider the equation* (1.1) *under the conditions* (1.4) *and* (1.5)*. Let $x(t)$ be a nonoscillatory solution of* (1.1) *such that $x(t) > 0$ for $t \geq T$ $(\geq t_0)$. Then, $x'(t) > 0$ for $t \geq T$.*

The above fact is easily deduced from the generalized Riccati integral equation associated with (1.1). See Lemma 2.3 in the next section.

Together with the equation (1.1), we consider the equation of the same type

$$(p(t)\Phi_\alpha(x'))' + q_0(t)\Phi_\alpha(x) = 0, \quad t \geq t_0, \tag{1.6}$$

where $q_0(t)$ is a real-valued continuous function on $[t_0, \infty)$. The equation (1.1) is regarded as a perturbation of the equation (1.6). In this paper it will be assumed that (1.6) has a nonoscillatory solution $x = x_0(t)$ such that

$$x_0(t) > 0, \quad x_0'(t) > 0 \quad \text{for } t \geq T \tag{1.7}$$

and

$$\int_{T}^{\infty} \frac{1}{p(t)x_0(t)^2 x_0'(t)^{\alpha-1}} dt = \infty. \tag{1.8}$$

The condition (1.8) is closely related to an integral characterization of the principal solution of (1.6). For the concept of the principal solution, see Došlý and Řehák [8, Section 4.2]. The following result is known.

Theorem 1.2 (Došlý and Elbert [5] and Došlá and Došlý [1, Proposition 2]). *Suppose that $x = x_0(t)$ is a nonoscillatory solution of (1.6) satisfying (1.7).*

(i) *Let $0 < \alpha \leq 1$. If (1.8) is satisfied, then $x_0(t)$ is the principal solution of (1.6).*

(ii) *Let $\alpha \geq 1$. If $x_0(t)$ is the principal solution of (1.6), then (1.8) holds.*

(iii) *Let $\alpha \geq 1$, and suppose that the conditions (1.4) and*

$$\int_{t_0}^{\infty} q_0(s)ds \quad \text{exists and} \quad \int_{t}^{\infty} q_0(s)ds \geq 0, \not\equiv 0 \quad \text{eventually} \tag{1.9}$$

are satisfied. Then, $x_0(t)$ is the principal solution of (1.6) if and only if (1.8) holds.

Note that the part (iii) of Theorem 1.2 is stated in [5, Theorem 3.3] and [8, Theorem 4.2.8] without the condition $\alpha \geq 1$. The part (iii) of Theorem 1.2 may fail to hold for $0 < \alpha < 1$ (see [1, Example 1]).

As an important oscillatory result the following theorem is known.

Theorem 1.3 (Došlý and Lomtatidze [7, Theorem 1]). *Suppose that the equation (1.6) is nonoscillatory and let $x = x_0(t)$ be the principal solution of (1.6) satisfying $x_0(t) > 0$ for $t \geq T$. If*

$$\int_{T}^{\infty} x_0(t)^{\alpha+1} [q(t) - q_0(t)] \, dt = \infty,$$

then the equation (1.1) is oscillatory.

Now let us consider the case where the equation (1.6) has a nonoscillatory solution $x = x_0(t)$ satisfying (1.7), (1.8) and

$$\int_{T}^{\infty} x_0(t)^{\alpha+1} [q(t) - q_0(t)] \, dt \quad \text{is convergent.} \tag{1.10}$$

It is not assumed that $x = x_0(t)$ is principal. Then we set

$$P(t) = p(t)x_0(t)^2 x_0'(t)^{\alpha-1} \quad \text{and} \quad Q(t) = x_0(t)^{\alpha+1} [q(t) - q_0(t)] \,. \tag{1.11}$$

Note that (1.7) implies $P(t) > 0$ ($t \geq T$).

The condition

$$\liminf_{t\to\infty} p(t)x_0(t)x_0'(t)^{\alpha} > 0 \tag{1.12}$$

also plays an important part. The following nonoscillatory result has been showed by Došlý and Fišnarová.

Theorem 1.4 (Došlý and Fišnarová [6, Theorem 3]). *Suppose that the equation (1.6) has a nonoscillatory solution $x = x_0(t)$ satisfying (1.7), (1.8) and (1.12). Suppose moreover that (1.10) holds. If there exists $\varepsilon > 0$ such that the linear equation*

$$(P(t)x')' + (1 + \varepsilon)\frac{\alpha + 1}{2\alpha} Q(t)x = 0 \tag{1.13}$$

is nonoscillatory, then the equation (1.1) is nonoscillatory.

The following corollary is obtained by applying the classical Hille–Nehari nonoscillation criterion to the linear equation (1.13).

Corollary 1.5 (Došlý and Fišnarová [6, Corollary 1 (i)]). *Suppose that* (1.6) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.7), (1.8) *and* (1.12). *Suppose moreover that* (1.10) *holds. If*

$$-\frac{3\alpha}{2(\alpha+1)} < \liminf_{t\to\infty}\left(\int_T^t \frac{1}{P(s)}ds\right)\left(\int_t^\infty Q(s)ds\right) \le \limsup_{t\to\infty}\left(\int_T^t \frac{1}{P(s)}ds\right)\left(\int_t^\infty Q(s)ds\right) < \frac{\alpha}{2(\alpha+1)},$$

then the equation (1.1) *is nonoscillatory.*

In this paper the following theorem will be proved.

Theorem 1.6. *Suppose that* $p(t)$ *and* $q(t)$ *satisfy* (1.4) *and* (1.5), *respectively. Suppose that the equation* (1.6) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.7), (1.8) *and* (1.12). *Suppose moreover that* (1.10) *holds. If there exists a number* ε *with* $0 < \varepsilon < 1$ *such that the linear equation*

$$(P(t)x')' + (1-\varepsilon)\frac{\alpha+1}{2\alpha}Q(t)x = 0 \tag{1.14}$$

is oscillatory, then the equation (1.1) *is oscillatory.*

Theorem 1.6 was proved in [3, Theorem 1] under the restricted condition that

$$\lim_{t\to\infty} p(t)x_0(t)x_0'(t)^\alpha \quad \text{exists and is a positive finite value.}$$

Theorem 1.6 gives a partial extension of Theorem 4 in [6]. Applying the classical Hille–Nehari oscillation criterion to the linear equation (1.14), we have the following corollary.

Corollary 1.7. *Suppose that* $p(t)$ *and* $q(t)$ *satisfy* (1.4) *and* (1.5), *respectively. Suppose that* (1.6) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.7), (1.8) *and* (1.12). *Suppose moreover that* (1.10) *holds. If*

$$\liminf_{t\to\infty}\left(\int_T^t \frac{1}{P(s)}ds\right)\left(\int_t^\infty Q(s)ds\right) > \frac{\alpha}{2(\alpha+1)}, \tag{1.15}$$

then the equation (1.1) *is oscillatory.*

Corollary 1.7 is a new result, while it is similar to Corollary 1 (ii) in [6].

Now, let

$$E(\alpha) = \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^\alpha, \quad \mu(\alpha) = \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^\alpha, \tag{1.16}$$

and

$$\log_0 t = t, \quad \log_k t = \log(\log_{k-1} t), \quad \mathrm{Log}_k t = \prod_{j=1}^k \log_j t \quad (k = 1,2,3,\ldots).$$

Then, consider the half-linear equation

$$(\Phi_\alpha(x'))' + \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^n \frac{1}{(\mathrm{Log}_j t)^2} + c(t)\right)\Phi_\alpha(x) = 0, \tag{1.17}$$

where $c(t)$ is a continuous function on an interval $[t_0, \infty)$ with sufficiently large t_0. The equation (1.17) is regarded as a perturbation of the half-linear Riemann–Weber (sometimes also called Euler–Weber) differential equation

$$(\Phi_\alpha(x'))' + \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^n \frac{1}{(\mathrm{Log}_j t)^2}\right)\Phi_\alpha(x) = 0. \tag{1.18}$$

It is known that (1.18) is nonoscillatory. Moreover, the asymptotic forms of (nonoscillatory) solutions of (1.18) are investigated by Elbert and Schneider [9, Corollary 1]. In this paper we pay attention to the fact that (1.18) has a nonoscillatory solution $x(t)$ such that

$$x(t) \sim t^{\alpha/(\alpha+1)}(\operatorname{Log}_n t)^{1/(\alpha+1)} \quad (t \to \infty). \tag{1.19}$$

We can prove the following theorem.

Theorem 1.8. *If*

$$\int_{t_0}^{\infty} t^{\alpha}(\operatorname{Log}_n t)c(t)dt = \infty, \tag{1.20}$$

then (1.17) *is oscillatory.*

The case $n = 1$ in Theorem 1.8 was obtained by Došlý [2, Corollary 1]. Theorem C in Elbert and Schneider [9] can be regarded as the case $n = 0$ in Theorem 1.8.

Next, consider the case where

$$\int_{t_0}^{\infty} t^{\alpha}(\operatorname{Log}_n t)c(t)dt \quad \text{is convergent.} \tag{1.21}$$

The following theorem is known.

Theorem 1.9 (Došlý [4, Theorem 3.3 (i)])**.** *Consider the equation* (1.17) *under the condition* (1.21)*. If*

$$\begin{aligned} -3\mu(\alpha) &< \liminf_{t\to\infty}\,(\log_{n+1} t)\int_t^{\infty} s^{\alpha}(\operatorname{Log}_n s)c(s)ds \\ &\le \limsup_{t\to\infty}\,(\log_{n+1} t)\int_t^{\infty} s^{\alpha}(\operatorname{Log}_n s)c(s)ds < \mu(\alpha), \end{aligned}$$

then (1.17) *is nonoscillatory.*

In the present paper the following theorem will be proved.

Theorem 1.10. *Consider the equation* (1.17) *under the condition* (1.21)*. If*

$$\liminf_{t\to\infty}\,(\log_{n+1} t)\int_t^{\infty} s^{\alpha}(\operatorname{Log}_n s)c(s)ds > \mu(\alpha), \tag{1.22}$$

then (1.17) *is oscillatory.*

Theorem 1.10 gives an improvement of Theorem 3.3 (ii) in [4].

Theorem 5 in [9] can be regarded as the case $n = 0$ in Theorems 1.9 and 1.10. Note that Theorem 5 in [9] is restricted to the case $n = 0$ and

$$\int_t^{\infty} s^{\alpha}c(s)ds \ge 0 \quad \text{for all large } t.$$

In the next section we state several basic (non)oscillatory results for the half-linear differential equation (1.1). The proofs are contained in the book of Došlý and Řehák [8]. For the proof of Theorem 1.6 we need some estimates for the function $F(u, v)$ which appears in the modified Riccati equation associated with (1.1). In Section 3 we state and prove the estimates for $F(u, v)$. The proof of Theorem 1.6 is given in Section 4. The proofs of Theorems 1.8 and 1.10 are presented in Section 5.

2 Basic results

For the convenience of the reader we summarize basic (non)oscillatory results for the half-linear differential equation (1.1). As usual, we use the asterisk notation

$$\xi^{\alpha *} = \Phi_\alpha(\xi) = |\xi|^\alpha \operatorname{sgn} \xi, \quad \xi \in \mathbb{R}, \quad \alpha > 0.$$

Then it is easy to see that, for $\xi, \eta \in \mathbb{R}$ and $\alpha, \alpha_1, \alpha_2 > 0$,

- $(\xi\eta)^{\alpha *} = \xi^{\alpha *}\eta^{\alpha *}, \quad (-\xi)^{\alpha *} = -\xi^{\alpha *};$
- $(\xi^{\alpha_1 *})^{\alpha_2 *} = \xi^{(\alpha_1\alpha_2)*}, \quad (\xi^{\alpha *})^{(1/\alpha)*} = \xi, \quad \left(\xi^{(1/\alpha)*}\right)^{\alpha *} = \xi;$
- $\xi^{\alpha *} \le \eta^{\alpha *}$ if and only if $\xi \le \eta$; $\xi^{\alpha *} < \eta^{\alpha *}$ if and only if $\xi < \eta$;
  $\xi^{\alpha *} = \eta^{\alpha *}$ if and only if $\xi = \eta$.

With this asterisk notation, the equation (1.1) is rewritten as

$$(p(t)(x')^{\alpha *})' + q(t)x^{\alpha *} = 0, \quad t \ge t_0.$$

Lemma 2.1. *The equation (1.1) is nonoscillatory if and only if there is a continuously differentiable function $y(t)$ which satisfies the generalized Riccati differential inequality*

$$y' + q(t) + \alpha p(t)^{-1/\alpha}|y|^{(\alpha+1)/\alpha} \le 0$$

on an interval $[T, \infty)$, $T \ge t_0$.

In what follows we consider the equation (1.1) under the condition (1.4). The next theorem is a half-linear extension of the classical Wintner oscillation criterion for (1.2).

Lemma 2.2. *Suppose that (1.4) holds. If*

$$\int_{t_0}^{\infty} q(t)dt = \infty,$$

then (1.1) is oscillatory.

As a next step we consider the case where (1.4) holds and

$$\int_{t_0}^{\infty} q(t)dt \quad \text{is convergent.} \tag{2.1}$$

Lemma 2.3. *Consider the equation (1.1) under the conditions (1.4) and (2.1). Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) > 0$ for $t \ge T$ $(\ge t_0)$. Then*

$$p(t)\left(\frac{x'(t)}{x(t)}\right)^{\alpha *} = \int_t^{\infty} q(s)ds + \alpha \int_t^{\infty} p(s)\left|\frac{x'(s)}{x(s)}\right|^{\alpha+1} ds, \quad t \ge T.$$

If the additional condition

$$\int_t^{\infty} q(s)ds \ge 0, \not\equiv 0 \quad \text{on} \quad [T^+, \infty) \quad \text{for any} \quad T^+ \ge T$$

is satisfied, then $x'(t) > 0$ for $t \ge T$.

The following lemma is the Hille–Nehari type (non)oscillation criteria for the equation (1.1).

Lemma 2.4. *Consider the equation* (1.1) *under the conditions* (1.4) *and* (2.1)*. Let $E(\alpha)$ be the constant defined by the former part of* (1.16)*.*

(i) *The equation* (1.1) *is nonoscillatory provided*

$$\begin{aligned} -(2\alpha+1)E(\alpha) &< \liminf_{t\to\infty}\left(\int_{t_0}^{t} p(s)^{-1/\alpha}ds\right)^{\alpha}\left(\int_{t}^{\infty} q(s)ds\right) \\ &\leq \limsup_{t\to\infty}\left(\int_{t_0}^{t} p(s)^{-1/\alpha}ds\right)^{\alpha}\left(\int_{t}^{\infty} q(s)ds\right) < E(\alpha). \end{aligned}$$

(ii) *The equation* (1.1) *is oscillatory provided*

$$\liminf_{t\to\infty}\left(\int_{t_0}^{t} p(s)^{-1/\alpha}ds\right)^{\alpha}\left(\int_{t}^{\infty} q(s)ds\right) > E(\alpha).$$

The results mentioned here are half-linear extensions of the classical results for the linear equation (1.2). For the proofs, see [8].

3 Lemmas

It is known that the function

$$F(u,v) = |u+v|^{(\alpha+1)/\alpha} - |v|^{(\alpha+1)/\alpha} - \frac{\alpha+1}{\alpha}v^{(1/\alpha)*}u, \quad u,v\in\mathbb{R}, \tag{3.1}$$

plays a crucial role in the study of the oscillation and nonoscillation of (1.1).

Lemma 3.1 (see, e.g., Došlý and Fišnarová [6, Lemma 4])**.** *Let $x = x(t)$ and $x = x_0(t)$ be nonoscillatory solutions of* (1.1) *and* (1.6)*, respectively. Suppose that $x(t) > 0$ and $x_0(t) > 0$ for $t \geq T$ $(\geq t_0)$. Then the function*

$$u(t) = p(t)x_0(t)^{\alpha+1}\left[\left(\frac{x'(t)}{x(t)}\right)^{\alpha*} - \left(\frac{x_0'(t)}{x_0(t)}\right)^{\alpha*}\right], \quad t\geq T, \tag{3.2}$$

is a solution of the modified Riccati differential equation

$$\begin{aligned} &u'(t) + x_0(t)^{\alpha+1}[q(t)-q_0(t)] \\ &\quad + \alpha p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F(u(t), p(t)x_0(t)x_0'(t)^{\alpha*}) = 0, \quad t\geq T, \end{aligned} \tag{3.3}$$

where $F(u,v)$ is defined by (3.1)*.*

Lemma 3.2. *Let $F(u,v)$ be the function which is defined by* (3.1)*.*

(i) *$F(u,v) \geq 0$ for all $u,v\in\mathbb{R}$; $F(u,v) = 0$ if and only if $u = 0$.*

(ii) *Let $k > 0$ be a constant. Then there are constants $L_1(k) > 0$ and $L_2(k) > 0$ such that*

$$L_1(k)|v|^{(1/\alpha)-1}u^2 \leq F(u,v) \leq L_2(k)|v|^{(1/\alpha)-1}u^2 \tag{3.4}$$

for $v > 0$ and $-v < u \leq kv$.

(iii) *Let k_1 and k_2 be constants satisfying $0 < k_1 < k_2$. Then there is a constant $L(k_1, k_2) > 0$ such that $F(u,v)$ can be expressed in the following form*

$$F(u,v) = \frac{\alpha+1}{2\alpha^2}|v|^{(1/\alpha)-1}u^2(1 + R(u,v))$$

with

$$|R(u,v)| \le \frac{|\alpha-1|}{3\alpha}L(k_1,k_2)|u|$$

for $v > 0$ and $|u| \le k_1 < k_2 \le v$.

Proof. It is obvious that $F(0,v) = 0$. Differentiating the function $F(u,v)$ with respect to u, we obtain

$$\begin{aligned}
F_u(u,v) &= \frac{\alpha+1}{\alpha}(u+v)^{(1/\alpha)*} - \frac{\alpha+1}{\alpha}v^{(1/\alpha)*}, \quad u,v \in \mathbb{R},\\
F_{uu}(u,v) &= \frac{\alpha+1}{\alpha^2}|u+v|^{(1/\alpha)-1}, \quad u > -v,\\
F_{uuu}(u,v) &= \frac{(\alpha+1)(-\alpha+1)}{\alpha^3}(u+v)^{[(1/\alpha)-2]*}, \quad u > -v.
\end{aligned}$$

Then, $F_u(0,v) = 0$ ($v \in \mathbb{R}$) and $F_{uu}(0,v) = [(\alpha+1)/\alpha^2]|v|^{(1/\alpha)-1}$ ($v > 0$).

(i) Let $v \in \mathbb{R}$ be fixed. It is seen that $F_u(u,v) < 0$ for $u < 0$, $F_u(u,v) > 0$ for $u > 0$ and $F_u(0,v) = 0$. This means that $F(u,v)$ is strictly decreasing on $(-\infty, 0)$ and $F(u,v)$ is strictly increasing on $(0,\infty)$. Then, since $F(0,v) = 0$, it is clear that $F(u,v) \ge 0$ for $u \in \mathbb{R}$ and $F(u,v) = 0$ if and only if $u = 0$.

(ii) Let $v > 0$ and $-v < u \le kv$. By Taylor's theorem with integral remainder we have

$$F(u,v) = F(0,v) + F_u(0,v)u + \int_0^u (u-s)F_{uu}(s,v)ds.$$

Hence

$$\begin{aligned}
F(u,v) &= \frac{\alpha+1}{\alpha^2}\int_0^u (u-s)|s+v|^{(1/\alpha)-1}ds\\
&= \frac{\alpha+1}{\alpha^2}\int_0^1 (u-u\sigma)|u\sigma+v|^{(1/\alpha)-1}u\,d\sigma\\
&= \frac{\alpha+1}{\alpha^2}|v|^{(1/\alpha)-1}u^2\int_0^1 (1-\sigma)\left|\frac{u}{v}\sigma+1\right|^{(1/\alpha)-1}d\sigma.
\end{aligned}$$

Then, noting

$$0 \le -\sigma+1 \le \frac{u}{v}\sigma+1 \le k\sigma+1 \quad (0 \le \sigma \le 1),$$

we find that, for the case $0 < \alpha \le 1$,

$$\begin{aligned}
\frac{\alpha}{\alpha+1} = \int_0^1 (1-\sigma)^{1/\alpha}d\sigma &\le \int_0^1 (1-\sigma)\left|\frac{u}{v}\sigma+1\right|^{(1/\alpha)-1}d\sigma\\
&\le \int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma;
\end{aligned}$$

and, for the case $\alpha > 1$,

$$\begin{aligned}
\int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma &\le \int_0^1 (1-\sigma)\left|\frac{u}{v}\sigma+1\right|^{(1/\alpha)-1}d\sigma\\
&\le \int_0^1 (1-\sigma)^{1/\alpha}d\sigma = \frac{\alpha}{\alpha+1}.
\end{aligned}$$

This shows that (3.4) holds with positive constants $L_1(k)$ and $L_2(k)$ such that, for the case $0 < \alpha \le 1$,

$$L_1(k) = \frac{1}{\alpha} \quad \text{and} \quad L_2(k) = \frac{\alpha+1}{\alpha^2}\int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma;$$

and, for the case $\alpha > 1$,

$$L_1(k) = \frac{\alpha+1}{\alpha^2}\int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma \quad \text{and} \quad L_2(k) = \frac{1}{\alpha}.$$

(iii) Let $v > 0$ and $|u| \le k_1 < k_2 \le v$. By Taylor's theorem there is θ such that $0 < \theta < 1$ and

$$F(u,v) = F(0,v) + F_u(0,v)u + \frac{F_{uu}(0,v)}{2!}u^2 + \frac{F_{uuu}(\theta u, v)}{3!}u^3.$$

Hence

$$\begin{aligned} F(u,v) &= \frac{\alpha+1}{2\alpha^2}|v|^{(1/\alpha)-1}u^2 + \frac{(\alpha+1)(-\alpha+1)}{6\alpha^3}(\theta u + v)^{[(1/\alpha)-2]*}u^3 \\ &= \frac{\alpha+1}{2\alpha^2}|v|^{(1/\alpha)-1}u^2\left[1 + \frac{-\alpha+1}{3\alpha}|v|^{-(1/\alpha)+1}(\theta u+v)^{[(1/\alpha)-2]*}u\right]. \end{aligned}$$

Notice here that

$$\theta u + v \ge -|u| + v \ge -k_1 + k_2 > 0 \quad (0 < \theta < 1)$$

and

$$0 < -\frac{k_1}{k_2} + 1 \le \frac{u}{v}\theta + 1 = \frac{\theta u + v}{v} \le \frac{k_1}{k_2} + 1 \quad (0 < \theta < 1).$$

Put

$$R(u,v) = \frac{-\alpha+1}{3\alpha}|v|^{-(1/\alpha)+1}(\theta u+v)^{[(1/\alpha)-2]*}u.$$

Then it is easy to see that

$$|R(u,v)| \le \frac{|\alpha-1|}{3\alpha}\left|\frac{\theta u+v}{v}\right|^{(1/\alpha)-1}|\theta u+v|^{-1}|u| \le \frac{|\alpha-1|}{3\alpha}L(k_1,k_2)|u|,$$

where $L(k_1,k_2)$ is given by

$$L(k_1,k_2) = \begin{cases} \left(1+\dfrac{k_1}{k_2}\right)^{(1/\alpha)-1}(k_2-k_1)^{-1} & (0<\alpha\le 1), \\ \left(1-\dfrac{k_1}{k_2}\right)^{(1/\alpha)-1}(k_2-k_1)^{-1} & (\alpha > 1). \end{cases}$$

This proves the part (iii) of Lemma 3.2. □

4 Proofs of the results

Proof of Theorem 1.6. Suppose that there is $\varepsilon \in (0,1)$ such that (1.14) is oscillatory. Assume, by contradiction, that the equation (1.1) has a nonoscillatory solution $x(t)$. We may suppose that $x(t) > 0$ for $t \ge T$. Then, we define the function $u(t)$ by (3.2). By Lemma 3.1, $u(t)$ satisfies (3.3). Integrating (3.3) from T to t, we obtain

$$\begin{aligned} &u(t) - u(T) + \int_T^t x_0(s)^{\alpha+1}[q(s)-q_0(s)]ds \\ &\quad + \alpha\int_T^t p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F(u(s), p(s)x_0(s)x_0'(s)^{\alpha})ds = 0 \end{aligned} \tag{4.1}$$

for $t \geq T$. Since the integrand of the last integral in the left-hand side of (4.1) is nonnegative for $t \geq T$ (see Lemma 3.2 (i)), we have either

$$\int_T^\infty p(s)^{-1/\alpha} x_0(s)^{-(\alpha+1)/\alpha} F(u(s), p(s)x_0(s)x_0'(s)^\alpha) ds = \infty \tag{4.2}$$

or

$$\int_T^\infty p(s)^{-1/\alpha} x_0(s)^{-(\alpha+1)/\alpha} F(u(s), p(s)x_0(s)x_0'(s)^\alpha) ds < \infty. \tag{4.3}$$

Suppose first that (4.2) holds. Since (1.10) is assumed to hold, it follows from (4.1) that $u(t) \to -\infty$ as $t \to \infty$. We may suppose that $u(t) < 0$ for $t \geq T$. By Lemma 2.3 we have $x'(t) > 0$ for $t \geq T$. Hence, by (3.2), we get

$$-p(t)x_0(t)x_0'(t)^\alpha = -p(t)x_0(t)^{\alpha+1}\left(\frac{x_0'(t)}{x_0(t)}\right)^{\alpha*} < u(t) \leq p(t)x_0(t)x_0'(t)^\alpha, \quad t \geq T.$$

Applying Lemma 3.2 (ii) to the case $k = 1$, $u = u(t)$ and $v = p(t)x_0(t)x_0'(t)^\alpha$, we find that there are constants $L_1 = L_1(1) > 0$ and $L_2 = L_2(1) > 0$ such that

$$\begin{aligned} &L_1 p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2 \\ &\quad \leq p(t)^{-1/\alpha} x_0(t)^{-(\alpha+1)/\alpha} F(u(t), p(t)x_0(t)x_0'(t)^\alpha) \\ &\quad \leq L_2 p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2, \quad t \geq T. \end{aligned}$$

Therefore, (3.3) yields

$$u'(t) + x_0(t)^{\alpha+1}[q(t) - q_0(t)] + \alpha L_1 p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2 \leq 0$$

for $t \geq T$, and (4.2) gives

$$\int_T^\infty p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2 dt = \infty.$$

Thus we obtain

$$u'(t) + Q(t) + \alpha L_1 P(t)^{-1} u(t)^2 \leq 0, \quad t \geq T, \tag{4.4}$$

and

$$\int_T^\infty P(t)^{-1} u(t)^2 dt = \infty. \tag{4.5}$$

Here the functions $P(t)$ and $Q(t)$ are given by (1.11).

Put

$$\varphi(t) = \int_T^t P(s)^{-1} ds, \quad t \geq T. \tag{4.6}$$

It follows from (4.4) that

$$\begin{aligned} &\int_T^t (\varphi(t) - \varphi(s))^2 u'(s) ds + \int_T^t (\varphi(t) - \varphi(s))^2 Q(s) ds \\ &\quad + \alpha L_1 \int_T^t (\varphi(t) - \varphi(s))^2 P(s)^{-1} u(s)^2 ds \leq 0, \quad t \geq T. \end{aligned} \tag{4.7}$$

Denote by $I(t)$ the first term of the left-hand side of (4.7). Then, it is seen that

$$I(t) = -u(T)\varphi(t)^2 + 2\int_T^t (\varphi(t) - \varphi(s)) P(s)^{-1} u(s) ds, \quad t \geq T,$$

and so, by the Cauchy–Schwarz inequality, we find that

$$|I(t)| \leq |u(T)|\varphi(t)^2 + 2\left(\int_T^t P(s)^{-1}ds\right)^{1/2}\left(\int_T^t (\varphi(t)-\varphi(s))^2 P(s)^{-1}u(s)^2 ds\right)^{1/2}$$

for $t \geq T$. Therefore, (4.6) and (4.7) yield

$$\begin{aligned}
\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 Q(s)ds &\leq |u(T)| \\
&+ \frac{2}{\varphi(t)^{1/2}}\left(\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 P(s)^{-1}u(s)^2 ds\right)^{1/2} \\
&- \alpha L_1\left(\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 P(s)^{-1}u(s)^2 ds\right), \quad t > T.
\end{aligned} \tag{4.8}$$

It follows from (1.8) and (4.6) that

$$\lim_{t\to\infty}\varphi(t) = \int_T^\infty P(s)^{-1}ds = \int_T^\infty p(s)^{-1}x_0(s)^{-2}x_0'(s)^{-\alpha+1}ds = \infty,$$

and, by L'Hospital's rule and (4.5), we get

$$\lim_{t\to\infty}\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 P(s)^{-1}u(s)^2 ds = \int_T^\infty P(s)^{-1}u(s)^2 ds = \infty.$$

Let β be a constant such that $0 < \beta < \alpha L_1$. Then it is easy to check that (4.8) yields

$$\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 Q(s)ds \leq -\frac{\beta}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 P(s)^{-1}u(s)^2 ds$$

for all large t, and consequently,

$$\lim_{t\to\infty}\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 Q(s)ds = -\infty.$$

On the other hand, the condition (1.10), i.e., the condition

$$\lim_{t\to\infty}\int_T^t Q(s)ds = \int_T^\infty Q(s)ds \quad \text{is convergent}$$

implies

$$\lim_{t\to\infty}\frac{1}{\varphi(t)^2}\int_T^t (\varphi(t)-\varphi(s))^2 Q(s)ds = \int_T^\infty Q(s)ds \in \mathbb{R}.$$

This is a contradiction. Therefore, (4.2) does not occur.

Next suppose that (4.3) holds. Using (1.10), (4.1) and (4.3), we see that $\lim_{t\to\infty} u(t)$ exists and is finite. Put $\lim_{t\to\infty} u(t) = \ell\ (\in \mathbb{R})$. Integrating the equality (3.3) from t to τ $(T \leq t \leq \tau)$ and letting $\tau \to \infty$, we obtain

$$\begin{aligned}
u(t) = \ell &+ \int_t^\infty x_0(s)^{\alpha+1}[q(s)-q_0(s)]ds \\
&+ \alpha\int_t^\infty p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F(u(s), p(s)x_0(s)x_0'(s)^\alpha)ds
\end{aligned}$$

for $t \geq T$. By Lemma 2.3 we have $x'(t) > 0$ for $t \geq T$. Hence, by (1.7), (3.2) and (1.12), there is a positive constant k such that

$$-p(t)x_0(t)x_0'(t)^\alpha < u(t) \leq kp(t)x_0(t)x_0'(t)^\alpha \tag{4.9}$$

for all large t. We may suppose that (4.9) is valid for $t \geq T$. Applying Lemma 3.2 (ii) to the case $u = u(t)$ and $v = p(t)x_0(t)x_0'(t)^\alpha$, we find that there is a constant $L_1(k) > 0$ such that

$$\begin{aligned} &L_1(k)p(t)^{-1}x_0(t)^{-2}x_0'(t)^{-\alpha+1}u(t)^2 \\ &\qquad \leq p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F(u(t), p(t)x_0(t)x_0'(t)^\alpha), \quad t \geq T. \end{aligned}$$

Hence, (4.3) gives

$$\int_T^\infty p(t)^{-1}x_0(t)^{-2}x_0'(t)^{-\alpha+1}u(t)^2 dt < \infty.$$

If $\lim_{t\to\infty} u(t) = \ell \neq 0$, then the above fact contradicts the condition (1.8). Therefore we see that $\ell = 0$.

Since

$$\lim_{t\to\infty} u(t) = \ell = 0, \tag{4.10}$$

we find from (1.12) that there are positive constants k_1 and k_2 such that

$$|u(t)| \leq k_1 < k_2 \leq p(t)x_0(t)x_0'(t)^\alpha \tag{4.11}$$

for all large t. We may suppose that (4.11) holds for $t \geq T$. Then, applying Lemma 3.2 (iii) to the case $u = u(t)$ and $v = p(t)x_0(t)x_0'(t)^\alpha$, we deduce that $F(u(t), p(t)x_0(t)x_0'(t)^\alpha)$ is expressed as

$$F(u(t), p(t)x_0(t)x_0'(t)^\alpha) = \frac{\alpha+1}{2\alpha^2}|p(t)x_0(t)x_0'(t)^\alpha|^{(1/\alpha)-1}u(t)^2(1 + R(t)) \tag{4.12}$$

with

$$|R(t)| \leq \frac{|\alpha-1|}{3\alpha}L(k_1, k_2)|u(t)| \tag{4.13}$$

for $t \geq T$. Here, $L(k_1, k_2)$ is the constant appearing in Lemma 3.2 (iii). Then, (4.12) gives

$$\begin{aligned} &p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F(u(t), p(t)x_0(t)x_0'(t)^\alpha) \\ &\quad = \frac{\alpha+1}{2\alpha^2}p(t)^{-1}x_0(t)^{-2}x_0'(t)^{-\alpha+1}u(t)^2(1 + R(t)), \quad t \geq T. \end{aligned} \tag{4.14}$$

By (4.10) and (4.13), we have $\lim_{t\to\infty} R(t) = 0$, and so

$$R(t) \geq -\varepsilon \quad \text{for all large } t, \tag{4.15}$$

where $\varepsilon \in (0, 1)$ is the number in the statement of Theorem 1.6. Then, by (3.3), (4.14) and (4.15), we find that

$$u'(t) + Q(t) + (1-\varepsilon)\frac{\alpha+1}{2\alpha}P(t)^{-1}u(t)^2 \leq 0 \quad \text{for all large } t.$$

Therefore the function

$$y(t) = (1-\varepsilon)\frac{\alpha+1}{2\alpha}u(t) \quad \text{with } 0 < \varepsilon < 1$$

satisfies

$$y'(t) + (1-\varepsilon)\frac{\alpha+1}{2\alpha}Q(t) + P(t)^{-1}y(t)^2 \leq 0 \quad \text{for all large } t.$$

Hence, Lemma 2.1 of the case $\alpha = 1$ implies that the linear equation (1.14) is nonoscillatory. This is a contradiction to the assumption that (1.14) is oscillatory. Therefore, (4.3) also does not occur. Consequently the equation (1.1) is oscillatory. The proof of Theorem 1.6 is complete. □

Proof of Corollary 1.7. Corollary 1.7 is a simple combination of Theorem 1.6 and Lemma 2.4 (ii) with $\alpha = 1$. □

5 Proofs of the results (continued)

In this section we prove Theorem 1.8 and Theorem 1.10. It is known that the half-linear Riemann–Weber differential equation (1.18) has a nonoscillatory solution $x(t)$ satisfying (1.19) (see Elbert and Schneider [9, Corollary 1]). Put

$$x_0(t) = t^{\alpha/(\alpha+1)}(\mathrm{Log}_n\, t)^{1/(\alpha+1)}, \quad t \geq T, \tag{5.1}$$

where T is taken sufficiently large such that t and $\mathrm{Log}_j\, t$ ($j = 1, 2, \ldots, n$) are positive for $t \geq T$. It is trivial that $x = x_0(t)$ is a positive solution of the equation

$$(\Phi_\alpha(x'))' - \frac{(\Phi_\alpha(x_0'(t)))'}{\Phi_\alpha(x_0(t))}\Phi_\alpha(x) = 0$$

on $[T, \infty)$. We define the function $c_0(t)$ by

$$c_0(t) = -\frac{(\Phi_\alpha(x_0'(t)))'}{\Phi_\alpha(x_0(t))} - \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2}\right). \tag{5.2}$$

Then the function $x_0(t)$ is a positive solution of

$$(\Phi_\alpha(x'))' + \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + c_0(t)\right)\Phi_\alpha(x) = 0. \tag{5.3}$$

In the equations (1.1) and (1.6), let $p(t) \equiv 1$ and

$$q(t) = \frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + c(t) \tag{5.4}$$

and

$$q_0(t) = \frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + c_0(t). \tag{5.5}$$

Then, the equations (1.1) and (1.6) become (1.17) and (5.3), respectively. The key idea of the proofs of Theorems 1.8 and 1.10 is to use the equation (5.3), not the equation (1.18).

From the calculation in [4] we see that

$$x_0'(t) = \frac{\alpha}{\alpha+1}t^{-1/(\alpha+1)}(\mathrm{Log}_n\, t)^{1/(\alpha+1)}\left(1 + \frac{1}{\alpha}\sum_{j=1}^{n}\frac{1}{\mathrm{Log}_j\, t}\right), \tag{5.6}$$

and

$$\begin{aligned}(\Phi_\alpha(x_0'(t)))' = &-t^{-(2\alpha+1)/(\alpha+1)}(\mathrm{Log}_n\, t)^{\alpha/(\alpha+1)} \\ &\times\left(\alpha E(\alpha) + \mu(\alpha)\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + O\left(\frac{1}{(\log t)^3}\right)\right)\end{aligned}$$

as $t \to \infty$. Therefore, the function $c_0(t)$ defined by (5.2) satisfies

$$c_0(t) = O\left(\frac{1}{t^{\alpha+1}(\log t)^3}\right) \quad (t \to \infty). \tag{5.7}$$

By (5.7) it is clear that

$$\int_T^\infty c_0(s)ds \quad \text{is convergent}$$

and

$$\lim_{t\to\infty} t^\alpha \int_t^\infty c_0(s)ds = 0.$$

Therefore, in the present case, we find that

$$\int_T^\infty q_0(s)ds \quad \text{is convergent}$$

and

$$\int_t^\infty q_0(s)ds = \frac{E(\alpha)}{t^\alpha} + \mu(\alpha)\sum_{j=1}^n \int_t^\infty \frac{1}{s^{\alpha+1}(\mathrm{Log}_j\, s)^2}ds + \int_t^\infty c_0(s)ds.$$

Then it is easy to see that

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q_0(s)ds = E(\alpha) > 0.$$

Consequently, the condition (1.9) is satisfied.

By (5.1) and (5.6), the condition (1.7) is satisfied. Furthermore it is easily checked that

$$x_0(t)^{-2}x_0'(t)^{-\alpha+1} \sim \left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1} \frac{1}{t\,\mathrm{Log}_n\, t} \quad (t\to\infty).$$

Note here that

$$\frac{d}{dt}\log_{n+1} t = \frac{1}{t\,\mathrm{Log}_n\, t},$$

and so

$$\int_T^t \frac{1}{s\,\mathrm{Log}_n\, s}ds = \log_{n+1} t - \log_{n+1} T.$$

This implies

$$\int_T^t x_0(s)^{-2}x_0'(s)^{-\alpha+1}ds \sim \left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1} \log_{n+1} t \quad (t\to\infty), \tag{5.8}$$

which yields (1.8) with $p(t) \equiv 1$.

We have

$$\int_T^t x_0(s)^{\alpha+1}\left[q(s) - q_0(s)\right]ds = \int_T^t s^\alpha(\mathrm{Log}_n\, s)\left[c(s) - c_0(s)\right]ds.$$

Došlý [4] showed that

$$\int_T^\infty \frac{\mathrm{Log}_n\, s}{s(\log s)^3}ds < \infty$$

and

$$\lim_{t\to\infty}(\log_{n+1} t)\int_t^\infty \frac{\mathrm{Log}_n\, s}{s(\log s)^3}ds = 0.$$

Therefore we deduce from (5.7) that

$$\int_T^\infty s^\alpha(\mathrm{Log}_n\, s)c_0(s)ds \quad \text{is convergent} \tag{5.9}$$

and

$$\lim_{t\to\infty}(\log_{n+1} t)\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c_0(s)ds = 0. \tag{5.10}$$

We are now ready to prove Theorems 1.8 and 1.10.

Proof of Theorem 1.8. We apply Theorem 1.3 with $p(t) \equiv 1$ to the equations (1.17) and (5.3). Let $q(t)$ and $q_0(t)$ be the functions defined by (5.4) and (5.5), respectively. Since (1.8) $(p(t) \equiv 1)$ and (1.9) are satisfied, the function $x_0(t)$ which is defined by (5.1) is the principal solution of (5.3). In fact, this can be derived from a direct application of Theorem 1.2 with $p(t) \equiv 1$. For the case $0 < \alpha \leq 1$, use the part (i), and for the case $\alpha \geq 1$ the part (iii). If (1.20) is satisfied, then (5.9) gives

$$\int_T^\infty x_0(s)^{\alpha+1}\,[q(s) - q_0(s)]\,ds = \int_T^\infty s^\alpha(\operatorname{Log}_n s)\,[c(s) - c_0(s)]\,ds = \infty.$$

Therefore it follows from Theorem 1.3 with $p(t) \equiv 1$ that if (1.20) is satisfied, then the equation (1.17) is oscillatory. The proof of Theorem 1.8 is complete. □

Proof of Theorem 1.10. We apply Corollary 1.7 with $p(t) \equiv 1$ to the equations (1.17) and (5.3). Let $q(t)$ and $q_0(t)$ be the functions defined by (5.4) and (5.5), respectively. We first show that if (1.21) holds, then (1.5) is satisfied. To see this, note that

$$\frac{d}{dt}\operatorname{Log}_n t = \frac{\operatorname{Log}_n t}{t}\left(\sum_{j=1}^n \frac{1}{\operatorname{Log}_j t}\right)$$

and

$$\begin{aligned}
\int_T^t c(s)ds &= \int_T^t \frac{1}{s^\alpha \operatorname{Log}_n s} s^\alpha(\operatorname{Log}_n s)c(s)ds \\
&= -\frac{1}{t^\alpha \operatorname{Log}_n t}\int_t^\infty r^\alpha(\operatorname{Log}_n r)c(r)dr + \frac{1}{T^\alpha \operatorname{Log}_n T}\int_T^\infty r^\alpha(\operatorname{Log}_n r)c(r)dr \\
&\quad - \int_T^t \left[\frac{\alpha}{s^{\alpha+1}\operatorname{Log}_n s} + \frac{1}{s^{\alpha+1}\operatorname{Log}_n s}\left(\sum_{j=1}^n \frac{1}{\operatorname{Log}_j s}\right)\right]\left(\int_s^\infty r^\alpha(\operatorname{Log}_n r)c(r)dr\right)ds.
\end{aligned}$$

Therefore we deduce that

$$\lim_{t\to\infty}\int_T^t c(s)ds \quad \text{exists and is finite}$$

and

$$\begin{aligned}
\int_t^\infty c(s)ds &= \frac{1}{t^\alpha \operatorname{Log}_n t}\int_t^\infty r^\alpha(\operatorname{Log}_n r)c(r)dr \\
&\quad - \int_t^\infty \left[\frac{\alpha}{s^{\alpha+1}\operatorname{Log}_n s} + \frac{1}{s^{\alpha+1}\operatorname{Log}_n s}\left(\sum_{j=1}^n \frac{1}{\operatorname{Log}_j s}\right)\right]\left(\int_s^\infty r^\alpha(\operatorname{Log}_n r)c(r)dr\right)ds
\end{aligned}$$

for $t \geq T$. Then it is easy to find that

$$\lim_{t\to\infty} t^\alpha \int_t^\infty c(s)ds = 0.$$

Since

$$\int_t^\infty q(s)ds = \frac{E(\alpha)}{t^\alpha} + \mu(\alpha)\sum_{j=1}^n \int_t^\infty \frac{1}{s^{\alpha+1}(\operatorname{Log}_j s)^2}ds + \int_t^\infty c(s)ds,$$

we obtain

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = E(\alpha)\ (>0).$$

Thus we see that the condition (1.5) is satisfied.

As mentioned before, the conditions (1.7) and (1.8) with $p(t) \equiv 1$ are satisfied. By (5.1) and (5.6), we have

$$x_0(t)x_0'(t)^\alpha = \left(\frac{\alpha}{\alpha+1}\right)^\alpha (\mathrm{Log}_n\, t)\left(1+\frac{1}{\alpha}\sum_{j=1}^{n}\frac{1}{\mathrm{Log}_j\, t}\right)^\alpha,$$

and so $\lim_{t\to\infty} x_0(t)x_0'(t)^\alpha = \infty$. Hence the condition (1.12) with $p(t) \equiv 1$ is also satisfied. By the definition of $P(t)$ and $Q(t)$ and the properties (5.8) and (5.10), we have

$$\begin{aligned}
&\left(\int_T^t \frac{1}{P(s)}ds\right)\left(\int_t^\infty Q(s)ds\right)\\
&\quad = \left(\int_T^t x_0(s)^{-2}x_0'(s)^{-\alpha+1}ds\right)\left(\int_t^\infty x_0(s)^{\alpha+1}[q(s)-q_0(s)]ds\right)\\
&\quad = \varepsilon_1(t)\left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1}(\log_{n+1} t)\left(\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)[c(s)-c_0(s)]ds\right)\\
&\quad = \varepsilon_1(t)\left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1}(\log_{n+1} t)\left(\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c(s)ds\right)+\varepsilon_2(t),
\end{aligned}$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are functions such that

$$\lim_{t\to\infty}\varepsilon_1(t) = 1 \quad \text{and} \quad \lim_{t\to\infty}\varepsilon_2(t) = 0,$$

respectively. Then it is easy to see that (1.22) implies (1.15). Thus, by Corollary 1.7, we can conclude that if (1.22) holds, then the equation (1.17) is oscillatory. The proof of Theorem 1.10 is complete. □

Finally we present an equation whose oscillation follows from Theorem 1.10 and does not follow from Theorem 3.3 (ii) in [4].

Example 5.1. Consider the equation (1.17) of the case

$$c(t) = \mu(\alpha)\frac{k+\sin(2k\log_{n+2} t)-2k\cos(2k\log_{n+2} t)}{t^{\alpha+1}(\mathrm{Log}_{n+1}\, t)^2}, \tag{5.11}$$

where k is a constant satisfying $k > 2$. In this case it can be shown without difficulty that the condition (1.21) is satisfied and

$$\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c(s)ds = \mu(\alpha)\frac{k+\sin(2k\log_{n+2} t)}{\log_{n+1} t}.$$

Therefore we have

$$\liminf_{t\to\infty}(\log_{n+1} t)\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c(s)ds = \mu(\alpha)(k-1) > \mu(\alpha).$$

Hence, by Theorem 1.10, we can conclude that, for any $\alpha > 0$, the equation (1.17) with (5.11) is oscillatory.

Theorem 3.3 (ii) in [4] requires the condition that there is a constant γ satisfying

$$t^{\alpha+1}(\log t)^3c(t) \ge \gamma > \frac{2(\alpha+1)(\alpha-1)}{3\alpha^2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

for all large t. If $\alpha \geq 1$, this condition leads to $c(t) > 0$ for all large t. There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim t_i = \infty$ $(i \to \infty)$ and

$$k \log_{n+2} t_i = \pi i \quad \text{for all large } i.$$

Then, for the function $c(t)$ given by (5.11),

$$c(t_i) = \mu(\alpha) \frac{-k}{t_i^{\alpha+1} (\operatorname{Log}_{n+1} t_i)^2} < 0 \quad \text{for all large } i.$$

Therefore, if $\alpha \geq 1$, then we cannot apply Theorem 3.3 (ii) in [4] to the present case.

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