Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **41**, 1–17; https://doi.org/10.14232/ejqtde.2024.1.41 www.math.u-szeged.hu/ejqtde/

Interval of the existence of positive solutions for a boundary value problem for system of three second-order differential equations

Aleksejs Antoņuks✉[1] and **Sergey Smirnov**[1,2]

[1]Faculty of Physics, Mathematics and Optometry, University of Latvia, Jelgavas iela 3, Riga, LV-1004, Latvia
[2]Institute of Mathematics and Computer Science, University of Latvia, Raina bulvaris 29, Riga, LV-1459, Latvia

Received 6 March 2024, appeared 3 August 2024
Communicated by Jeff R. L. Webb

Abstract. The aim of this paper is to estimate an interval of the existence of positive solutions for a boundary value problem for the system of three nonlinear second-order ordinary differential equations. Krasnosel'skiĭ–Precup fixed point theorem is used to determine this interval theoretically. For Dirichlet boundary conditions theoretical result is compared with result obtained numerically.

Keywords: boundary value problem, system of second-order ODEs, interval of the existence of solutions, positive solutions, Krasnosel'skiĭ–Precup fixed point theorem.

2020 Mathematics Subject Classification: 34B15, 34B18.

1 Introduction

Systems of three second-order ordinary differential equations emerge naturally from the application of Newton's laws in modeling three body interaction: each equation represents the acceleration of a body in response to the forces exerted by the other two bodies. Such systems have a vital role in modeling problems of mechanics and oscillations.

In this paper, we investigate the interval of the existence of (strictly) positive solutions, i.e. we determine real positive τ for which at least one positive solution exists, for the following system of nonlinear second-order differential equations

$$x_i''(t) + f_i(t, x_1(t), x_2(t), x_3(t)) = 0, \quad t \in (0, \tau), \, i = 1, 2, 3, \tag{1.1}$$

coupled with nonlocal boundary conditions

$$x_i(0) = \varphi_i[x_i] + a_i, \quad x_i(\tau) = \psi_i[x_i] + b_i, \tag{1.2}$$

✉Corresponding author. Email: aleksey.antonyuk1@gmail.com

where $a_i, b_i \geq 0$, $f_i : [0,\tau] \times [0,+\infty)^3 \to [0,+\infty)$ are continuous, $\varphi_i[x] = \int_0^\tau x(t)\mathrm{d}\Phi_i(t)$ and $\psi_i[x] = \int_0^\tau x(t)\mathrm{d}\Psi_i(t)$ are linear functionals defined via Riemann–Stieltjes integrals, where $\Phi_i, \Psi_i : [0,\tau] \to \mathbb{R}$ are functions of bounded variation.

We write $\varphi_i[\mathrm{Id}]$ and $\varphi_i[\tau]$ to denote φ_i applied to the identity function and constant function with value τ, respectively. The notation $|A|$ denotes the determinant of a square matrix A. Throughout the paper, we assume that

(A1) $0 \leq \varphi_i[\mathrm{Id}]$, $0 \leq \varphi_i[\tau - \mathrm{Id}] \leq \tau$ and $0 \leq \psi_i[\tau - \mathrm{Id}]$, $0 \leq \psi_i[\mathrm{Id}] \leq \tau$,

(A2) $0 < D_i = \begin{vmatrix} \tau - \varphi_i[\tau - \mathrm{Id}] & -\varphi_i[\mathrm{Id}] \\ -\psi_i[\tau - \mathrm{Id}] & \tau - \psi_i[\mathrm{Id}] \end{vmatrix}$,

are valid for every $i = 1,2,3$.

By a positive solution of problem (1.1), (1.2) we understand $(x_1, x_2, x_3) \in \left(C^2[0,\tau]\right)^3$, which satisfies system of differential equations (1.1), boundary conditions (1.2) and positive *coexistence* condition, i.e. $x_i(t) > 0$ for all $t \in (0,\tau)$ and every $i = 1,2,3$.

We do not assume $\varphi_i[x_i] \geq 0$ and $\psi_i[x_i] \geq 0$ for all $x_i \geq 0$, but we allow $\mathrm{d}\Phi_i$ and $\mathrm{d}\Psi_i$ to be signed measures. For details on signed measure and Riemann–Stieltjes integrals we refer reader, for instance, to [17–19]. But we require $\varphi_i[x_i] \geq 0$ and $\psi_i[x_i] \geq 0$ for corresponding component of the positive solution (x_1, x_2, x_3).

The term "coexistence" was introduced by Lan [11] in context of fixed points in product Banach spaces. Coexistence fixed point denotes a fixed point with all the components different from zero. The common approach to obtain solutions of operator equation is to seek the fixed points. The best-known fixed point theorems for positive solutions are Krasnosel'skiĭ's fixed point theorem in cones [10] and its generalizations, for instance, Krasnosel'skiĭ–Benjamin fixed point theorem [1], where conditions are weakened, and Guo–Krasnosel'skiĭ fixed point theorem [3], where considered region is more general. But, as it was mentioned in [12, 13, 15], these theorems cannot guarantee the coexistence fixed point. Motivated by this, Precup [12, 13] established (2-dimensional) vector version of Krasnosel'skiĭ's fixed point theorem, which allows to localize fixed point in the component-wise manner. Recently, Rodríguez-López [15] showed an alternative proof via fixed point index theory. As it was pointed out in [15], the result by Precup remains valid for n-dimensions. For multiplicity result of positive solutions by vector version of Krasnosel'skiĭ's fixed point theorem we refer reader to [8, 14].

Generalized version of problem (1.1), (1.2) with $\tau = 1$ and $i = 1,2$, was studied by Henderson and Luca [5, 6]. In [5] was considered problem (in our notations)

$$(a_i(t)x_i(t)')' - b_i(t)x_i(t) + \lambda_i p_i(t) f_i(t, x_1(t), x_2(t)) = 0, \quad t \in (0,1),\ i = 1,2, \tag{1.3}$$

$$\alpha_i x_i(0) - \beta_i a(0) x_i'(0) = \varphi_i[x_i], \quad \gamma_i x_i(1) + \delta_i a(1) x_i'(1) = \psi_i[x_i], \tag{1.4}$$

and sufficient conditions on λ_i and f_i were given such that non-negative solutions of problem (1.3), (1.4) exist. The result was based on Guo–Krasnosel'ski fixed point theorem. In [6] by applying fixed point index theory results on existence and multiplicity of positive solutions were obtained for the slightly modified problem (1.3), (1.4): the functions f_i depended on only one unknown $x_{j\neq i}$, i.e. $f_i(t, x_{j\neq i}(t))$.

In this paper, we apply two methods that allow us to obtain an interval of the existence of positive solutions for the problem (1.1), (1.2). First we find τ by solving system of inequalities, which is based on Green's functions of problem (1.1), (1.2) and behavior of functions f_i. To prove that for these τ there exist positive solutions we apply vector version of Krasnosel'skiĭ's fixed point theorem, or Krasnosel'skiĭ–Precup fixed point theorem. Let us recall this result here. A nonempty closed convex subset $K \subset X$ of normed space $(X, \|\cdot\|)$ is called a cone if $\lambda x \in K$ for every $x \in K$ and for all $\lambda \geq 0$, and $K \cap (-K) = 0$.

Theorem 1.1 (Krasnosel'skiĭ–Precup, [12, 15]). *Let $(X, \|\cdot\|)$ be a normed space, $K_1, \dots, K_n$ cones in X, $K = K_1 \times \cdots \times K_n$, $r = (r_1, \dots, r_n)$, $R = (R_1, \dots, R_n)$, with $0 < r_i < R_i$ for $i \in \{1, \dots, n\}$, and*

$$\overline{K}_{r,R} = \{x = (x_1, \dots, x_n) \in K : \forall\, i \in \{1, \dots, n\} \quad r_i \leq \|x_i\| \leq R_i\}.$$

Assume that $T = (T_1, .., T_n) : \overline{K}_{r,R} \to K$ is a completely continuous map and for each $i \in \{1, .., n\}$ there exists $\mathrm{h}_i \in K_i \setminus \{0\}$ such that one of the following conditions is satisfied in $\overline{K}_{r,R}$:

(i) $T_i x + \mu \mathrm{h}_i \neq x_i$ *if* $\|x_i\| = r_i$ *and* $\mu > 0$, *and* $T_i x \neq \lambda x_i$ *if* $\|x_i\| = R_i$ *and* $\lambda > 1$;

(ii) $T_i x \neq \lambda x_i$ *if* $\|x_i\| = r_i$ *and* $\lambda > 1$, *and* $T_i x + \mu \mathrm{h}_i \neq x_i$ *if* $\|x_i\| = R_i$ *and* $\mu > 0$.

Then T has at least one fixed point $x \in K$ with $r_i \leq \|x_i\| \leq R_i$, $i \in \{1, \dots, n\}$.

Conditions (i) and (ii) are called compression type and expansion type condition, respectively.

To satisfy compression and expansion type conditions various authors considered asymptotic behavior of f/x at zero and infinity. This approach is widely used in case of one differential equation or systems in which all f_i depend on only one unknown $x_{j \neq i}$ (see, for instance, [2, 6, 7, 9, 18, 19]). The idea is to use limits

$$\lim_{x \to 0} \sup_t \frac{f(t,x)}{x}, \qquad \lim_{x \to \infty} \sup_t \frac{f(t,x)}{x}$$
$$\lim_{x \to 0} \inf_t \frac{f(t,x)}{x}, \qquad \lim_{x \to \infty} \inf_t \frac{f(t,x)}{x}.$$

In [9] the case where the above limits were zero or infinity was studied. In [2, 7] the limits were allowed to be small or large enough, in a sense that necessary inequalities hold. In the case of systems of differential equations in which f_i depend on all unknowns many authors require additional assumptions on f_i to construct similar limits. For instance, f_i is monotone with respect to x_j, see [12, 13], or bounded with respect to x_j, see [4].

If we let $\varphi_i \equiv 0$ and $\psi_i \equiv 0$, then boundary conditions (1.2) become Dirichlet boundary conditions. For such problem we compare the theoretical result with the result based on built-in functions of program *Mathematica* [20]. The numerical result is obtained by shooting method: we consider the initial value problem for system of differential equations and determine τ.

The outline of the rest of the paper is as follows. In Section 2, we rewrite boundary value problem (1.1), (1.2) as an equivalent system of integral equations by constructing the Greens functions and show the estimations of Greens functions. We prove the existence of positive solutions by applying Krasnosel'skiĭ–Precup fixed point theorem in Section 3 and formulate main result of this article in Theorem 3.7. Finally, in Section 4, we compare theoretical and numerical results for problem (1.1) with the boundary conditions $x_i(0) = a_i$, $x_i(\tau) = b_i$.

2 Construction and estimation of Green's functions

Standard approach is to rewrite problem (1.1), (1.2) as an equivalent system of integral equations via corresponding Green's functions. Results of this section are well-known and for details we refer reader to [16, 18, 19].

The Green's function G_0 corresponding to problem $x''(t) + h(t) = 0$, $x(1) = 0 = x(\tau)$, is given by

$$G_0(t,s) = \frac{1}{\tau}\begin{cases} s(\tau - t), & 0 \le s \le t \le \tau, \\ t(\tau - s), & 0 \le t \le s \le \tau. \end{cases} \tag{2.1}$$

We denote $\mathcal{G}_{\varphi_i}(s) = \int_0^\tau G_0(t,s)\mathrm{d}\Phi_i(t)$, $\mathcal{G}_{\psi_i}(s) = \int_0^\tau G_0(t,s)\mathrm{d}\Psi_i(t)$ and in addition to (A1) and (A2) we assume

(A3) $\mathcal{G}_{\varphi_i}(s) \ge 0$ and $\mathcal{G}_{\psi_i}(s) \ge 0$ for all $s \in [0,\tau]$ and every $i = 1,2,3$.

Recall that D_i is given by (A2) and $|A|$ denotes the determinant of a square matrix A.

Proposition 2.1. *A triple (x_1, x_2, x_3) is a solution of boundary value problem (1.1), (1.2) if and only if (x_1, x_2, x_3) is a solution of the system of integral equations*

$$x_i(t) = \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s))\mathrm{d}s + g_i(t), \quad t \in [0,\tau],\ i = 1,2,3, \tag{2.2}$$

where

$$G_i(t,s) = \frac{1}{D_i}\begin{vmatrix} \tau - t & \tau - \varphi_i[\tau - \mathrm{Id}] & -\psi_i[\tau - \mathrm{Id}] \\ t & -\varphi_i[\mathrm{Id}] & \tau - \psi_i[\mathrm{Id}] \\ G_0(t,s) & -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\psi_i}(s) \end{vmatrix} \tag{2.3}$$

and

$$g_i(t) = \frac{1}{D_i}\begin{vmatrix} \tau - t & \tau - \varphi_i[\tau - \mathrm{Id}] & -\psi_i[\tau - \mathrm{Id}] \\ t & -\varphi_i[\mathrm{Id}] & \tau - \psi_i[\mathrm{Id}] \\ 0 & -a_i & -b_i \end{vmatrix}. \tag{2.4}$$

Proof. Let (x_1, x_2, x_3) be a solution of boundary value problem (1.1), (1.2). For every $i = 1,2,3$, integrating (1.1) twice from 0 to t and applying boundary conditions (1.2), we get

$$x_i(t) = \int_0^\tau G_0(t,s) f_i(s, x_1(s), x_2(s), x_3(s))\mathrm{d}s + \frac{t}{\tau}(b_i + \psi_i[x_i]) + \frac{\tau - t}{\tau}(a_i + \varphi_i[x_i]). \tag{2.5}$$

Let us denote $(Fx_i)(t) = \int_0^\tau G_0(t,s) f_i(s, x_1(s), x_2(s), x_3(s))\mathrm{d}s$. Applying φ_i and ψ_i to (2.5), we get

$$\begin{aligned} \varphi_i[x_i](\tau - \varphi_i[\tau - \mathrm{Id}]) - \varphi_i[\mathrm{Id}]\psi_i[x_i] &= \tau\,\varphi_i[Fx_i] + b_i\varphi_i[\mathrm{Id}] + a_i\varphi_i[\tau - \mathrm{Id}], \\ \psi_i[x_i](\tau - \psi_i[\mathrm{Id}]) - \psi_i[\tau - \mathrm{Id}]\varphi_i[x_i] &= \tau\,\psi_i[Fx_i] + b_i\psi_i[\mathrm{Id}] + a_i\psi_i[\tau - \mathrm{Id}]. \end{aligned} \tag{2.6}$$

We rewrite (2.6) in matrix form

$$\begin{pmatrix} \tau - \varphi_i[\tau - \mathrm{Id}] & -\varphi_i[\mathrm{Id}] \\ -\psi_i[\tau - \mathrm{Id}] & \tau - \psi_i[\mathrm{Id}] \end{pmatrix}\begin{pmatrix} \varphi_i[x_i] \\ \psi_i[x_i] \end{pmatrix} = \begin{pmatrix} \tau\,\varphi_i[Fx_i] \\ \tau\,\psi_i[Fx_i] \end{pmatrix} + \begin{pmatrix} \varphi_i[\tau - \mathrm{Id}] & \varphi_i[\mathrm{Id}] \\ \psi_i[\tau - \mathrm{Id}] & \psi_i[\mathrm{Id}] \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

By assumption (A2), $D_i > 0$ and it follows

$$\begin{aligned} \begin{pmatrix} \varphi_i[x_i] \\ \psi_i[x_i] \end{pmatrix} &= \frac{1}{D_i}\begin{pmatrix} \tau - \psi_i[\mathrm{Id}] & \varphi_i[\mathrm{Id}] \\ \psi_i[\tau - \mathrm{Id}] & \tau - \varphi_i[\tau - \mathrm{Id}] \end{pmatrix}\begin{pmatrix} \tau\,\varphi_i[Fx_i] \\ \tau\,\psi_i[Fx_i] \end{pmatrix} \\ &+ \frac{1}{D_i}\begin{pmatrix} \tau - \psi_i[\mathrm{Id}] & \varphi_i[\mathrm{Id}] \\ \psi_i[\tau - \mathrm{Id}] & \tau - \varphi_i[\tau - \mathrm{Id}] \end{pmatrix}\begin{pmatrix} \varphi_i[\tau - \mathrm{Id}] & \varphi_i[\mathrm{Id}] \\ \psi_i[\tau - \mathrm{Id}] & \psi_i[\mathrm{Id}] \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix}. \end{aligned} \tag{2.7}$$

Substituting $\varphi_i[x_i]$ and $\psi_i[x_i]$ from (2.7) in (2.5), we get

$$\begin{aligned} x_i(t) &= (Fx_i)(t) - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -\varphi_i[Fx_i] & -\psi_i[Fx_i]\end{vmatrix} + \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -\varphi_i[Fx_i] & -\psi_i[Fx_i]\end{vmatrix} \\ &\quad - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -a_i & -b_i\end{vmatrix} + \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -a_i & -b_i\end{vmatrix} \\ &= \frac{1}{D_i}\begin{vmatrix} \tau-t & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ t & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ (Fx_i)(t) & -\varphi_i[Fx_i] & -\psi_i[Fx_i]\end{vmatrix} + \frac{1}{D_i}\begin{vmatrix} \tau-t & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ t & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ 0 & -a_i & -b_i\end{vmatrix} \\ &= \int_0^\tau G_i(t,s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + g_i(t), \end{aligned}$$

where G_i is given by (2.3) and g_i is given by (2.4).

Now, let (x_1,x_2,x_3) satisfy system of integral equations (2.2). It follows that each x_i also satisfies (2.5). By differentiating (2.5) twice, it is easy to see that (x_1,x_2,x_3) satisfies (1.1), (1.2) and $(x_1,x_2,x_3) \in \left(C^2[0,\tau]\right)^3$. □

Remark 2.2. Note that $G_i \geq 0$ and $g_i \geq 0$ for every $i = 1,2,3$. Indeed, expansion of (2.3) and (2.4) along the first column is

$$\begin{aligned} G_i(t,s) &= \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\varphi_i}(s)\end{vmatrix} - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\varphi_i}(s)\end{vmatrix} + G_0(t,s) \\ &= \frac{\tau-t}{D_i}\begin{vmatrix} \varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & \mathcal{G}_{\varphi_i}(s)\end{vmatrix} + \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & \psi_i[\tau-\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & \mathcal{G}_{\varphi_i}(s)\end{vmatrix} + G_0(t,s) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} g_i(t) &= \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -a_i & -b_i\end{vmatrix} - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -a_i & -b_i\end{vmatrix} \\ &= \frac{\tau-t}{D_i}\begin{vmatrix} \varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -a_i & b_i\end{vmatrix} + \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & \psi_i[\tau-\mathrm{Id}] \\ -a_i & b_i\end{vmatrix}. \end{aligned} \tag{2.9}$$

By assumptions (A1)–(A3), and $a_i, b_i \geq 0$, and fact that G_0, given by (2.1), is non-negative, determinants in last parts of (2.8) and (2.9) are non-negative for all $(t,s) \in [0,\tau] \times [0,\tau]$ and $t \in [0,\tau]$, respectively.

Let $m(t) = \min\{t/\tau,\ 1 - t/\tau\}$. It is known that Green's function G_0 satisfies

$$m(t)G_0(s,s) \leq G_0(t,s) \leq G_0(s,s), \quad (t,s) \in [0,\tau] \times [0,\tau].$$

Proposition 2.3. *Green's function G_i, given by (2.3), satisfies*

$$m(t)H_i(s) \leq G_i(t,s) \leq H_i(s), \quad (t,s) \in [0,\tau] \times [0,\tau],$$

where

$$H_i(s) = \frac{1}{D_i}\begin{vmatrix} \tau & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ \tau & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ G_0(s,s) & -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\psi_i}(s)\end{vmatrix}.$$

Proof. Expansion of $G_i(t,s)$ along the first column is given by (2.8). We replace $\tau - t$ with τ in first determinant, t with τ in second determinant, $G_0(t,s)$ with $G_0(s,s)$ in third determinant and get $H_i(s)$. Therefore, $H_i \geq 0$ by the same argument as $G_i \geq 0$.

We get inequality $G_i(t,s) \leq H_i(s)$ by estimating $\tau - t \leq \tau$, $t \leq \tau$ and $G_0(t,s) \leq G_0(s,s)$.

It is clear that $1 - t/\tau \geq m(t)$ and $t/\tau \geq m(t)$ for all $t \in [0,\tau]$. We get inequality $G_i(t,s) \geq m(t)H_i(s)$ by estimating $\tau - t \geq m(t)\tau$, $t \geq m(t)\tau$ and $G_0(t,s) \geq m(t)G_0(s,s)$. □

Observe that if $a_i = b_i = 0$, then $g_i \equiv 0$. By (2.9), it is easy to see that $g_i(t)$ is a polynomial with degree at most one. Hence g_i is concave. Concavity of g_i implies

$$g_i(t) \geq m(t)\, g_i(t_0), \quad (t,t_0) \in [0,\tau] \times [0,\tau]. \tag{2.10}$$

For every $c \in (0,\tau/2)$ inequality $\tau c \leq m(t)$ holds for $t \in [c, \tau - c]$. As it was mentioned in [18, 19], for Green's function G_0 optimal constant is $c = \tau/4$. Optimal in a sense that $\inf\{\int_c^{\tau-c} G_0(t,s)\mathrm{d}s : t \in [c,\tau-c]\}$ is maximal.

3 Theoretical result on the existence of a positive solution

Consider Banach space $C[0,\tau]$ endowed with the norm $\|x\| = \max\{|x(t)| : t \in [0,\tau]\}$. We define cone k_i by

$$k_i = \left\{ u \in C[0,\tau] : u(t) \geq 0 \text{ for } t \in [0,\tau],\ \min_{t\in[\tau/4,3\tau/4]} u(t) \geq \frac{1}{4}\|u\|,\ \varphi_i[u] \geq 0,\ \psi_i[u] \geq 0 \right\}.$$

Let $K = k_1 \times k_2 \times k_3$, $x = (x_1,x_2,x_3)$ and $T = (T_1,T_2,T_3) : K \to (C[0,\tau])^3$ be an operator defined by

$$(T_i x)(t) = \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s))\mathrm{d}s + g_i(t), \tag{3.1}$$

where G_i is given by (2.3) and g_i is given by (2.4).

Observe that T is a completely continuous operator. Indeed, g_i is obviously completely continuous and $T_i x - g_i$ is completely continuous by application of Arzelà–Ascoli theorem. Boundary value problem (1.1), (1.2) has a non-negative solution if and only if operator T has a fixed point in K. To prove that maximal value of each x_i is positive, and hence the solution is positive, we apply Krasnosel'skiĭ–Precup fixed point theorem (Theorem 1.1). Now, we show that T maps K into itself.

Proposition 3.1. *Operator T, given by (3.1), satisfies $T(K) \subset K$.*

Proof. It is obvious that $T_i x \geq 0$ for each $i = 1,2,3$.

Let $T_i x$ achieve maximum value at point t_0, i.e. $(T_i x)(t_0) = \|T_i x\|$. By Proposition 2.3 and (2.10), for every $t \in [\tau/4, 3\tau/4]$ we have

$$\begin{aligned}
(T_i x)(t) &= \int_0^\tau G_i(t,s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + g_i(t) \\
&\geq m(t)\int_0^\tau H_i(s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + m(t) g_i(t_0) \\
&\geq \frac{1}{4}\Big(\int_0^\tau G_i(t_0,s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + g_i(t_0)\Big) \\
&= \frac{1}{4}\|T_i x\|.
\end{aligned}$$

Next, consider

$$\varphi_i[Tx_i] = \int_0^\tau \left(\int_0^\tau G_i(t,s) \mathrm{d}\Phi_i(t) \right) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + \varphi_i[g_i].$$

By (A1)–(A3), we get $\int_0^\tau G_i(t,s)\mathrm{d}\Phi_i(t) \geq 0$ and $\varphi_i[g_i] \geq 0$. Hence $\varphi_i[Tx_i] \geq 0$. Similarly $\psi_i[Tx_i] \geq 0$. Therefore, $T(K) \subset K$. □

Now, we briefly describe the main result. First, we show that if certain conditions on f_i hold, then T_i satisfies compression type condition (i) or expansion type condition (ii) of Krasnosel'skiĭ–Precup fixed point theorem (Theorem 1.1). Then we choose r and R such that each T_i satisfies either condition (i) or (ii) for all $x \in \overline{K}_{r,R}$. Finally, we conclude that at least one positive solution of problem (1.1), (1.2) exists.

Let us introduce notations

$$A_i = \inf_{t \in [\tau/4, 3\tau/4]} \int_{\tau/4}^{3\tau/4} G_i(t,s) \mathrm{d}s, \quad B_i = \sup_{t \in [0,\tau]} \int_0^\tau G_i(t,s) \mathrm{d}s.$$

To prove the following Lemma 3.2 (and Proposition 3.6) we use standard techniques. See, for instance, [2,7,9,18,19].

Lemma 3.2. *Operator T_i satisfies compression type condition (i) if there exist constants $0 < q < Q$ such that*

$$q < \min_{\substack{t \in [\tau/4, 3\tau/4] \\ x_i \in [q/4, q] \\ x_{j \neq i} \in [q/4, Q]^2}} f_i(t,x) \cdot A_i \quad \text{and} \quad \max_{\substack{t \in [0,\tau] \\ x \in [0,Q]^3}} f_i(t,x) \cdot B_i + \|g_i\| < Q, \tag{3.2}$$

and T_i satisfies expansion type condition (ii) if there exist constants $0 < q < Q$ such that

$$Q < \min_{\substack{t \in [\tau/4, 3\tau/4] \\ x_i \in [Q/4, Q] \\ x_{j \neq i} \in [q/4, Q]^2}} f_i(t,x) \cdot A_i \quad \text{and} \quad \max_{\substack{t \in [0,\tau] \\ x_i \in [0,q] \\ x_{j \neq i} \in [0,Q]^2}} f_i(t,x) \cdot B_i + \|g_i\| < q. \tag{3.3}$$

Proof. Let $\overline{K}_{q,Q} = \{x \in K : q \leq \|x_i\| \leq Q,\ i = 1,2,3\}$. We show a proof for compression type condition. Proof for expansion type condition is similar.

Let $\|x_i\| = Q$ and $\Omega = [0,\tau] \times [0,Q]^3$. We show that $\|T_i x\| \leq \|x_i\|$. It is known that this implies $T_i x \neq \lambda x_i$ for $\lambda > 1$. Consider

$$\begin{aligned} \|T_i x\| &\leq \max_{t \in [0,\tau]} \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + \|g_i\| \\ &\leq \max_{(t,x) \in \Omega} f_i(t,x) \cdot B_i + \|g_i\| < Q = \|x_i\|. \end{aligned}$$

Now, suppose to contrary that there exists x_i with $\|x_i\| = q$ such that $T_i x + \mu \mathrm{h} = x_i$ for $\mu > 0$ and $\mathrm{h} : t \mapsto 1$. Since $x \in \overline{K}_{q,Q}$, we have

$$x_j(t) \geq \frac{1}{4} \|x_j\| \geq \frac{1}{4} q, \quad t \in [\tau/4, 3\tau/4],\ j = 1,2,3.$$

Let $\omega = [\tau/4, 3\tau/4] \times [q/4, q] \times [q/4, Q]^2$. We get

$$\begin{aligned} x_i(t) &= \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + g_i(t) + \mu \\ &\geq \int_{\tau/4}^{3\tau/4} G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + g_i(t) + \mu \\ &\geq \min_{(t, x_i, x_{j\neq i}) \in \omega} f_i(t,x) \cdot A_i + g_i(t) + \mu \\ &> q + g_i(t) + \mu, \end{aligned}$$

which gives contradiction. □

Let us show examples of f_i, $i = 1,2,3$, that satisfy (3.2) and (3.3) for sufficiently small q and sufficiently large Q, i.e. there exist $q_i < Q_i$ such that f_i satisfies (3.2) or (3.3) for $0 < q \leq q_i$ and $Q_i \leq Q < +\infty$. The ability to choose such q and Q is used to define proper $\overline{K}_{r,R}$ in the proof of the main result.

Let us define

$$u_{ij}^{w} = \begin{cases} u, & i = j, \\ w, & i \neq j. \end{cases}$$

We use notation u_{ij}^w to denote that i-th element of a triple $(u_{i1}^w, u_{i2}^w, u_{i3}^w)$ is u and j-th element $(j \neq i)$ is w, e.g. $(u_{11}^w, u_{12}^w, u_{13}^w) = (u, w, w)$ and $(u_{21}^0, u_{22}^0, u_{23}^0) = (0, u, 0)$.

Example 3.3. Let f_i be non-decreasing with respect to all x_i, $i = 1,2,3$. Function f_i satisfies (3.2) for sufficiently small q and sufficiently large Q if

$$1 < \lim_{u \to 0+} \inf_{t \in [\tau/4, 3\tau/4]} \frac{f_i(t,u,u,u)}{u} \cdot \frac{A_i}{4}, \qquad \lim_{u \to +\infty} \sup_{t \in [0,\tau]} \frac{f_i(t,u,u,u)}{u} \cdot B_i < 1,$$

and satisfies (3.3) for sufficiently small q and sufficiently large Q if $a_i = b_i = 0$ and

$$\begin{aligned} \forall w \in [0, +\infty) \ \lim_{u \to 0+} \sup_{t \in [0,\tau]} \frac{f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w)}{u} &= 0, \\ 1 < \lim_{u \to +\infty} \inf_{t \in [\tau/4, 3\tau/4]} \frac{f_i(t, u_{i1}^0, u_{i2}^0, u_{i3}^0)}{u} &\cdot \frac{A_i}{4}. \end{aligned} \tag{3.4}$$

For proof see Proposition 3.6.

Example 3.4. Let f_i be bounded with respect to x_i and non-decreasing with respect to every $x_{j\neq i}$, $j = 1,2,3$. Function f_i satisfies (3.2) for sufficiently small q and sufficiently large Q if

$$1 < \lim_{w \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ u \in [0, +\infty)}} \frac{f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w)}{w} \cdot \frac{A_i}{4}, \qquad \lim_{w \to +\infty} \sup_{\substack{t \in [0,\tau] \\ u \in [0,+\infty)}} \frac{f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w)}{w} \cdot B_i < 1,$$

and satisfies (3.3) for sufficiently small q and sufficiently large Q if $a_i = b_i = 0$ and (3.4).

Example 3.5. Let f_i be bounded with respect to every $x_{j\neq i}$, $j = 1,2,3$. Function f_i satisfies (3.2) for sufficiently small q and sufficiently large Q if

$$1 < \lim_{x_i \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ x_{j\neq i} \in [0,+\infty)^2}} \frac{f_i(t, x_1, x_2, x_3)}{x_i} \cdot \frac{A_i}{4}, \quad \lim_{x_i \to +\infty} \sup_{\substack{t \in [0,\tau] \\ x_{j \neq i} \in [0, +\infty)^2}} \frac{f_i(t, x_1, x_2, x_3)}{x_i} \cdot B_i < 1,$$

and satisfies (3.3) for sufficiently small q and sufficiently large Q if $a_i = b_i = 0$ and

$$\lim_{x_i \to 0+} \sup_{\substack{t\in[0,\tau] \\ x_{j\neq i}\in[0,+\infty)^2}} \frac{f_i(t,x_1,x_2,x_3)}{x_i} \cdot B_i < 1, \quad 1 < \lim_{x_i \to +\infty} \inf_{\substack{t\in[\tau/4,3\tau/4] \\ x_{j\neq i}\in[0,+\infty)^2}} \frac{f_i(t,x_1,x_2,x_3)}{x_i} \cdot \frac{A_i}{4}.$$

Proposition 3.6. *Function f_i from Example 3.3 satisfies inequalities (3.2) and (3.3) for sufficiently small q and sufficiently large Q.*

Proof. First, we show that f_i satisfies (3.2). Let us denote

$$\underline{f_0} = \lim_{u\to 0+} \inf_{t\in[\tau/4,3\tau/4]} \frac{f_i(t,u,u,u)}{u}, \quad \overline{f_\infty} = \lim_{u\to +\infty} \sup_{t\in[0,\tau]} \frac{f_i(t,u,u,u)}{u}.$$

Choose $\varepsilon > 0$ and $\delta > 0$ such that

$$1 < (\underline{f_0} - \varepsilon)A_i/4 \quad \text{and} \quad (\overline{f_\infty} + \delta)B_i < 1.$$

Then there exist positive constants r and p such that

$$\begin{aligned} f_i(t,u,u,u) &\geq (\underline{f_0} - \varepsilon)u, \quad (t,u) \in [\tau/4, 3\tau/4] \times (0,r], \\ f_i(t,u,u,u) &\leq (\overline{f_\infty} + \delta)u, \quad (t,u) \in [0,\tau] \times [p,+\infty). \end{aligned}$$

We denote $M = \max\{f_i(t,u,u,u) : t \in [0,\tau],\ u \in [0,p]\}$. Then

$$f_i(t,u,u,u) \leq M + (\overline{f_\infty} + \delta)u, \quad (t,u) \in [0,\tau] \times [0,+\infty).$$

We choose $q \in (0,r]$, define

$$Q = \frac{B_i M + \|g_i\|}{1 - B_i(\overline{f_\infty} + \delta)} + q$$

and let $\overline{K}_{q,Q} = \{x \in K : q \leq \|x_i\| \leq Q,\ i = 1,2,3\}$. Observe that

$$\begin{aligned} \max_{\substack{t\in[0,\tau] \\ x\in[0,Q]^3}} f_i(t,x)\cdot B_i + \|g_i\| &\leq \max_{t\in[0,\tau]} f_i(t,Q,Q,Q)\cdot B_i + \|g_i\| \leq B_i M + B_i(\overline{f_\infty} + \delta)Q + \|g_i\| \\ &= (B_i M + \|g_i\|) + \frac{(B_i M + \|g_i\|)B_i(\overline{f_i^\infty} + \delta)}{1 - B_i(\overline{f_\infty} + \delta)} + q\, B_i(\overline{f_\infty} + \delta) \\ &= \frac{B_i M + \|g_i\|}{1 - B_i(\overline{f_\infty} + \delta)} + q\, B_i(\overline{f_\infty} + \delta) < Q \end{aligned}$$

and

$$\min_{\substack{t\in[\tau/4,3\tau/4] \\ x_i\in[q/4,q] \\ x_{j\neq i}\in[q/4,Q]^2}} f_i(t,x)\cdot A_i \geq \min_{t\in[\tau/4,3\tau/4]} f_i(t,q/4,q/4,q/4)\cdot A_i \geq A_i(\underline{f_0} - \varepsilon)q/4 > q.$$

Now, we show that f_i satisfies (3.3). Recall that $a_i = b_i = 0$ implies $\|g_i\| = 0$. Let us denote

$$\underline{f_\infty} = \lim_{u\to +\infty} \inf_{t\in[\tau/4,3\tau/4]} \frac{f_i(t,u^0_{i1},u^0_{i2},u^0_{i3})}{u}.$$

Choose $\varepsilon > 0$ such that $(\overline{f_\infty} - \varepsilon)A_i/4 > 1$. Then there exist positive constants r and p such that

$$f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w) \le B_i^{-1}u, \quad (t,u,w) \in [0,\tau] \times (0,r] \times [0,+\infty)^2,$$
$$f_i(t, u_{i1}^0, u_{i2}^0, u_{i3}^0) \ge (\underline{f_\infty} - \varepsilon)u, \quad (t,u) \in [\tau/4, 3\tau/4] \times [p,+\infty),$$

We choose $q \in (0,r]$, $Q \in [4p+q,+\infty)$ and let $\overline{K}_{q,Q} = \{x \in K : q \le \|x_i\| \le Q,\ i = 1,2,3\}$. Observe that

$$\max_{\substack{t\in[0,\tau] \\ x_i \in [0,q] \\ x_{j\ne i} \in [0,Q]^2}} f_i(t,x) \cdot B_i \le \max_{t\in[0,\tau]} f_i(t, q_{i1}^Q, q_{i2}^Q, q_{i3}^Q) \cdot B_i < q$$

and

$$\min_{\substack{t\in[\tau/4,3\tau/4] \\ x_i \in [Q/4,Q] \\ x_{j\ne i} \in [q/4,Q]^2}} f_i(t,x) \cdot A_i \ge \min_{t\in[\tau/4,3\tau/4]} f_i\Big(t, (Q/4)_{i1}^0, (Q/4)_{i2}^0, (Q/4)_{i3}^0\Big) \cdot A_i \ge (\underline{f_\infty} - \varepsilon)A_iQ/4 > Q.$$

Finally, note that constants q and Q could be chosen as small and as large as desired, respectively. □

The main result of this paper is following.

Theorem 3.7. *If for every f_i, $i = 1,2,3$, exist $q_i < Q_i$ such that f_i satisfies (3.2) or (3.3) for $0 < q \le q_i$ and $Q_i \le Q < +\infty$, then boundary value problem (1.1), (1.2) has at least one positive solution.*

Proof. We denote $r = \min\{q_i : i = 1,2,3\}$, $R = \max\{Q_i : i = 1,2,3\}$ and let

$$\overline{K}_{r,R} = \{x \in K : r \le \|x_i\| \le R,\ i = 1,2,3\}.$$

By Lemma 3.2, each T_i satisfies compression type condition (i) or expansion type condition (ii) in $\overline{K}_{r,R}$. Therefore, by Krasnosel'skiĭ–Precup fixed point theorem, operator T has a fixed point in $\overline{K}_{r,R}$, which implies that boundary value problem (1.1), (1.2) has at least one positive solution. □

Let us show applicability of Theorem 3.7 in following example. Here and in Section 4, we round numbers to three decimal places unless we can calculate the numbers exactly.

Example 3.8. Consider system of differential equations

$$\begin{aligned} x_1'' + x_1^2(t + x_2x_3)^3 &= 0, \quad t \in (0,\tau), \\ x_2'' + \Big(x_1 t + x_3^{1/3}\Big)\frac{\exp(-x_2) + 1}{2} &= 0, \quad t \in (0,\tau), \\ x_3'' + \frac{80x_3t}{x_3^3+1} + 7\sin(x_1 - x_2) + 7 &= 0, \quad t \in (0,\tau), \end{aligned} \tag{3.5}$$

with boundary conditions

$$\begin{aligned} x_1(0) &= 3x_1(1/5) - x_1(1/2), & x_1(\tau) &= \frac{1}{2}\int_0^\tau t^2 x_1(t)\mathrm{d}t, \\ x_2(0) &= a_2, & x_2(\tau) &= \int_0^\tau (\tau - t)x_2(t)\mathrm{d}t + b_2, \\ x_3(0) &= x_3(1/2) + a_3, & x_3(\tau) &= b_3, \end{aligned} \tag{3.6}$$

where $a_2,\ b_2,\ a_3,\ b_3 \geq 0$. Observe that $1/5$ and $1/2$ appear in the multi-point boundary conditions in first and third line of (3.6). Hence τ is greater than $1/2$.

In this example, the Green's functions (and intervals where assumptions (A1)–(A3) are valid) are as follows (recall that G_0 is given by (2.1)):

$$G_1(t,s) = \frac{1}{\tau/10 - \tau^4/60}\begin{vmatrix} \tau - t & 1/10 & -\tau^4/24 \\ t & -1/10 & \tau - \tau^4/8 \\ G_0(t,s) & G_0(1/2,s) - 3G_0(1/5,s) & -(s\tau^3 - s^4)/24 \end{vmatrix}, \tau \in (1/2, 6^{1/3}),$$

$$G_2(t,s) = \frac{1}{\tau^2(1-\tau^2/6)}\begin{vmatrix} \tau - t & \tau & -\tau^3/3 \\ t & 0 & \tau - \tau^3/6 \\ G_0(t,s) & 0 & -(s^3 - 3s^2\tau + 2s\tau^2)/6 \end{vmatrix}, \quad \tau \in (1/2, \sqrt{6}),$$

$$G_3(t,s) = \frac{2}{\tau}\begin{vmatrix} \tau - t & 1/2 & 0 \\ t & -1/2 & \tau \\ G_0(t,s) & -G_0(1/2,s) & 0 \end{vmatrix}, \quad \tau \in (1/2, +\infty).$$

Observe that $f_1(t,x) = x_1^2(t + x_2x_3)^3$ is non-decreasing with respect to all x_i, $i = 1,2,3$, and $a_1 = b_1 = 0$, and

$$\forall w \in [0,+\infty)\ \lim_{u\to 0+}\sup_{t\in[0,\tau]} \frac{u^2(t+w^2)^3}{u} = 0, \qquad \lim_{u\to+\infty}\inf_{t\in[\tau/4,3\tau/4]} \frac{u^2(t+0)^3}{u} = +\infty.$$

We do not need to calculate B_1 and A_1. But we need $A_1 > 0$, which is true for $\tau \in (1/2, 6^{1/3})$. Therefore (see Example 3.3), f_1 satisfies (3.3) for $\tau \in (1/2, 6^{1/3})$.

Next, $f_2(t,x) = (x_1 t + x_3^{1/3})(\exp(-x_2) + 1)/2$ is bounded with respect to x_2, non-decreasing with respect to x_1, x_3 and

$$\lim_{w\to 0+}\inf_{\substack{t\in[\tau/4,3\tau/4]\\ u\in[0,+\infty)}} \left(wt + w^{1/3}\right)\frac{\exp(-u)+1}{2w} = +\infty,$$

$$\lim_{w\to+\infty}\sup_{\substack{t\in[0,\tau]\\ u\in[0,+\infty)}} \left(wt + w^{1/3}\right)\frac{\exp(-u)+1}{2w} = \tau.$$

We expand $G_2(t,s)$ along the second column and consider

$$\begin{aligned} B_2 &= \sup_{t\in[0,\tau]} \int_0^\tau G_2(t,s)\mathrm{d}s \\ &= \sup_{t\in[0,\tau]} \frac{1}{\tau^2(1-\tau^2/6)}\int_0^\tau \left(\frac{\tau\, t(s^3 - 3s^2\tau + 2s\tau^2)}{6} + \tau\left(\tau - \frac{\tau^3}{6}\right)G_0(t,s)\right)\mathrm{d}s \\ &= \sup_{t\in[0,\tau]} \frac{\tau\, t(\tau^2 - 12) - 2t^2(\tau^2-6)}{4(\tau^2-6)} = \begin{cases} (144\tau^2 - 24\tau^4 + \tau^6)/(32(\tau^2-6)^2), & 1/2 < \tau < 2, \\ \tau^4/(4(6-\tau^2)), & 2 \leq \tau < \sqrt{6}. \end{cases} \end{aligned}$$

Calculations show that $\tau B_2 < 1$ for $\tau \in (1/2, 1.612)$. Therefore (see Example 3.4), f_2 satisfies (3.2) for $\tau \in (1/2, 1.612)$.

Next, $f_3(t,x) = 80x_3t/(x_3^3+1) + 7\sin(x_1 - x_2) + 7$ is bounded with respect to x_1, x_2 and

$$\lim_{x_3\to 0+}\inf_{\substack{t\in[\tau/4,3\tau/4]\\ x_1,x_2\in[0,+\infty)}} \frac{80x_3t}{(x_3^3+1)x_3} + \frac{7\sin(x_1-x_2)+7}{x_3} = 20\tau,$$

$$\lim_{x_3\to+\infty} \sup_{\substack{t\in[0,\tau]\\ x_1,x_2\in[0,+\infty)}} \frac{80x_3t}{(x_3^3+1)x_3} + \frac{7\sin(x_1-x_2)+7}{x_3} = 0.$$

We expand $G_3(t,s)$ along the third column and consider

$$\begin{aligned} A_3 &= \inf_{t\in[\tau/4,3\tau/4]} \int_{\tau/4}^{3\tau/4} G_3(t,s)\mathrm{d}s = \inf_{t\in[\tau/4,3\tau/4]} \frac{2}{\tau}\int_{\tau/4}^{3\tau/4} \Big(\tau(\tau-t)G_0(1/2,s)+\frac{\tau}{2}G_0(t,s)\Big)\,\mathrm{d}s \\ &= \inf_{t\in[\tau/4,3\tau/4]} \frac{-16t^2-\tau(8-15\tau+2\tau^2)+2t(4+\tau^2)}{32} \\ &= \begin{cases} (-12\tau+28\tau^2-3\tau^3)/64, & 1/2<\tau\le 2(2-\sqrt{3}) \text{ or } 2(2+\sqrt{3})<\tau, \\ (-4\tau+12\tau^2-\tau^3)/64, & 2(2-\sqrt{3})<\tau\le 2(2+\sqrt{3}). \end{cases} \end{aligned}$$

Calculations show that $1 < 5\tau A_3$ for $\tau \in (1.197, 8.877)$. Therefore (see Example 3.5), f_3 satisfies (3.2) for $\tau \in (1.197, 8.877)$.

Finally, we consider interval

$$(1/2, 6^{1/3}) \cap (1/2, 1.612) \cap (1.197, 8.877) = (1.197, 1.612).$$

Each f_i satisfies either (3.2) or (3.3) for sufficiently small q, sufficiently large Q and $\tau \in (1.197, 1.612)$. Therefore, by Theorem 3.7, boundary value problem (4.3), (4.4) has at least one positive solution for $\tau \in (1.197, 1.612)$.

4 Numerical result for Dirichlet boundary conditions

In this section, we consider problem (1.1) with boundary conditions

$$x_i(0) = a_i,\ x_i(\tau) = b_i, \tag{4.1}$$

and show examples where is compared theoretical estimation of τ with result obtained numerically. Note that here $G_i = G_0$, $A_i = \tau^2/16$ and $B_i = \tau^2/8$ for every $i = 1, 2, 3$.

For numerical result let us consider the initial conditions

$$x_i(0) = a_i, \quad x_i'(0) = c_i \in \mathbb{R}, \quad i = 1, 2, 3. \tag{4.2}$$

Let $c = (c_1, c_2, c_3)$ and $x^c = (x_1^c, x_2^c, x_3^c)$ be a solution of initial value problem (1.1), (4.2). We denote by $t_1(c) > 0$ the positive argument for which $x^c(t_1(c)) = (b_1, b_2, b_3)$ holds for the first time. Such $t_1(c)$ exists if and only if boundary value problem (1.1), (4.1) has a solution for $t_1(c) = \tau$. Thus set of values of the map $c \mapsto t_1(c)$ determines values of the τ. We assume that if there is no c such that $t_1(c) = \tau > 0$, then $t_1(c) = 0$.

In case of one equation this method is known as the shooting method. We do the following in our case. We fix c_1 and consider $t_1(c_1, \cdot, \cdot)$. If problem (1.1), (4.1) has a solution for $t_1(c) = \tau$, then $t_1(c_1, \cdot, \cdot)$ is everywhere zero except one point.

To obtain the result we are using “brute force”, i.e. go through all possible choices. To make count of choices less, we consider meshes with step sizes θ_i for c_i, $i = 1, 2, 3$. To make sure to “shoot somewhere”, we consider weakened conditions

$$\Big(x_2(t_1(c)) - b_2\Big)^2 + \Big(x_3(t_1(c)) - b_3\Big)^2 < \varepsilon^2.$$

Thus for every c_1 there exists a set $\Omega_\varepsilon \subset \mathbb{R}^2$ such that $t_1(c) > 0$ for $(c_2, c_3) \in \Omega_\varepsilon$. We denote $t_M(c_1) = \max\{t_1(c) : (c_2, c_3) \in \mathbb{R}^2\}$. Numerical result is a discrete plot $c_1 \mapsto t_M(c_1)$.

In the following examples we compare the results. Examples emphasize that Theorem 3.7 gives sufficient conditions.

Example 4.1. Consider system of differential equations

$$\begin{aligned} x_1'' + (x_1 t^3 + x_3)^{1/2}\, x_2^{1/3} &= 0, \quad t \in (0, \tau), \\ x_2'' + (x_1 t^3 + x_3^{1/2}) \frac{\exp(-x_2) + 1}{10} &= 0, \quad t \in (0, \tau), \\ x_3'' + \frac{16^2 x_3^4 + x_3}{1 + x_3^3}\Big(2 + \sin(x_1 t + x_2)\Big) &= 0, \quad t \in (0, \tau), \end{aligned} \tag{4.3}$$

with boundary conditions

$$\begin{aligned} x_1(0) &= 0.2, & x_1(\tau) &= 0, \\ x_2(0) &= 0, & x_2(\tau) &= 0.2, \\ x_3(0) &= 0, & x_3(\tau) &= 0. \end{aligned} \tag{4.4}$$

Here $f_1(t, x) = (x_1 t^3 + x_3)^{1/2}\, x_2^{1/3}$ is non-decreasing with respect to all x_i, $i = 1, 2, 3$, and

$$\lim_{u \to 0+} \inf_{t \in [\tau/4, 3\tau/4]} \frac{(ut^3 + u)^{1/2}\, u^{1/3}}{u} = +\infty, \qquad \lim_{u \to +\infty} \sup_{t \in [0, \tau]} \frac{(ut^3 + u)^{1/2} u^{1/3}}{u} = 0.$$

Therefore, f_1 satisfies (3.2) for $\tau \in (0, +\infty)$.

Next, $f_2(t, x) = (x_1 t^3 + x_3^{1/2})(\exp(-x_2) + 1)/10$ is bounded with respect to x_2, non-decreasing with respect to x_1, x_3 and

$$\lim_{w \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ u \in [0, +\infty)}} \Big(w t^3 + w^{1/2}\Big) \frac{\exp(-u) + 1}{10 w} = +\infty,$$

$$\lim_{w \to +\infty} \sup_{\substack{t \in [0, \tau] \\ u \in [0, +\infty)}} \Big(w t^3 + w^{1/2}\Big) \frac{\exp(-u) + 1}{10 w} = \frac{\tau^3}{5}.$$

Calculations show that $B_2 \tau^3/5 < 1$ for $\tau \in (0, 40^{1/5})$. Therefore, f_2 satisfies (3.2) for $\tau \in (0, 40^{1/5})$.

Next, $f_3(t, x) = (16^2 x_3^4 + x_3)(2 + \sin(x_1 t + x_2))/(1 + x_3^3)$ is bounded with respect to x_1, x_2, and

$$\lim_{x_3 \to 0+} \sup_{\substack{t \in [0, \tau] \\ x_1, x_2 \in [0, +\infty)}} \frac{16^2 x_3^4 + x_3}{(1 + x_3^3) x_3}\Big(2 + \sin(x_1 t + x_2)\Big) = 3,$$

$$\lim_{x_3 \to +\infty} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ x_1, x_2 \in [0, +\infty)}} \frac{16^2 x_3^4 + x_3}{(1 + x_3^3) x_3}\Big(2 + \sin(x_1 t + x_2)\Big) = 16^2.$$

Calculations show that $3B_3 < 1 < 16^2 A_3/4$ for $\tau \in (1/2, 2\sqrt{2/3})$. Therefore, f_3 satisfies (3.3) for $\tau \in (1/2, 2\sqrt{2/3})$.

Each f_i satisfies either (3.2) or (3.3) for sufficiently small q, sufficiently large Q and $\tau \in (1/2, 2\sqrt{2/3})$. Therefore, by Theorem 3.7, the theoretical result is $1/2 < \tau < 2\sqrt{2/3}$, or approximately $1/2 < \tau < 1.633$.

Since $a_2 \leq b_2$ and $a_3 \leq b_3$, we consider non-negative c_2 and c_3. For numerical result we make meshes in interval $[-1, 1]$ for c_1, $[0, 2]$ for c_2 and $[0.001, 2.001]$ for c_3 with step sizes $\theta_1 = \theta_2 = \theta_3 = 0.1$ and $\varepsilon = 0.1$. The result is illustrated in Figure 4.1. Numerical result shows that $0.200 \leq \tau \leq 2.579$.

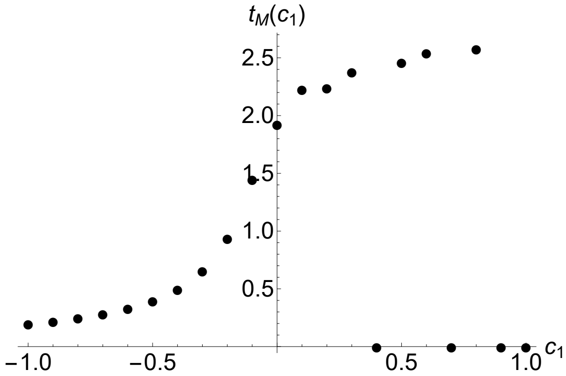

Figure 4.1: Graph of the $c_1 \mapsto t_M(c_1)$ for problem (4.3), (4.4).

Example 4.2. Consider system of differential equations

$$\begin{aligned} x_1'' + (x_1x_2x_3)^{1/4} &= 0, \quad t \in (0, \tau), \\ x_2'' + \frac{1}{1 + tx_2} + \frac{1}{1 + x_1x_3} &= 0, \quad t \in (0, \tau), \\ x_3'' + + \frac{(15 + 4t)x_3}{1 + x_3^2} + \cos x_1 \sin x_2 + 1 &= 0, \quad t \in (0, \tau), \end{aligned} \tag{4.5}$$

with boundary conditions

$$\begin{aligned} x_1(0) &= 1, \quad & x_1(\tau) &= 0, \\ x_2(0) &= 0, \quad & x_2(\tau) &= 1, \\ x_3(0) &= 1, \quad & x_3(\tau) &= 1. \end{aligned} \tag{4.6}$$

Here $f_1(t, x) = (x_1x_2x_3)^{1/4}$ is non-decreasing with respect to all x_i, $i = 1, 2, 3$, and

$$\lim_{u \to 0+} \inf_{t \in [\tau/4, 3\tau/4]} \frac{u^{3/4}}{u} = +\infty, \qquad \lim_{u \to +\infty} \sup_{t \in [0, \tau]} \frac{u^{3/4}}{u} = 0.$$

Therefore, f_1 satisfies (3.2) for $\tau \in (0, +\infty)$.

Next, $f_2(t, x) = (1 + tx_2)^{-1} + (1 + x_1x_3)^{-1}$ is bounded with respect to x_1, x_3 and

$$\lim_{x_2 \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ x_1, x_3 \in [0, +\infty)}} \frac{1}{(1 + tx_2)x_2} + \frac{1}{(1 + x_1x_3)x_2} = +\infty,$$

$$\lim_{x_2 \to +\infty} \sup_{\substack{t \in [0, \tau] \\ x_1, x_3 \in [0, +\infty)}} \frac{1}{(1 + tx_2)x_2} + \frac{1}{(1 + x_1x_3)x_2} = 0.$$

Therefore, f_2 satisfies (3.2) for $\tau \in (0, +\infty)$.

Next, $f_3(t, x) = (15 + 4t)x_3/(1 + x_3^2) + \cos x_1 \sin x_2 + 1$ is bounded with respect to x_1, x_2, and

$$\lim_{x_3 \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ x_1, x_2 \in [0, +\infty)}} \frac{(15 + 4t)x_3}{(1 + x_3^2)x_3} + \frac{\cos x_1 \sin x_2 + 1}{x_3} = 15 + \tau,$$

$$\lim_{x_3\to+\infty} \sup_{\substack{t\in[0,\tau]\\ x_1,x_2\in[0,+\infty)}} \frac{(15+4t)x_3}{(1+x_3^2)x_3} + \frac{\cos x_1 \sin x_2 + 1}{x_3} = 0.$$

Calculations show that $1 < (15+\tau)A_3/4$ for $\tau \in (1.944, +\infty)$. Therefore, f_3 satisfies (3.2) for $\tau \in (1.944, +\infty)$.

Each f_i satisfies (3.2) for sufficiently small q, sufficiently large Q and $\tau \in (1.944, +\infty)$. Therefore, by Theorem 3.7, the theoretical result is $\tau > 1.944$.

For numerical result we make meshes in interval $[-7, 0]$ for c_1, $[0, 7]$ for c_2 and c_3 with step sizes $\theta_1 = 1$, $\theta_2 = \theta_3 = 0.1$ and $\varepsilon = 0.1$. The result is illustrated in Figure 4.2. Numerical result shows that τ could be less than 1.944.

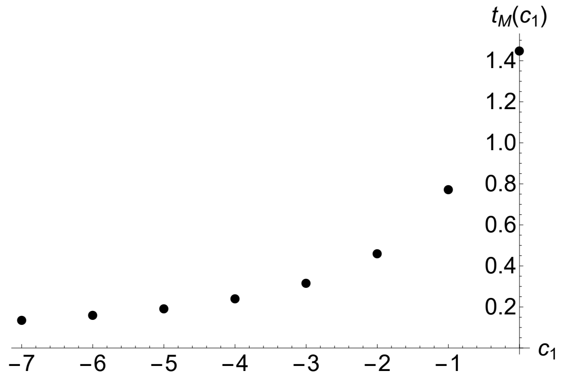

Figure 4.2: Graph of the $c_1 \mapsto t_M(c_1)$ for problem (4.5), (4.6).

Remark 4.3. There is no ground to say that this method is not suitable for nonlocal conditions, for instance, $x(0) = \varphi_i[x_i] + a_i$, $x(\tau) = b_i$. But, since we are using "brute force" (which is long itself), in case of nonlocal conditions program needs much smaller step size to get nonzero $t_M(c_1)$, and hence much more time to run, which makes the program inefficient.

Acknowledgements

The authors would like to thank the referee for making several suggestions that significantly improved an earlier version of this paper.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18**(1976), No. 4, 620–709. https://doi.org/10.1137/1018114; MR0415432; Zbl 0345.47044

[2] J. R. Graef, B. Yang, Existence and nonexistence of positive solutions of a nonlinear third order boundary value problem, *Electron. J. Qual. Theory Differ. Equ., Proc. 8'th Coll. Qualitative Theory of Diff. Equ.* **2007**, No. 9, 1–13. https://doi.org/10.14232/ejqtde.2007.7.9; MR2509167; Zbl 1213.34036

[3] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, 1988. MR0959889; Zbl 0661.47045

[4] X. Hao, L. Liu, Y. Wu, Positive solutions for second order differential systems with nonlocal conditions, *Fixed Point Theory* **13**(2012), No. 2, 507–5016. MR3024335; Zbl 1286.34043

[5] J. Henderson, R. Luca, Positive solutions for systems of second-order integral boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 70, 1–21. https://doi.org/10.14232/ejqtde.2013.1.70; MR3151717; Zbl 1340.34097

[6] J. Henderson, R. Luca, Existence of positive solutions for a system of nonlinear second-order integral boundary value problems, *Discrete Contin. Dyn. Syst., Dynamical systems, differential equations and applications*, 10th AIMS Conference. Suppl. (2015), 596–604. https://doi.org/10.3934/proc.2015.0596; MR3462494; Zbl 1341.34032

[7] J. Henderson, S. K. Ntouyas, Positive solutions for systems of nth order three-point nonlocal boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* **2007**, No. 18, 1–12. https://doi.org/10.14232/ejqtde.2007.1.18; MR2336606; Zbl 1182.34029

[8] D.-R. Herlea, D. O'Regan, R. Precup, Harnack type inequalities and multiple solutions in cones of nonlinear problems, *Z. Anal. Anwend.* **39**(2020), No. 2, 151–170. https://doi.org/10.4171/ZAA/1655; MR4082922; Zbl 1444.47065

[9] L. Hu, L. Wang, Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations, *J. Math. Anal. Appl.* **335**(2007), No. 2, 1052–1060. https://doi.org/10.1016/j.jmaa.2006.11.031; MR2346890; Zbl 1127.34010

[10] M. A. Krasnosel'skiĭ, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964. MR0181881; Zbl 0121.10604

[11] K. Lan, Coexistence fixed point theorems in product Banach spaces and applications, *Math. Methods Appl. Sci.* **44**(2021), No. 5, 3960–3984. https://doi.org/10.1002/mma.7001; MR4227968; Zbl 1509.47080

[12] R. Precup, A vector version of Krasnoselskiĭs fixed point theorem in cones and positive periodic solutions of nonlinear systems, *J. Fixed Point Theory Appl.* **2**(2007), 141–151. https://doi.org/10.1007/s11784-007-0027-4; MR2336504; Zbl 1134.47041

[13] R. Precup, Componentwise compressionexpansion conditions for systems of nonlinear operator equations and applications, in: *Mathematical models in engineering, biology and medicine*, AIP Conference Proceedings, Vol. 1124, American Institute of Physics, Melville, NY, 2009, pp. 284–293. MR2657155

[14] R. Precup, J. Rodríguez-López, Multiplicity results for operator systems via fixed point index, *Results Math.* **74**(2019), No. 1, Paper No. 25. https://doi.org/10.1007/s00025-019-0955-5; MR3897520; Zbl 1407.34036

[15] J. Rodríguez-López, A fixed point index approach to Krasnoselskiĭ–Precup fixed point theorem in cones and applications, *Nonlinear Anal.* **226**(2023), No. 113138. https://doi.org/10.1016/j.na.2022.113138; MR4484853; Zbl 7608944

[16] S. Roman, Linear differential equation with additional conditions and formulae for Greens function, *Math. Model. Anal.* **16**(2011), No. 3, 401–417. https://doi.org/10.3846/13926292.2011.602125; MR2834941

[17] S. Smirnov, Existence of multiple positive solutions for a third-order boundary value problem with nonlocal conditions, *Nonlinear Anal. Model. Control* **28**(2023), No. 3, 597–612. https://doi.org/10.15388/namc.2023.28.32123; MR4585533; Zbl 1525.34054

[18] J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.* **74**(2006), 673–693. https://doi.org/10.1112/S0024610706023179; MR2286439; Zbl 1115.34028

[19] J. R. L. Webb, G. Infante, Nonlocal boundary value problems of arbitrary order, *J. London Math. Soc.* **79**(2009), 238–258. https://doi.org/10.1112/jlms/jdn066; MR2472143; Zbl 1165.34010

[20] Wolfram Research, Inc., *Mathematica*, Version 10.4, Champaign, Illinois, 2016.