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Ground-state solutions of a Hartree–Fock type system involving critical Sobolev exponent

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Abstract. In this paper, ground-state solutions to a Hartree–Fock type system with a critical growth are studied. Firstly, instead of establishing the local Palais–Smale (P.–S.) condition and estimating the mountain-pass critical level, a perturbation method is used to recover compactness and obtain the existence of ground-state solutions. To achieve this, an important step is to get the right continuity of the mountain-pass level on the coefficient in front of perturbing terms. Subsequently, depending on the internal parameters of coupled nonlinearities, whether the ground state is semi-trivial or vectorial is proved.

Keywords: Hartree–Fock systems, ground-state solutions, critical growth.

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1 Introduction

In this paper, we will study the following class of Hartree–Fock (HF) system

$$\begin{cases} -\Delta u + u + \phi_{u,v} u = |u|^{2q-2}u + \beta|v|^q|u|^{q-2}u + \mu(u^5 + \alpha|v|^3|u|u), & x \in \mathbb{R}^3, \\ -\Delta v + v + \phi_{u,v} v = |v|^{2q-2}v + \beta|u|^q|v|^{q-2}v + \mu(v^5 + \alpha|u|^3|v|v), & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the Coulomb term $\phi_{u,v}$ has the following form

$$\phi_{u,v}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)+v^2(y)}{|x-y|}\mathrm{d}y, \qquad x \in \mathbb{R}^3, \tag{1.2}$$

$\alpha, \beta, \mu \in \mathbb{R}_+ := [0, \infty)$ are parameters and $q \in (2, 3)$.

It is well known that the (HF) equation is one of the most important equations in quantum physics, condensed matter physics and quantum chemistry. For example, in the study of a molecular system composed of M atomic nucleus interacting with N electrons through Coulomb potential, the (HF) equation is used as an approximation to describe the stationary state, and one can refer to [5] for the specific process of derivation. According to [5], in the system (1.1), $-\Delta u, -\Delta v$ represent the kinetic part of the electronic system, Vu, Vv denote potentials of the action on electronic system by nucleus, $\phi_{u,v}u, \phi_{u,v}v$ represent the electron-electron

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Coulomb interactions, and the power-type nonlinearity describes the effects of exchange and correlation among electrons. For more details on the physical aspects of the Hartree–Fock system, we refer readers to [1,2,8–11] and the references therein.

In mathematics, a particular case of system (1.1), when $\mu = 0$, leads to the following class of Hartree–Fock type system with a cooperative pure power and subcritical nonlinearity

$$\begin{cases} -\Delta u + u + \phi_{u,v} u = |u|^{2q-2}u + \beta |v|^q |u|^{q-2} u, & x \in \mathbb{R}^3, \\ -\Delta v + v + \phi_{u,v} v = |v|^{2q-2}v + \beta |u|^q |v|^{q-2} v, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

which has been studied by d'Avenia, Maia and Siciliano in [5]. In the case that $q \in (3/2, 3)$, they showed the existence of semitrivial and vectorial ground state depending on parameters involved. Furthermore, they also derived the asymptotic behavior of ground states with respect to the parameter β.

In view of conclusions obtained in [5], we considered the Sobolev critical case $q = 3$. However, combining the Pohozaev identity and Nehari manifold, it could be proved that the system (1.3) has no nontrivial solution when $q = 3$. Motivated by the above facts, we would like to consider the system (1.1), which is obtained through a Sobolev perturbation basing on the above system (1.3). It is well known that since Brezis and Nirenberg published their famous paper [3] in 1983, elliptic equations or systems with Sobolev critical growth have been researched extensively. The usual strategy to achieve the ground-state solution to these critical problems is establishing the local (P.-S.) condition and verifying that the ground-state energy belongs to the interval where the (P.-S.) condition holds. Differently, in this paper, we will achieve the existence of ground-state solutions to the system (1.1) with a perturbation method.

Before stating our main results, we introduce the variational setting used in this paper. Firstly, let $H_r^1(\mathbb{R}^3) = \{w \in H^1(\mathbb{R}^3) : w(x) = w(|x|)\}$ and $\|w\|_1^2 = \int_{\mathbb{R}^3} [|\nabla w|^2 + w^2]$ for $w \in H_r^1(\mathbb{R}^3)$. Then our working space is $H := H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$ endowed with the norm

$$\|(u,v)\| = \left(\|u\|_1^2 + \|v\|_1^2\right)^{1/2}, \qquad (u,v) \in H.$$

It is well known that the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2,6]$ and compact for $s \in (2,6)$. Hence the same conclusions hold for the embedding $H \hookrightarrow L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$ for $s \in [2,6]$. Throughout this paper, denote the norm endowed in $L^s(\mathbb{R}^3)$ by $|\cdot|_s$: $|w|_s = \left[\int_{\mathbb{R}^3} |w|^s\right]^{1/s}$ for $w \in L^s(\mathbb{R}^3)$. While the norm of $L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$ is $|(u,v)|_s = \left(|u|_s^s + |v|_s^s\right)^{1/s}$ for $(u,v) \in L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$. Subsequently, we will give the energy functional corresponding to the system (1.1). According to the Hardy–Littlewood–Sobolev inequality, the nonlocal term $\int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2)$ is well defined in H. Therefore, we could define the energy functional related to the system (1.1) as

$$\begin{aligned} J_\mu(u,v) &= \frac{1}{2}\|(u,v)\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) - \frac{1}{2q}\left[|u|_{2q}^{2q} + |v|_{2q}^{2q} + 2\beta \int_{\mathbb{R}^3} |u|^q |v|^q\right] \\ &\quad - \frac{\mu}{6}\left[|u|_6^6 + |v|_6^6 + 2\alpha \int_{\mathbb{R}^3} |u|^3 |v|^3\right] \\ &=: \frac{1}{2}A(u,v) + \frac{1}{4}B(u,v) - \frac{1}{2q}C(u,v) - \frac{\mu}{6}D(u,v), \quad (u,v) \in H. \end{aligned} \tag{1.4}$$

Via a standard proof, there also holds that $J_\mu \in C^1(H,\mathbb{R})$ with

$$\begin{aligned}\left\langle J'_\mu(u,v),(\varphi,\psi)\right\rangle = &\int_{\mathbb{R}^3}(\nabla u\cdot\nabla\varphi+\nabla v\cdot\nabla\psi+u\varphi+v\psi)+\int_{\mathbb{R}^3}\phi_{u,v}(u\varphi+v\psi)\\ &-\int_{\mathbb{R}^3}\left[|u|^{2q-2}u\varphi+|v|^{2q-2}v\psi+\beta(|v|^q|u|^{q-2}u\varphi+|u|^q|v|^{q-2}v\psi)\right]\\ &-\mu\int_{\mathbb{R}^3}\left[u^5\varphi+v^5\psi+\alpha(|v|^3|u|u\varphi+|u|^3|v|v\psi)\right], \qquad (u,v),(\varphi,\psi)\in H.\end{aligned}$$

Hence, finding solutions of the system (1.1) is equivalent to seeking critical points of the functional J_μ in H. Furthermore, to achieve the ground-state solution to the system (1.1), we may consider the ground state of the energy functional J_μ, and the Nehari manifold is used in this paper. Now, let I_μ be the related Nehari functional, that is, $I_\mu(u,v) := \langle J'_\mu(u,v),(u,v)\rangle, (u,v)\in H$. Then adopting notations given in (1.4) it could be rewritten as

$$I_\mu(u,v) = A(u,v)+B(u,v)-C(u,v)-\mu D(u,v), \qquad (u,v)\in H. \tag{1.5}$$

Let us denote by $\mathcal{N}_\mu$ the Nehari manifold associated to the functional J_μ, namely

$$\mathcal{N}_\mu = \{(u,v)\in H\setminus\{(0,0)\} : I_\mu(u,v)=0\},$$

and define the ground-state energy as

$$d(\mu) = \inf_{\mathcal{N}_\mu} J_\mu.$$

In this context, the ground-state solution to be found in this paper is a radial ground state whose energy is minimal among all other radial ones.

Now, we formulate our first result for the system (1.1).

Theorem 1.1. *Assume that $q\in(2,3)$. Then for any given $\alpha,\beta\in\mathbb{R}_+$, there exists $\mu_0>0$ such that the system (1.1) has a ground-state solution $(u_\mu,v_\mu)\neq(0,0)$ for all $\mu\in[0,\mu_0)$.*

An important step to prove Theorem 1.1 via perturbation methods is estimating the distance between the (P.–S.)$_{m(\mu)}$ sequence of the functional J_μ and the ground-state critical points set of the functional J_0 for μ small enough. Here $m(\mu)$ is the mountain-pass level of the functional J_μ. To achieve this, we first verify the fact that $m(\mu)=d(\mu)$ and get the right continuity of $m(\cdot)$ at $\mu=0$ by showing that $\lim_{\mu\to 0^+} d(\mu)=d(0)$ subsequently, where the implicit function theorem is used.

Basing on the existence of ground-state radial solutions, motivated by [4] and [5], we also consider whether the ground state obtained above is semitrivial or vectorial and get the following conclusion. Here we say that $(u,v)\neq(0,0)$ is semitrivial if $u=0$ or $v=0$, and (u,v) is vectorial if $u\neq0$ and $v\neq0$.

Theorem 1.2. *Assume that $q\in(2,3)$ and $\mu\in[0,\mu_0)$, where μ_0 is given by Theorem 1.1. Let (u_μ,v_μ) be the ground state achieved in Theorem 1.1.*

(i) If $0\leqslant\alpha<3, 0\leqslant\beta<2^{q-1}-1$, then (u_μ,v_μ) is semitrivial.

(ii) If $\alpha>3,\beta>2^{q-1}-1$, then (u_μ,v_μ) is vectorial.

In view of Theorem 1.2, there is an open question that whether the ground state obtained in Theorem 1.1 is semitrivial or vectorial in the cases that $(\alpha,\beta) \in (0,3] \times [2^{q-1}-1,\infty)$ or $(\alpha,\beta) \in [3,\infty) \times (0,2^{q-1}-1]$. This is caused by the non-homogeneity of the nonlinearity in the system (1.1).

This paper is organized as follows. In Section 2, we give some preliminaries to get the existence of ground state via the perturbation method, subsequently, Theorems 1.1 and 1.2 are proved in Sections 3 and 4 respectively. Throughout this paper, $C_i (i = 0,1,2,\dots)$ represent some positive constants which may be different from line to line.

2 Preliminaries

In this section, we first give some inequalities about the four functionals A, B, C, and D by the following lemma.

Lemma 2.1. *There exist some constants C_0, C_1, C_2 independent of μ such that for any $(u,v) \in H$, the following inequalities hold*

$$B(u,v) \leqslant C_0[A(u,v)]^2, \tag{2.1}$$

$$C(u,v) \leqslant C_1|(u,v)|_{2q}^{2q} \leqslant C_2[A(u,v)]^q, \tag{2.2}$$

$$D(u,v) \leqslant C_1|(u,v)|_6^6 \leqslant C_2[A(u,v)]^3. \tag{2.3}$$

Proof. For (2.1), it follows from (1.2) that $\phi_{u,v} \in D^{1,2}(\mathbb{R}^3)$ is a weak solution to the equation $-\Delta\phi_{u,v} = u^2 + v^2$ for all $(u,v) \in H$. Consequently,

$$B(u,v) = \int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) = \int_{\mathbb{R}^3} |\nabla\phi_{u,v}|^2.$$

By the Hölder inequality and the Sobolev embedding, there exists a constant $C_0 > 0$ independent of (u,v) such that

$$\int_{\mathbb{R}^3} \phi_{u,v}u^2 \leqslant |\phi_{u,v}|_6 |u|_{12/5}^2 \leqslant C_0|\nabla\phi_{u,v}|_2 \|u\|_1^2.$$

Similarly, we get

$$\int_{\mathbb{R}^3} \phi_{u,v}v^2 \leqslant C_0|\nabla\phi_{u,v}|_2 \|v\|_1^2.$$

Thus

$$|\nabla\phi_{u,v}|_2^2 = \int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) \leqslant C_0|\nabla\phi_{u,v}|_2 \|(u,v)\|^2 = C_0|\nabla\phi_{u,v}|_2 A(u,v),$$

which implies that (2.1) holds.

By the Hölder inequality and the embedding that $H \hookrightarrow L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$ for $s \in [2,6]$,

$$C(u,v) \leqslant |u|_{2q}^{2q} + |v|_{2q}^{2q} + 2\beta|u|_{2q}^q|v|_{2q}^q \leqslant \max\{\beta,1\}\left(|u|_{2q}^q + |v|_{2q}^q\right)^2 \leqslant C_1|(u,v)|_{2q}^{2q} \leqslant C_2[A(u,v)]^q.$$

Hence (2.2) holds. Similarly, (2.3) holds. □

Next, we prove that the functional J_μ has a mountain pass geometry structure for all $\mu \in \mathbb{R}_+$. Let

$$\Gamma_\mu = \{\gamma \in C([0,1],H) : \gamma(0) = 0, J_\mu(\gamma(1)) < 0\},$$

and

$$m(\mu) = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} J_\mu(\gamma(t)).$$

Then we could prove that both Γ_μ and $m(\mu)$ are well defined.

Lemma 2.2. *Assume $\mu \in \mathbb{R}_+$. Then $\Gamma_\mu \neq \emptyset$ and $m(\mu) > 0$.*

Proof. First, for any $(u,v) \in H\backslash\{(0,0)\}$, we define a fiber mapping corresponding to the functional J_μ as follows:

$$\begin{aligned} g_{u,v}(t) &= J_\mu(t(u,v)) \\ &= \frac{t^2}{2}A(u,v) + \frac{t^4}{4}B(u,v) - \frac{t^{2q}}{2q}C(u,v) - \frac{\mu}{6}t^6 D(u,v), \quad t \in \mathbb{R}. \end{aligned} \tag{2.4}$$

Since $q \in (2,3)$, there exists a sufficiently small positive number δ depending on μ such that $g_{u,v}(t) > 0, t \in (0,\delta)$. Moreover, note that $g_{u,v}(t) \to -\infty$ as $t \to \infty$. Then there exists $t_0 > 0$ such that $J_\mu(t_0(u,v)) = g_{u,v}(t_0) < 0$. Let $\gamma_0(t) = tt_0(u,v), t \in [0,1]$. Then $\gamma_0 \in \Gamma_\mu$.

For $\mu \in \mathbb{R}_+$, it follows from inequalities (2.2) and (2.3) that

$$J_\mu(u,v) \geqslant \frac{1}{2}A(u,v) - \frac{1}{2q}C_2[A(u,v)]^q - \frac{1}{6}\mu C_2[A(u,v)]^3, \quad (u,v) \in H.$$

Therefore, there exists $\rho > 0$ depending on μ such that if $0 < \|(u,v)\|^2 = A(u,v) < \rho^2$, then $J_\mu(u,v) > 0$. Moreover,

$$\alpha_\mu := \inf_{\|(u,v)\|=\rho} J_\mu(u,v) > 0.$$

Furthermore, by the standard process one can deduce that $m(\mu) \geqslant \alpha_\mu > 0$. □

Lemma 2.3. *Suppose that $(u,v) \in H\backslash\{(0,0)\}$. Then the following conclusions hold:*

(i) for any $\mu \in \mathbb{R}_+$, there exists a unique $t(\mu) > 0$ such that $t(\mu)(u,v) \in \mathcal{N}_\mu$, $I_\mu(t(u,v)) > 0, t \in (0,t(\mu))$ and $I_\mu(t(u,v)) < 0, t \in (t(\mu),\infty)$. Furthermore,

$$J_\mu(t(\mu)(u,v)) = \max_{t\in\mathbb{R}_+} J_\mu(t(u,v));$$

(ii) the function $t(\cdot)$: $\mathbb{R}_+ \to (0,\infty)$ is differentiable and

$$t'(\mu) = -\frac{t^5(\mu)D(u,v)}{2A(u,v) + 4t^2(\mu)B(u,v) + 2qt^{2q-2}(\mu)C(u,v) + 6\mu t^4(\mu)D(u,v)}. \tag{2.5}$$

Moreover, $t(\cdot)$ is decreasing in μ.

Proof. (i) Assume $\mu \in \mathbb{R}_+$. For each $(u,v) \in H\backslash\{(0,0)\}$, recall the definition of $g_{u,v}$ given in (2.4). Then

$$g'_{u,v}(t) = tA(u,v) + t^3B(u,v) - t^{2q-1}C(u,v) - \mu t^5 D(u,v), \qquad t \in \mathbb{R}_+, \tag{2.6}$$

which yields that

$$g'_{u,v}(t)/t \to A(u,v) > 0, t \to 0^+, \qquad g'_{u,v}(t) \to -\infty,\ t \to \infty. \tag{2.7}$$

Therefore, there exists $t(\mu) > 0$ satisfying $g'_{u,v}(t(\mu)) = 0$, and so $t(\mu)u \in \mathcal{N}_\mu$. Furthermore, it follows from (2.6) that

$$t^{-2}(\mu)A(u,v) - t^{2q-4}(\mu)C(u,v) - \mu t^2(\mu)D(u,v) = -B(u,v).$$

Because the function $t \mapsto t^{-2}A(u,v) - t^{2q-4}C(u,v) - \mu t^2 D(u,v)$ is decreasing in t, then $g'_{(u,v)}(t) = 0$ has a unique positive root. Hence, $t(\mu)$ is the unique positive critical point of $g_{u,v}$. Combining this with (2.7) and (1.5), we know that (i) holds.

(ii) Let $H(t,\mu) = I_\mu(t(u,v)), (t,\mu) \in (-\delta,\infty) \times (-\delta,\infty)$ for some $\delta > 0$. Then it follows from (1.5) that

$$H(t,\mu) = t^2 A(u,v) + t^4 B(u,v) - t^{2q} C(u,v) - \mu t^6 D(u,v), \qquad (t,\mu) \in (-\delta,\infty) \times (-\delta,\infty).$$

For any $(t,\mu) \in (-\delta,\infty) \times (-\delta,\infty)$ for some $\delta > 0$, we have

$$\frac{\partial H}{\partial t}(t,\mu) = 2tA(u,v) + 4t^3 B(u,v) - 2qt^{2q-1}C(u,v) - 6\mu t^5 D(u,v) \tag{2.8}$$

and

$$\frac{\partial H}{\partial \mu}(t,\mu) = -t^6 D(u,v).$$

Note that $H(t(\mu),\mu) = 0$ i.e. $I_\mu(t(\mu)(u,v)) = 0$ for $\mu \in [0,1]$. Then it could be derived from (2.8) and (1.5) that

$$\frac{\partial H}{\partial t}(t(\mu),\mu) = -2t(\mu)A(u,v) - (2q-4)t^{2q-1}(\mu)C(u,v) - 2\mu t^5(\mu)D(u,v) < 0.$$

Therefore, by the implicit function theorem, we can obtain that $t(\cdot) : \mathbb{R}_+ \to (0,\infty)$ is continuous and differentiable, and (2.5) holds. It then follows from (2.5) directly that for given $(u,v) \in H\backslash\{(0,0)\}$, $t(\cdot)$ is decreasing in μ. □

Lemma 2.4. $\inf_{\mu\in[0,1]} \operatorname{dist}(0,\mathcal{N}_\mu) > 0$.

Proof. Assume $\mu \in [0,1]$. Given $(u,v) \in \mathcal{N}_\mu$, it follows from (1.5), (2.2) and (2.3) that

$$\begin{aligned} A(u,v) &\leqslant A(u,v) + B(u,v) \\ &= C(u,v) + \mu D(u,v) \\ &\leqslant C_2 \left[[A(u,v)]^q + [A(u,v)]^3 \right]. \end{aligned}$$

Therefore there exists a $\sigma > 0$ independent of μ such that $\|(u,v)\|^2 = A(u,v) \geqslant \sigma$ for all $(u,v) \in \mathcal{N}_\mu$. The proof is complete. □

In view of Lemma 2.4, since $2q > 4$, then for any $(u,v) \in \mathcal{N}_\mu$, it holds that

$$J_\mu(u,v) = J_\mu(u,v) - \frac{1}{4} I_\mu(u,v) \geqslant \frac{1}{4}\|(u,v)\|^2, \qquad (u,v) \in \mathcal{N}_\mu, \tag{2.9}$$

from which one can also derive that $d(\mu)$ is well defined for all $\mu \in \mathbb{R}_+$. Moreover, recall the definition of $m(\mu)$. Then by Lemma 2.2 and (i) of Lemma 2.3, we can prove that following lemma via a standard process similarly to the proof of [12, Theorem 4.2, p. 73].

Lemma 2.5. *For any $\mu \in \mathbb{R}_+$, it holds that $m(\mu) = d(\mu)$.*

Note that due to definitions of J_μ, Γ_μ and $m(\mu)$ for $\mu \in \mathbb{R}_+$, it could be concluded that $m(\cdot)$ is decreasing on $\mathbb{R}_+$. Then by Lemma 2.5, $d(\cdot)$ is also decreasing on $\mathbb{R}_+$. Now, we will prove the continuity of $m(\cdot)$ at $\mu = 0$. Actually, by the above lemma, it is sufficient to illustrate the right continuity of $d(\cdot)$ at $\mu = 0$. Hence, we have the following lemma.

Lemma 2.6. *$d(\cdot)$ is right continuous at $\mu = 0$.*

Proof. Assume $\{\mu_n\} \subset [0,1]$ satisfies that $\mu_n \to 0^+$ as $n \to \infty$. Then, according to the definition of $d(\mu_n)$, for each $\varepsilon \in (0, d(0))$, there exists $(u_n, v_n) \in \mathcal{N}_{\mu_n}$ such that

$$J_{\mu_n}(u_n, v_n) \leqslant d(\mu_n) + \varepsilon/2, \qquad n \in \mathbb{N}. \tag{2.10}$$

Combining this with (2.9), one gets that

$$A(u_n, v_n) = \|(u_n, v_n)\|^2 \leqslant 4d(\mu_n) + 2\varepsilon < 6d(0). \tag{2.11}$$

Thus, there exist $(u, v) \in H$ and a subsequence of $\{(u_n, v_n)\}$ (still denoted by $\{(u_n, v_n)\}$) such that $(u_n, v_n) \rightharpoonup (u, v)$. Moreover, $(u, v) \neq (0, 0)$. Otherwise, it follows from $(u_n, v_n) \in \mathcal{N}_{\mu_n}$, (2.2), the compact embedding $H \hookrightarrow L^{2q}(\mathbb{R}^3) \times L^{2q}(\mathbb{R}^3)$, (2.3) and (2.11) that

$$A(u_n, v_n) + B(u_n, v_n) = C(u_n, v_n) + \mu_n D(u_n, v_n) \leqslant C_1 |(u_n, v_n)|_{2q}^{2q} + \mu_n C_2 [A(u_n, v_n)]^3 \to 0,$$

which contradicts to Lemma 2.4. Hence $(u, v) \neq (0, 0)$. Consequently, noting $(u_n, v_n) \to (u, v)$ in $L^{2q}(\mathbb{R}^3) \times L^{2q}(\mathbb{R}^3)$, there exists some $N_0 > 0$ such that for $n > N_0$,

$$C(u_n, v_n) \geqslant |(u_n, v_n)|_{2q}^{2q} \geqslant |(u, v)|_{2q}^{2q}/2 > 0,$$

which implies that for some positive number C_3 independent of n such that

$$C(u_n, v_n) \geqslant C_3 > 0, \qquad n \in \mathbb{N} \tag{2.12}$$

Now, for given $n > N_0$, according to (i) of Lemma 2.3, let $t_n(\mu)$ satisfy that $t_n(\mu)(u_n, v_n) \in \mathcal{N}_\mu$ for all $\mu \in [0, \mu_n]$, and define

$$h_n(\mu) = J_\mu(t_n(\mu)(u_n, v_n)), \qquad \mu \in [0, \mu_n].$$

Since $t_n(\mu)(u_n, v_n) \in \mathcal{N}_\mu$, one could derive that

$$\begin{aligned} h_n'(\mu) &= \left\langle J_\mu'(t_n(\mu)(u_n, v_n)), (u_n, v_n) \right\rangle t_n'(\mu) - \frac{1}{6} t_n^6(\mu) \left[|u_n|_6^6 + |v_n|_6^6 + 2\alpha \int_{\mathbb{R}^3} |u_n|^3 |v_n|^3 \right] \\ &= -\frac{1}{6} t_n^6(\mu) D(u_n, v_n), \quad \mu \in [0, \mu_n]. \end{aligned}$$

Hence, from $t_n(\mu_n) = 1$, (ii) of Lemma 2.3, (2.3) and (2.11), we arrive at

$$\begin{aligned} &J_0(t_n(0)(u_n, v_n)) - J_{\mu_n}(u_n, v_n) \\ &\quad = h_n(0) - h_n(\mu_n) \\ &\quad = -\int_0^{\mu_n} h_n'(s) \mathrm{d}s \\ &\quad = \frac{1}{6} \int_0^{\mu_n} t_n^6(s) D(u_n, v_n) \mathrm{d}s \\ &\quad \leqslant \frac{1}{6} t_n^6(0) \mu_n C_2 [A(u_n, v_n)]^3 \leqslant 36 t_n^6(0) \mu_n C_2 d^3(0). \end{aligned} \tag{2.13}$$

Next, we shall illustrate that $\{t_n(0)\}$ is bounded. Indeed, due to $I_0(u_n, v_n) > I_\mu(u_n, v_n) = 0$ and (i) of Lemma 2.3, it holds that $t_n(0) > 1$. Moreover, by (2.12), (2.1) and (2.11) one can deduce that

$$C_3 t_n^{2q}(0) \leqslant t_n^{2q}(0) C(u_n, v_n) = t_n^2(0) A(u_n, v_n) + t_n^4(0) B(u_n, v_n) \leqslant 6d(0) t_n^2(0) + 36 C_2 d^2(0) t_n^4(0).$$

Since $q > 2$, this implies that there exists some C_4 independent of n such that $1 < t_n(0) \leqslant C_4$ for $n \in \mathbb{N}$.

Subsequently, combining this with (2.13), it holds that

$$J_0(t_n(0)(u_n, v_n)) - J_{\mu_n}(u_n, v_n) \leqslant 36C_4^6 \mu_n C_2 d^3(0).$$

Furthermore, as a consequence of Lemma 2.5, the fact that $m(0) \geqslant m(\mu)$ for $\mu \geqslant 0$ and (2.10), we also get that

$$0 \leqslant d(0) - d(\mu_n) \leqslant J_0(t_n(0)(u_n, v_n)) - J_{\mu_n}(u_n, v_n) + \varepsilon/2 \leqslant 36C_4^6 \mu_n C_2 d^3(0) + \varepsilon/2.$$

Thus,

$$0 \leqslant \limsup_{n\to\infty} [d(0) - d(\mu_n)] \leqslant \varepsilon/2,$$

which yields that $d(\mu_n) \to d(0)$ as a consequence of the arbitrariness of ε. The proof is complete. □

3 Proof of Theorem 1.1

Lemma 3.1. *Assume $\mu \in (0, 1]$ and $\{(u_n^\mu, v_n^\mu)\}$ is a (P.–S.)$_{m(\mu)}$ sequence for the functional J_μ. Then*

$$\lim_{\mu\to 0} \lim_{n\to\infty} \operatorname{dist}\big((u_n^\mu, v_n^\mu), K\big) = 0,$$

where

$$K = \{(u, v) \in H : J_0'(u, v) = 0, J_0(u, v) = m(0)\}.$$

Proof. This proof is motivated by [13] and [6]. Firstly, by the mountain pass theorem and the fact that J_0 satisfies the (P.–S.) condition on H, it holds that $K \neq \emptyset$.

Secondly, for any $\mu \in (0, 1]$, since $\{(u_n^\mu, v_n^\mu)\}$ is a (P.–S.)$_{m(\mu)}$ sequence for the functional J_μ, we have

$$m(\mu) + 1 + \|(u_n^\mu, v_n^\mu)\| \geqslant J_\mu(u_n^\mu, v_n^\mu) - \frac{1}{4} I_\mu(u_n^\mu, v_n^\mu).$$

Thus, similarly to (2.9) we can derive that

$$m(0) + 1 + \|(u_n^\mu, v_n^\mu)\| \geqslant \frac{1}{4} \|(u_n^\mu, v_n^\mu)\|^2. \tag{3.1}$$

Therefore, there is a constant $C_5 > 0$ independent of μ and n such that $\|(u_n^\mu, v_n^\mu)\| \leqslant C_5$ for all $n \in \mathbb{N}$ and $\mu \in (0, 1]$.

Now, assume that $\{\mu_i\}$ satisfies $\mu_i \to 0$ as $i \to \infty$. Denote the (P.–S.)$_{m(\mu_i)}$ sequence of the functional J_{μ_i} by $\{(u_n^{\mu_i}, v_n^{\mu_i})\}$. Furthermore, for any given i, we could find $n_i > i$ such that

$$\left|J_{\mu_i}(u_{n_i}^{\mu_i}, v_{n_i}^{\mu_i}) - m(\mu_i)\right| \leqslant \frac{1}{i}, \qquad \left\|J_{\mu_i}'(u_{n_i}^{\mu_i}, v_{n_i}^{\mu_i})\right\| \leqslant \frac{1}{i}.$$

Denote $\{(u_{n_i}^{\mu_i}, v_{n_i}^{\mu_i})\}$ by $\{(u_i, v_i)\}$. Then by (2.3), the uniform boundedness of the sequence $\{(u_n^\mu, v_n^\mu)\}$, Lemma 2.6 and $\mu_i \to 0$, we can derive that

$$\begin{aligned} |J_0(u_i, v_i) - m(0)| &\leqslant |J_{\mu_i}(u_i, v_i) - m(\mu_i)| + \frac{\mu_i}{6}\left[|u_i|_6^6 + |v_i|_6^6 + 2\alpha \int_{\mathbb{R}^3} |u_i|^3 |v_i|^3\right] + m(0) - m(\mu_i) \\ &\leqslant \frac{1}{i} + \frac{\mu_i}{6} C_2 C_5^6 + m(0) - m(\mu_i) \to 0, \qquad i \to \infty. \end{aligned}$$

Similarly,

$$\|J_0'(u_i, v_i)\| \leqslant \|J_0'(u_i, v_i)\| + \mu_i C_6 \|(u_i, v_i)\|^5 \to 0, \qquad i \to \infty,$$

where C_6 is some positive constant independent of i. These yield that $\{(u_i, v_i)\}$ is a (P.-S.)$_{m(0)}$ sequence of J_0. Due to the fact that J_0 satisfies the (P.–S.) condition on H, then there exists $(u_0, v_0) \in K$ and up to a subsequence still denoted by $\{(u_i, v_i)\}$ such that $(u_i, v_i) \to (u_0, v_0)$ as $i \to \infty$. Thus, one can derive that

$$\lim_{i\to\infty} \lim_{n\to\infty} \operatorname{dist}\big((u_n^{\mu_i}, v_n^{\mu_i}), K\big) \leqslant \lim_{i\to\infty} \operatorname{dist}\big((u_i, v_i), K\big) = 0.$$

In the end, by the arbitrariness for $\{\mu_i\}$, the proof is complete. □

Proof of Theorem 1.1. Assume $\mu \in [0, 1]$ and $\{(u_n^\mu, v_n^\mu)\}$ is a (P.–S.)$_{m(\mu)}$ sequence of the functional J_μ. Similarly to (3.1), we can derive that $\{(u_n^\mu, v_n^\mu)\}$ is uniformly bounded for $\mu \in [0, 1]$, then there exists $(u_\mu, v_\mu) \in H$ and a subsequence for $\{(u_n^\mu, v_n^\mu)\}$ still denoted by $\{(u_{n_i}^\mu, v_{n_i}^\mu)\}$ such that $(u_{n_i}^\mu, v_{n_i}^\mu) \rightharpoonup (u_\mu, v_\mu)$ as $i \to \infty$ and $J_\mu'(u_\mu, v_\mu) = 0$.

In what follows, we will prove that there exists $\mu_0 > 0$, such that $(u_\mu, v_\mu) \neq (0, 0)$ for $\mu \in [0, \mu_0]$. Indeed, since $m(0) > 0$ and K is nonempty and compact, $\delta_0 := \operatorname{dist}((0, 0), K) = \min_{(u,v)\in K} \|(u, v)\| > 0$. According to Lemma 3.1,

$$\lim_{\mu\to 0} \lim_{i\to\infty} \operatorname{dist}\big((u_{n_i}^\mu, v_{n_i}^\mu), K\big) = 0.$$

Hence, for any given $\delta < \delta_0/2$, there exists some $\mu_0 = \mu_0(\delta)$ satisfying: for any $\mu \in (0, \mu_0)$ there is $i_0 = i_0(\mu)$ such that

$$\operatorname{dist}\big((u_{n_i}^\mu, v_{n_i}^\mu), K\big) < \delta, \qquad i > i_0.$$

Thus, for fixed $\mu \in (0, \mu_0)$, by the compactness of K, one can obtain a sequence $\{(u_i^\mu, v_i^\mu)\} \subset K$ such that $\|(u_{n_i}^\mu, v_{n_i}^\mu) - (u_i^\mu, v_i^\mu)\| \leqslant \delta$ for $i > i_0$. Moreover, noting that there is $(u_0^\mu, v_0^\mu) \in K$ such that $(u_i^\mu, v_i^\mu) \to (u_0^\mu, v_0^\mu)$ as $i \to \infty$, it also holds that $(u_{n_i}^\mu, v_{n_i}^\mu) \in B_{2\delta}(u_0^\mu, v_0^\mu)$ for i large. Therefore, the facts that $\overline{B_{2\delta}(u_0^\mu, v_0^\mu)}$ is closed weakly and $(u_{n_i}^\mu, v_{n_i}^\mu) \rightharpoonup (u_\mu, v_\mu)$ lead to

$$(u_\mu, v_\mu) \in \overline{B_{2\delta}(u_0^\mu, v_0^\mu)}.$$

Thereby, owing to the choosing of δ,

$$\|(u_\mu, v_\mu)\| \geqslant \|(u_0^\mu, v_0^\mu)\| - 2\delta > 0, \qquad \mu \in (0, \mu_0).$$

In the end, we will prove that (u_μ, v_μ) is a ground-state solution to the system (1.1). Actually, it is sufficient to prove that $J_\mu(u_\mu, v_\mu) = d(\mu)$ since $(u_\mu, v_\mu) \neq (0, 0)$ and $J_\mu'(u_\mu, v_\mu) = 0$. To achieve this, we calculate the following inequalities:

$$\begin{aligned} d(\mu) &\leqslant J_\mu(u_\mu, v_\mu) \\ &= J_\mu(u_\mu, v_\mu) - I_\mu(u_\mu, v_\mu)/4 \\ &= A(u_\mu, v_\mu)/4 + (q-2)C(u_\mu, v_\mu)/(4q) + \mu D(u_\mu, v_\mu)/12 \\ &\leqslant \liminf_{i\to\infty} \big[A(u_{n_i}^\mu, v_{n_i}^\mu)/4 + (q-2)C(u_{n_i}^\mu, v_{n_i}^\mu)/(4q) + \mu D(u_{n_i}^\mu, v_{n_i}^\mu)/12\big] \\ &= \liminf_{i\to\infty} [J_\mu(u_{n_i}^\mu, v_{n_i}^\mu) - I_\mu(u_{n_i}^\mu, v_{n_i}^\mu)/4] = m(\mu). \end{aligned}$$

Hence, it follows from Lemma 2.5 that $J_\mu(u_\mu, v_\mu) = d(\mu)$. Therefore, (u_μ, v_μ) is a ground-state solution to the system (1.1). □

4 Proof of Theorem 1.2

Lemma 4.1. *Let $p \in [2, \infty)$, and $\sigma \geqslant 0$. Define*

$$h_\sigma(s) = s^p + (1-s)^p + 2\sigma s^{p/2}(1-s)^{p/2}, \qquad s \in [0,1].$$

Then

(i) if $\sigma < 2^{p-1} - 1$, then $h_\sigma(s) < 1$ for all $s \in (0,1)$;

(ii) if $\sigma > 2^{p-1} - 1$, then $h_\sigma(1/2) > 1$.

Proof. For (i) one can refer to [4, Lemma 2.7] or [5, Lemma 2.4], while (ii) could be derived through a direct calculation. □

Lemma 4.2. *Assume that $q \in (2,3)$, $\mu > 0$, $0 \leqslant \alpha < 3, 0 \leqslant \beta < 2^{q-1} - 1$. If $(u,v) \in H$ is a ground-state radial solution to the system (1.1) with $J_\mu(u,v) = d(\mu)$, then $u = 0$ or $v = 0$.*

Proof. Suppose by contradiction, $u \neq 0$ and $v \neq 0$. Then replacing (u,v) with $(|u|,|v|)$, by a regularity process and the maximum principle, one could also assume that $u > 0$ and $v > 0$. Now, let (ρ, θ) be the polar form of (u,v), that is, that is,

$$(u,v) = (\rho\cos\theta, \rho\sin\theta), \qquad \rho = \rho(x) > 0, \qquad \theta = \theta(x) \in (0, \pi/2).$$

Then on one aspect, by the convexity inequality for gradients in [7], there also holds that $\rho = \sqrt{u^2+v^2} \in H_r^1(\mathbb{R}^3)$. On the other aspect, through calculations, we could get that

$$\nabla u = (\cos\theta)\nabla\rho - \rho(\sin\theta)\nabla\theta, \qquad \nabla v = (\sin\theta)\nabla\rho + \rho(\cos\theta)\nabla\theta.$$

Hence, it follows from definitions of functionals A, B, C and D given in (1.4),

$$\begin{aligned}
A(u,v) &= \int_{\mathbb{R}^3} \left[|\nabla\rho|^2 + \rho^2|\nabla\theta|^2 + \rho^2\right] = A(\rho,0) + \int_{\mathbb{R}^3} \rho^2|\nabla\theta|^2,\\
B(u,v) &= \lambda\int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) = \lambda\int_{\mathbb{R}^3} \phi_{\rho,0}\rho^2 = B(\rho,0),\\
C(u,v) &= \int_{\mathbb{R}^3} \rho^{2q}\left[\cos^{2q}\theta + \sin^{2q}\theta + 2\beta\cos^q\theta\sin^q\theta\right] = \int_{\mathbb{R}^3} \rho^{2q} h_\beta(\cos^2\theta),
\end{aligned}$$

and similarly there holds

$$D(u,v) = \int_{\mathbb{R}^3} \rho^6 h_\alpha(\cos^2\theta).$$

Furthermore, since $\theta \in (0,1)$, then by Lemma 4.1 it holds that

$$C(u,v) < |\rho|_{2q}^{2q} = C(\rho,0), \;\; D(u,v) < D(\rho,0).$$

Note that by (i) of Lemma 2.3 there exists some $t(\mu) > 0$ such that $t(\mu)(\rho,0) \in \mathcal{N}_\mu$. Then

$$\begin{aligned}
d(\mu) &\leqslant J_\mu(t(\mu)(\rho,0))\\
&= \frac{1}{2}t^2(\mu)A(\rho,0) + \frac{1}{4}t^4(\mu)B(\rho,0) - \frac{1}{2q}t^{2q}(\mu)C(\rho,0) - \frac{1}{6}\mu t^6(\mu)D(\rho,0)\\
&< \frac{1}{2}t^2(\mu)A(u,v) + \frac{1}{4}t^4(\mu)B(u,v) - \frac{1}{2q}t^{2q}(\mu)C(u,v) - \frac{1}{6}\mu t^6(\mu)D(u,v)\\
&= J_\mu(t(\mu)(u,v)) < J_\mu(u,v) = d(\mu).
\end{aligned}$$

This is absurd. Thus, it could only hold that $u = 0$ or $v = 0$. □

Lemma 4.3. *Assume that $q \in (2,3)$, $\mu > 0$, $\alpha > 3, \beta > 2^{q-1} - 1$. If $(u,v) \in H$ is a ground-state radial solution to the system* (1.1) *with $J_\mu(u,v) = d(\mu)$, then $u \neq 0$ and $v \neq 0$.*

Proof. Arguing by contradiction, we suppose that $v = 0$. Define $(u_0, v_0) = (u/\sqrt{2}, v/\sqrt{2})$. Then $(u_0, v_0) \in H \backslash \{(0,0)\}$. Similarly to the proof of Lemma 4.2, one could derive that

$$A(u_0, v_0) = A(u,0), \qquad B(u_0, v_0) = B(u,0)$$

and

$$C(u_0, v_0) = h_\beta(1/2) C(u,0), \qquad D(u_0, v_0) = h_\alpha(1/2) D(u,0).$$

Moreover, by (i) of Lemma 2.3, there exists a unique $t(\mu) > 0$ such that $t(\mu)(u_0, v_0) \in \mathcal{N}_\mu$. Now, we make the following calculation

$$\begin{aligned} & J_\mu(t(\mu)(u_0,v_0)) \\ &\quad = \frac{1}{2} A(u_0,v_0) t^2(\mu) + \frac{1}{4} B(u_0,v_0) t^4(\mu) - \frac{1}{2q} C(u_0,v_0) t^{2q}(\mu) - \frac{1}{6} \mu D(u_0,v_0) t^6(\mu) \\ &\quad = \frac{1}{2} A(u,0) t^2(\mu) + \frac{1}{4} B(u,0) t^4(\mu) - \frac{1}{2q} h_\beta(1/2) C(u,0) t^{2q}(\mu) - \frac{1}{6} \mu h_\alpha(1/2) D(u,0) t^6(\mu) \\ &\quad < \frac{1}{2} A(u,0) t^2(\mu) + \frac{1}{4} B(u,0) t^4(\mu) - \frac{1}{2q} C(u,0) t^{2q}(\mu) - \frac{1}{6} \mu D(u,0) t^6(\mu) = J_\mu(t(\mu)(u,0)). \end{aligned}$$

Consequently,

$$d(\mu) \leqslant J_\mu(t(\mu)(u_0,v_0)) < J_\mu(t(\mu)(u,0)) \leqslant J_\mu(u,0) = d(\mu),$$

which is a contraction. Thus, it holds that $u \neq 0$ and $v \neq 0$. □

Proof of Theorem 1.2. According to Lemmas 4.2 and 4.3, one can get Theorem 1.2 directly. □

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