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Ground state sign-changing solution for a logarithmic Kirchhoff-type equation in $\mathbb{R}^3$

Wei-Long Yang[1] and **Jia-Feng Liao**✉[1, 2]

[1]School of Mathematics and Information, China West Normal University,
Nanchong, Sichuan, 637002, P.R. China
[2]College of Mathematics Education, China West Normal University,
Nanchong, Sichuan, 637002, P.R. China

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Abstract. We investigate the following logarithmic Kirchhoff-type equation:

$$\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2+V(x)u^2dx\right)[-\Delta u+V(x)u]=|u|^{p-2}u\ln|u|,\qquad x\in\mathbb{R}^3,$$

where $a,b>0$ are constants, $4<p<2^*=6$. Under some appropriate hypotheses on the potential function V, we prove the existence of a positive ground state solution, a ground state sign-changing solution and a sequence of solutions by using the constraint variational methods, topological degree theory, quantitative deformation lemma and symmetric mountain pass theorem. Our results complete those of Gao et al. [*Appl. Math. Lett.* **139**(2023), 108539] with the case of $4<p<6$.

Keywords: Kirchhoff-type equation, logarithmic nonlinearity, ground state sign-changing solution, variational methods, topological degree theory.

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1 Introduction and main result

In this work, we are concerned with the existence of ground state sign-changing solutions for the following logarithmic Kirchhoff-type equation

$$\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2+V(x)u^2dx\right)[-\Delta u+V(x)u]=|u|^{p-2}u\ln|u|,\qquad x\in\mathbb{R}^3,\tag{1.1}$$

where $a,b>0$ are constants, $4<p<6$. Besides, we shall impose the following conditions on potential function V:

(V_1) $V\in C(\mathbb{R}^3,\mathbb{R})$ and $\lim_{|x|\to\infty}V(x)=+\infty$;

(V_2) There exists a constant V_0 such that $\inf_{x\in\mathbb{R}^3}V(x)\geq V_0>0$.

✉Corresponding author. E-mail address: liaojiafeng@163.com

As is known to all, Kirchhoff [12] first proposed the following Kirchhoff model given by the stationary analogue of equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ is the mass density, P_0 is the initial tension, h represents the area of the cross-section, E is the Young modulus of the material and L is the length of the string. The above model is an extension of the classical D'Alembert wave equation by taking into account the changes in the length of the string during the transverse vibrations. After that, Lions [13] derived the following Kirchhoff equation by using the functional analysis method

$$u_{tt} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u = f(x,u). \tag{1.2}$$

This model is used to describe the chord length variation of elastic strings caused by lateral vibration, where u is displacement, f is external force, b is initial tension force and a is related to inherent properties of strings (see [1, 2, 5, 7] and the references therein). The corresponding problem associated with equation (1.2) is called as the Kirchhoff-type problem.

In the past years, logarithmic nonlinearity appears frequently in partial differential equations, which has numerous applications to quantum optics, quantum mechanics, nuclear physics, transport and diffusion phenomenon etc (see [22] and the references therein). Therefore, many scholars studied the following Kirchhoff-type problem with logarithmic nonlinearity

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = |u|^{p-2} u \log u^2, \qquad x \in \mathbb{R}^3, \tag{1.3}$$

where $4 < p < 6$ and $V \in C(\mathbb{R}^3, \mathbb{R})$. By using the constrained variational method, deformation lemma and topological degree theory, Hu and Gao [10] proved that equation (1.3) owns both positive solution and sign-changing solution under different types of potential (coercive potential and periodic potential). Wen, Tang and Chen [20] verified that equation (1.3) in smooth bounded domain $\Omega \subset \mathbb{R}^3$ has a ground state solution and a ground state sign-changing solution, besides, the energy of sign-changing solution is larger than twice of the ground state energy.

In particular, letting $a = 1, b = 0$ and $p = 2$ in equation (1.3), it leads to the following logarithmic Schrödinger equation

$$-\Delta u + V(x)u = u \log u^2, \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Equation (1.4) has received much attention in mathematical analysis and applications. Ji and Szulkin [11] got infinitely many solutions by adapting some arguments of the fountain theorem when the potential is coercive (i.e. $\lim_{|x|\to\infty} V(x) = +\infty$), and in the case of bounded potential (i.e. $\lim_{|x|\to\infty} V(x) = V_\infty \in (-1, +\infty)$), they obtained a ground state solution. By using the direction derivative and constrained minimization method, Shuai [16] proved the existence of positive and sign-changing solutions of equation (1.4) under different types of potential (coercive potential and periodic potential). When the potential is radially symmetric, the author constructed infinitely many radial nodal solutions. Zhang and Zhang [24] proved the existence, uniqueness, non-degeneracy and some qualitative properties of positive solutions of equation (1.4) when the potential $V \in C^2(\mathbb{R}^N, \mathbb{R})$ is radially symmetric and allowed to be singular at $x = 0$ and repulsive at infinity(i.e. $\lim_{|x|\to\infty} V(x) = -\infty$). When potential

$V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\lim_{|x|\to\infty} V(x) = V_\infty$ and $V(x) < V_\infty + \log 2$, Feng, Tang and Zhang [8] proved that equation (1.4) has a positive bound state solution.

After that, inspired by [10], Gao, Jiang and Liu et al. [9] studied the existence of solutions to equation (1.1) for the first time, and proved that equation (1.1) has only trivial solution for large $b > 0$ and two positive solutions for small $b > 0$ and $2 < p < 4$. To the best of our knowledge, there is no result for the existence of positive ground state, ground state sign-changing solutions and sequence of solutions of equation (1.1) with $4 < p < 6$. Inspired by the above literature, we are interested in the existence of positive ground state solutions, ground state sign-changing solutions and sequence of solutions for equation (1.1).

Equation (1.1) is formally associated with the energy functional $I : H \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) = \frac{a}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx\right)^2 \\ - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p \ln|u|dx + \frac{1}{p^2}\int_{\mathbb{R}^3}|u|^p dx, \end{aligned} \tag{1.5}$$

with $I(0) = 0$, where Sobolev space H is defined as follows:

$$H := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty \right\}.$$

endowed with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + V(x)uv)dx, \qquad \forall u, v \in H$$

and endowed with the norm

$$\|u\|^2 := \langle u, u\rangle = \int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx.$$

Denote $|u|_k = \left(\int_{\mathbb{R}^3}|u|^k dx\right)^{1/k}$ the norm of $u \in L^k(\mathbb{R}^3)$ for $k \geq 1$, the $C, C_1, C_2, \ldots$ represent several different positive constants. A elementary computation, we have

$$\lim_{t\to 0}\frac{t^{p-1}\ln|t|}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{t^{p-1}\ln|t|}{t^{q-1}} = 0,$$

where $4 < p < q < 6$. Therefore, for arbitrarily $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t^{p-1}\ln|t|| \leq \varepsilon|t| + C_\varepsilon|t|^{q-1}, \qquad \forall\, t \in \mathbb{R}\backslash\{0\}. \tag{1.6}$$

By (1.6) and [19, Lemma 3.10], we get that $I \in C^1(H, \mathbb{R})$ and the Fréchet derivative of I is given by

$$\langle I'(u), v\rangle = (a + b\|u\|^2)\int_{\mathbb{R}^3}(\nabla u\nabla v + V(x)uv)dx - \int_{\mathbb{R}^3}|u|^{p-2}uv\ln|u|dx, \tag{1.7}$$

for all $u, v \in H$. $u \in H$ is a weak solution of equation (1.1) if and only if u is a critical point of I. Additional, if $u \in H$ is a weak solution of equation (1.1) with $u^\pm \neq 0$, then u is called a sign-changing solution of equation (1.1), where

$$u^+ := \max\{u(x), 0\}, \qquad u^- := \min\{u(x), 0\}.$$

From (1.7), we know

$$\langle I'(u), u\rangle = a\|u\|^2 + b\|u\|^4 - \int_{\mathbb{R}^3} |u|^p \ln|u| dx \tag{1.8}$$

and

$$\langle I'(u), u^\pm\rangle = (a + b\|u\|^2)\|u^\pm\|^2 - \int_{\mathbb{R}^3} |u^\pm|^p \ln|u^\pm| dx. \tag{1.9}$$

By virtue of (1.8) and (1.9), it is noticed that if $u \not\equiv 0$, then

$$\begin{aligned} I(u) &= I(u^+) + I(u^-) + \frac{b}{2}\|u^+\|^2\|u^-\|^2, \\ \langle I'(u), u^+\rangle &= \langle I'(u^+), u^+\rangle + b\|u^+\|^2\|u^-\|^2, \\ \langle I'(u), u^-\rangle &= \langle I'(u^-), u^-\rangle + b\|u^+\|^2\|u^-\|^2. \end{aligned}$$

In this paper, our main purpose is to seek the ground state sign-changing solution for equation (1.1). As we all known, there are some very interesting results for the existence and multiplicity of sign-changing solutions of the following Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \qquad x \in \mathbb{R}^N. \tag{1.10}$$

However, these methods of seeking sign-changing solutions dependent on the following decomposition

$$J(u) = J(u^+) + J(u^-), \tag{1.11}$$

and

$$\langle J'(u), u^+\rangle = \langle J'(u^+), u^+\rangle, \qquad \langle J'(u), u^-\rangle = \langle J'(u^-), u^-\rangle, \tag{1.12}$$

where J is the energy functional of equation (1.10) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} F(x, u) dx.$$

However, it follows from (1.5) that the energy functional I does not possess the same decompositions as (1.11) and (1.12). Indeed, a direct calculation yields that

$$I(u) > I(u^+) + I(u^-),$$

and

$$\langle I'(u), u^+\rangle > \langle I'(u^+), u^+\rangle, \qquad \langle I'(u), u^-\rangle > \langle I'(u^-), u^-\rangle$$

for $u^\pm \neq 0$. Therefore, the method of getting sign-changing solutions for the local problem (1.10) does not seem applicable to equation (1.1). In order to overcome this difficulty, we follow in [4] by the following Nehari manifold and the nodal Nehari sets respectively

$$\mathcal{N} := \left\{u \in H\backslash\{0\} : \langle I'(u), u\rangle = 0\right\},$$

and

$$\mathcal{M} := \left\{u \in H, u^\pm \neq 0 : \langle I'(u), u^\pm\rangle = 0\right\}.$$

It is well known that the existence of positive ground state and sign-changing solutions to equation (1.1) can be transformed into studying the following minimization problems respectively

$$c := \inf_{u\in\mathcal{N}} I(u) \quad \text{and} \quad m := \inf_{u\in\mathcal{M}} I(u).$$

Now, we state the main results.

Theorem 1.1. *Assume that* (V_1)–(V_2) *hold and* $4 < p < 6$*, then equation* (1.1) *possesses a positive ground state solution* $\bar{u} \in \mathcal{N}$ *such that* $I(\bar{u}) = c$*.*

Theorem 1.2. *Assume that* (V_1)–(V_2) *hold and* $4 < p < 6$*, then equation* (1.1) *has a ground state sign-changing solution* $u_* \in \mathcal{M}$ *with precisely two nodal domains such that* $I(u_*) = m$*. Moreover,* $m > 2c$*.*

Theorem 1.3. *Assume that* (V_1)–(V_2) *hold and* $4 < p < 6$*, then equation* (1.1) *owns a sequence of solutions of* $\{u_n\}$ *with* $I(u_n) \to +\infty$ *as* $n \to \infty$*.*

Remark 1.4. To our best knowledge, our results are up to date. Compared with [9], we study the case of $4 < p < 6$. Moreover, we consider the ground state sign-changing solution and a sequence of high energy solutions for equation (1.1).

The remaining of paper is organized as follows. In Section 2, we show some necessary logarithmic inequalities and important lemmas. In Section 3, we prove Theorems 1.1–1.3 by the maximum principle, quantitative deformation lemma, topological degree theory and symmetric mountain pass theorem.

2 Some preliminary results

Firstly, because of the existence of logarithmic nonlinearity, the following lemmas will be used to obtain vital estimates for our problem.

Lemma 2.1. *The following inequalities hold*

$$(1 - x^s) + sx^s \ln x > 0, \qquad \forall x \in (0,1) \cup (1,+\infty), s > 0; \tag{2.1}$$

$$\ln x \le \frac{1}{e\sigma} x^\sigma, \qquad \forall x \in (0,+\infty), \sigma > 0. \tag{2.2}$$

Proof. Define $f(x) := (1 - x^s) + sx^s \ln x$, then $f'(x) := s^2 x^{s-1} \ln x$, it's easy to see that the function $f(x)$ is decreasing on $(0,1)$ and increasing on $(1,+\infty)$. So $f(x) > f(1) = 0$, i.e.

$$(1 - x^s) + sx^s \ln x > 0.$$

Thus, (2.1) is true. The proof of (2.2) is similar to that of (2.1), here we omit it. □

Next, we give the following lemma by the conclusions of [3].

Lemma 2.2. *Under the assumptions* (V_1)–(V_2)*, then the embedding* $H \hookrightarrow L^q(\mathbb{R}^3)$ *is compact for* $q \in [2,6)$*.*

By virtue of Lemma 2.2, we define the following Sobolev embedding constants

$$S_q = \inf_{u \in H \setminus \{0\}} \frac{\|u\|^q}{|u|_q^q}, \qquad q \in [2,6]. \tag{2.3}$$

As is known to all, the logarithmic nonlinearity $|u|^{p-2}u \ln |u|$ satisfies neither the well-known Nehari type monotonicity condition in [23] nor (AR) condition in [17]. Therefore, we will establish an energy inequality related to $I(u)$, $I(su^+ + tu^-)$, $\langle I'(u), u^+ \rangle$ and $\langle I'(u), u^- \rangle$ in order to overcome the this difficulty.

Lemma 2.3. *For all $u \in H$ and $s,t \geq 0$, there holds*

$$I(u) \geq I(su^+ + tu^-) + \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle. \tag{2.4}$$

Proof. It follows from (1.9) that (2.4) holds for $u = 0$, then we only consider the case when $u \in H\backslash\{0\}$. Set

$$\Omega^+ = \{u \in \mathbb{R}^3 : u(x) \geq 0\}, \qquad \Omega^- = \{u \in \mathbb{R}^3 : u(x) < 0\}.$$

For all $u \in H\backslash\{0\}$ and $s,t \geq 0$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} |su^+ + tu^-|^p \ln|su^+ + tu^-| dx \\
&= \int_{\Omega^+} |su^+ + tu^-|^p \ln|su^+ + tu^-| dx + \int_{\Omega^-} |su^+ + tu^-|^p \ln|su^+ + tu^-| dx \\
&= \int_{\Omega^+} |su^+|^p \ln|su^+| dx + \int_{\Omega^-} |tu^-|^p \ln|tu^-| dx \\
&= \int_{\mathbb{R}^3} (|su^+|^p \ln|su^+| + |tu^-|^p \ln|tu^-|) dx.
\end{aligned} \tag{2.5}$$

It follows from (1.5), (1.8), (2.1) and (2.5) that

$$\begin{aligned}
&I(u) - I(su^+ + tu^-) \\
&= \frac{a}{2}\left(\|u\|^2 - \|su^+ + tu^-\|^2\right) + \frac{b}{4}\left(\|u\|^4 - \|su^+ + tu^-\|^4\right) \\
&\quad + \frac{1}{p^2}\int_{\mathbb{R}^3} \left(|u|^p - |su^+ + tu^-|^p\right) dx \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^3} \left(|u|^p \ln|u| - |su^+ + tu^-|^p \ln|su^+ + tu^-|\right) dx \\
&= \frac{a(1-s^2)}{2}\|u^+\|^2 + \frac{a(1-t^2)}{2}\|u^-\|^2 + \frac{b(1-s^4)}{4}\|u^+\|^4 + \frac{b(1-t^4)}{4}\|u^-\|^4 \\
&\quad + \frac{b(1-s^2t^2)}{2}\|u^+\|^2\|u^-\|^2 + \frac{(1-s^p)}{p^2}\int_{\mathbb{R}^3} |u^+|^p dx + \frac{(1-t^p)}{p^2}\int_{\mathbb{R}^3} |u^-|^p dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} \left(|u^+|^p \ln|u^+| - |su^+|^p \ln|u^+| - |su^+|^p \ln s\right) dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} \left(|u^-|^p \ln|u^-| - |tu^-|^p \ln|u^-| - |tu^-|^p \ln t\right) dx \\
&= \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle \\
&\quad + a\left[\left(\frac{1-s^2}{2} - \frac{1-s^p}{p}\right)\|u^+\|^2 + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p}\right)\|u^-\|^2\right] \\
&\quad + b\left[\left(\frac{1-s^4}{4} - \frac{1-s^p}{p}\right)\|u^+\|^4 + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)\|u^-\|^4\right] \\
&\quad + b\left(\frac{1-s^2t^2}{2} - \frac{1-s^p}{p} - \frac{1-t^p}{p}\right)\|u^+\|^2\|u^-\|^2 \\
&\quad + \frac{(1-s^p) + ps^p \ln s}{p^2}\int_{\mathbb{R}^3} |u^+|^p dx + \frac{(1-t^p) + pt^p \ln t}{p^2}\int_{\mathbb{R}^3} |u^-|^p dx.
\end{aligned}$$

Since the function $f(x) = \frac{1-a^x}{x}$ is monotonically decreasing on $(0, +\infty)$ for $a \in (0,1) \cup (1, +\infty)$. It follows from the above equation that

$$\begin{aligned}
&I(u) - I(su^+ + tu^-)\\
&\geq \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle + b\left(\frac{1-s^2t^2}{2} - \frac{1-s^p}{p} - \frac{1-t^p}{p}\right)\|u^+\|^2\|u^-\|^2\\
&= \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle\\
&\quad + b\left[\frac{(s^2-t^2)^2}{4} + \left(\frac{1-s^4}{4} - \frac{1-s^p}{p}\right) + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)\right]\|u^+\|^2\|u^-\|^2\\
&\geq \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle.
\end{aligned}$$

Hence, (2.4) holds for all $u \in H$ and $s, t \geq 0$. □

Let $s = t$ in (2.4), we can obtain the following corollary.

Corollary 2.4. *For all $u \in H$ and $t \geq 0$, there holds*

$$I(u) \geq I(tu) + \frac{1-t^p}{p}\langle I'(u), u\rangle.$$

According to Lemma 2.3 and Corollary 2.4, we have the following lemmas.

Lemma 2.5. *For all $u \in \mathcal{M}$, there holds $I(u) = \max_{s,t\geq 0} I(su^+ + tu^-)$.*

Lemma 2.6. *For all $u \in \mathcal{N}$, there holds $I(u) = \max_{t\geq 0} I(tu)$.*

By Lemmas 2.3 and 2.5, we have the following lemma.

Lemma 2.7. *For all $u \in \mathcal{M}$ and $s, t \geq 0$, there holds $I(u) \geq I(su^+ + tu^-)$, and the equality sign holds if and only if $s = t = 1$.*

Lemma 2.8. *For any $u \in H$ with $u^\pm \neq 0$, there exists a unique positive numbers pair (s_0, t_0) such that $s_0u^+ + t_0u^- \in \mathcal{M}$.*

Proof. We firstly prove that there exists positive numbers pair (s_0, t_0) such that $s_0u^+ + t_0u^- \in \mathcal{M}$. For any $u \in H$ with $u^\pm \neq 0$, let

$$g(s,t) = \langle I'(su^+ + tu^-), su^+\rangle, \qquad h(s,t) = \langle I'(su^+ + tu^-), tu^-\rangle.$$

From (1.9), one gets

$$g(s,t) = (a + b\|su^+ + tu^-\|^2)\|su^+\|^2 - \int_{\mathbb{R}^3} |su^+|^p \ln|su^+|dx; \tag{2.6}$$

$$h(s,t) = (a + b\|su^+ + tu^-\|^2)\|tu^-\|^2 - \int_{\mathbb{R}^3} |tu^-|^p \ln|tu^-|dx. \tag{2.7}$$

Let $t = s$ in (2.6), then

$$\begin{aligned}
g(s,s) &= (a + b\|su\|^2)\|su^+\|^2 - \int_{\mathbb{R}^3} |su^+|^p \ln|su^+|dx\\
&= as^2\|u^+\|^2 + bs^4\|u^+\|^4 + bs^4\|u^+\|^2\|u^-\|^2\\
&\quad - s^p\int_{\mathbb{R}^3} |u^+|^p \ln|u^+|dx - s^p \ln s\int_{\mathbb{R}^3} |u^+|^p dx.
\end{aligned}$$

Obviously $g(s,s)$ is continuous, it is easy to verify that $g(s,s) > 0$ when $0 < s < 1$ small enough and $g(s,s) < 0$ when $s > 1$ large enough. Similarly, $h(t,t) > 0$ when $0 < t < 1$ small enough and $h(t,t) < 0$ when $t > 1$ large enough. Therefore, there exists $0 < r < R$ such that

$$g(r,r) > 0, h(r,r) > 0; \qquad g(R,R) < 0, h(R,R) < 0. \tag{2.8}$$

It follows from (2.6)–(2.8) that we have

$$\begin{aligned} &g(r,t) > 0, \quad g(R,t) < 0, \quad \forall t \in [r,R], \\ &h(s,r) > 0, \quad h(s,R) < 0, \quad \forall s \in [r,R]. \end{aligned}$$

Based on Miranda's Theorem [14], there exist $r < s_0, t_0 < R$ such that $g(s_0,t_0) = h(s_0,t_0) = 0$, which implies that $s_0u^+ + t_0u^- \in \mathcal{M}$.

Next, we prove the uniqueness of (s_0,t_0). By contradiction, we suppose that there are two pairs positive numbers $(s_1,t_1),(s_2,t_2)$ with $s_1 \neq s_2, t_1 \neq t_2$ such that $g(s_1,t_1) = g(s_2,t_2) = 0, h(s_1,t_1) = h(s_2,t_2) = 0$. Let $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$, then $s \neq 1$ and $t \neq 1$. From Lemma 2.7, we know

$$I(s_1u^+ + t_1u^-) = I(s(s_2u^+) + t(t_2u^-)) < I(s_2u^+ + t_2u^-). \tag{2.9}$$

Similarly, one has

$$I(s_2u^+ + t_2u^-) < I(s_1u^+ + t_1u^-),$$

which contradicts (2.9). Therefore, (s_0,t_0) is unique. □

Lemma 2.9. *For any $u \in H$ with $u \neq 0$, there exists a unique positive number $t_0 > 0$ such that $t_0u \in \mathcal{N}$.*

Proof. Define a function $g(t) = \langle I'(tu), tu\rangle$ on $(0,+\infty)$, then

$$\begin{aligned} g(t) &= a\|tu\|^2 + b\|tu\|^4 - \int_{\mathbb{R}^3} |tu|^p \ln|tu| dx \\ &= at^2\|u\|^2 + bt^4\|u\|^4 - t^p \int_{\mathbb{R}^3} |u|^p \ln|u| dx - t^p \ln t \int_{\mathbb{R}^3} |u|^p dx. \end{aligned} \tag{2.10}$$

Combined with (1.8), we know

$$g(t) = t^p\langle I'(u), u\rangle + a(t^2 - t^p)\|u\|^2 + b(t^4 - t^p)\|u\|^4 - t^p \ln t \int_{\mathbb{R}^3} |u|^p dx.$$

If $u \in \mathcal{N}$, then $t_0 = 1$. Therefore, we only consider the existence of t_0 when $u \notin \mathcal{N}$. Since $4 < p < 6$ and in view of (2.3), we have $\int_{\mathbb{R}^3} |u|^p dx \leq S_p^{-1}\|u\|^p < +\infty$. It follows from (2.10) that $g(t) > 0$ for $0 < t < 1$ small enough and $g(t) < 0$ for $t > 1$ large enough. Since $g(t)$ is continuous, there exists $t_0 > 0$ such that $g(t_0) = \langle I'(t_0u), t_0u\rangle = 0$, i.e. $t_0u \in \mathcal{N}$. As a similar argument of Lemma 2.8, we can obtain the uniqueness of t_0. □

Lemma 2.10. *Assume there exists $u \in H$ with $u^\pm \neq 0$ such that $\langle I'(u), u^\pm\rangle \leq 0$, then the unique positive numbers pair (s_0,t_0) obtained in Lemma 2.8 satisfies $0 < s_0, t_0 \leq 1$.*

Proof. From Lemma 2.8, there exists a unique positive numbers pair (s_0,t_0) such that $s_0u^+ + t_0u^- \in \mathcal{M}$. Without loss of generally, we may suppose that $s_0 \geq t_0 > 0$. Since $s_0u^+ + t_0u^- \in \mathcal{M}$, we have

$$\begin{aligned} I'(s_0u^+ + t_0u^-), s_0u^+\rangle &= as_0^2\|u^+\|^2 + bs_0^4\|u^+\|^4 + bs_0^2t_0^2\|u^+\|^2\|u^-\|^2 \\ &\quad - \int_{\mathbb{R}^3} |s_0u^+|^p \ln|s_0u^+| dx. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3} |s_0 u^+|^p \ln |s_0 u^+| dx &= a s_0^2 \|u^+\|^2 + b s_0^4 \|u^+\|^4 + b s_0^2 t_0^2 \|u^+\|^2 \|u^-\|^2 \\ &\leq a s_0^2 \|u^+\|^2 + b s_0^4 \|u^+\|^4 + b s_0^4 \|u^+\|^2 \|u^-\|^2.\end{aligned} \tag{2.11}$$

Since $\langle I'(u), u^+ \rangle \leq 0$, one has

$$a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \leq \int_{\mathbb{R}^3} |u^+|^p \ln |u^+| dx.$$

Multiplying the both sides of the above equation with $-s_0^p$, then

$$-s_0^p \int_{\mathbb{R}^3} |u^+|^p \ln |u^+| dx \leq -a s_0^p \|u^+\|^2 - b s_0^p \|u^+\|^4 - b s_0^p \|u^+\|^2 \|u^-\|^2. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$s_0^p \ln s_0 \int_{\mathbb{R}^3} |u^+|^p dx \leq a(s_0^2 - s_0^p)\|u^+\|^2 + b(s_0^4 - s_0^p)\|u^+\|^4 + b(s_0^4 - s_0^p)\|u^+\|^2\|u^-\|^2.$$

Clearly, if $s_0 > 1$, the left-hand side of the above inequality is positive, while the right-hand side of the above inequality is always negative. This is a contradiction. Therefore, $s_0 \leq 1$. Similarly, we can also obtain $t_0 \leq 1$. □

Lemma 2.11. *The following minimax characterization hold*

$$\inf_{u \in \mathcal{N}} I(u) = c = \inf_{u \in H \setminus \{0\}} \max_{t \geq 0} I(tu),$$

and

$$\inf_{u \in \mathcal{M}} I(u) = m = \inf_{u \in H, u^\pm \neq 0} \max_{s,t \geq 0} I(su^+ + tu^-).$$

Moreover,

$$c > 0 \quad \textit{and} \quad m > 0 \quad \textit{are achieved.}$$

Proof. Firstly, we prove the second equality since the first equality is similar. On one hand, it follows from Lemma 2.5 that

$$\inf_{u \in H, u^\pm \neq 0} \max_{s,t \geq 0} I(su^+ + tu^-) \leq \inf_{u \in \mathcal{M}} \max_{s,t \geq 0} I(su^+ + tu^-) = \inf_{u \in \mathcal{M}} I(u) = m. \tag{2.13}$$

On the other hand, for all $u \in H$ with $u^\pm \neq 0$, Lemma 2.8 implies that there exists a unique positive numbers pair (s_0, t_0) such that $s_0 u^+ + t_0 u^- \in \mathcal{M}$. Let $v := s_0 u^+ + t_0 u^- \in \mathcal{M}$, we have

$$m = \inf_{v \in \mathcal{M}} I(v) \leq I(s_0 u^+ + t_0 u^-) \leq \max_{s,t \geq 0} I(su^+ + tu^-),$$

which implies that

$$m = \inf_{v \in \mathcal{M}} I(v) \leq \inf_{u \in H, u^\pm \neq 0} \max_{s,t \geq 0} I(su^+ + tu^-). \tag{2.14}$$

Thus, the conclusion directly follows from (2.13) and (2.14).

Next, we prove that $m > 0$ is achieved. Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence, i.e. $I(u_n) \to m$ as $n \to \infty$. In light of (1.5) and (1.8), one has

$$\begin{aligned} m + o(1) &= I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &= a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3} |u_n|^p dx \\ &\geq a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2. \end{aligned}$$

This implies that $\{u_n\}$ is bounded in H. Thus, up to a subsequence, there exists $u_* \in H$ such that

$$\begin{cases} u_n^{\pm} \rightharpoonup u_*^{\pm}, & \text{in } H, \\ u_n^{\pm} \to u_*^{\pm}, & \text{in } L^q(\mathbb{R}^3),\ 2 \leq q < 6, \\ u_n^{\pm}(x) \to u_*^{\pm}(x), & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Since $\{u_n\} \subset \mathcal{M}$, we have $\langle I'(u_n), u_n^{\pm}\rangle = 0$. In light of (1.9), (2.2) and (2.3), for all $q \in (p, 6)$ and taking $\sigma = q - p$ in (2.2), we have

$$\begin{aligned} aS_q^{2/q}\left(\int_{\mathbb{R}^3} |u_n^{\pm}|^q dx\right)^{2/q} &\leq a\|u_n^{\pm}\|^2 \leq a\|u_n^{\pm}\|^2 + b\|u_n\|^2\|u_n^{\pm}\|^2 \\ &\leq \int_{\mathbb{R}^3} (|u_n^{\pm}|^p \ln |u_n^{\pm}|)^+ dx \\ &\leq \frac{1}{e(q-p)}\int_{\mathbb{R}^3} |u_n^{\pm}|^q dx. \end{aligned} \tag{2.15}$$

Thus,

$$\int_{\mathbb{R}^3} |u_n^{\pm}|^q dx \geq C > 0.$$

By Lemma 2.2, we get

$$\int_{\mathbb{R}^3} |u_*^{\pm}|^q dx \geq C > 0, \tag{2.16}$$

which implies that $u_*^{\pm} \neq 0$.

Since $\langle I'(u_n), u_n\rangle = \langle I'(u_n), u_n^+\rangle + \langle I'(u_n), u_n^-\rangle = 0$, in view of (2.3) and (2.15), we have

$$a\|u_n\|^2 \leq a\|u_n\|^2 + b\|u_n\|^4 \leq \int_{\mathbb{R}^3} (|u_n|^p \ln |u_n|)^+ dx \leq C\int_{\mathbb{R}^3} |u_n|^q dx \leq CS_q^{-1}\|u_n\|^q, \tag{2.17}$$

which implies that

$$\|u_n\| \geq C > 0.$$

If $\|u_n\| \to 0$ as $n \to \infty$, from (2.17) we know $\int_{\mathbb{R}^3} |u_n|^q dx \to 0$. Using Lemma 2.2 we get $\int_{\mathbb{R}^3} |u_*|^q dx = 0$, which is in contradiction with (2.16). Therefore

$$m = \lim_{n\to\infty}\left[a(\frac{1}{2} - \frac{1}{p})\|u_n\|^2 + b(\frac{1}{4} - \frac{1}{p})\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3} |u_n|^p dx\right] \geq C > 0.$$

By the Lebesgue dominated convergence theorem and the weak semi-continuity of norm, we have

$$\begin{aligned} a\|u_*^{\pm}\|^2 + b\|u_*\|^2\|u_*^{\pm}\|^2 &\leq \liminf_{n\to\infty}\left(a\|u_n^{\pm}\|^2 + b\|u_n\|^2\|u_n^{\pm}\|^2\right) \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^3} |u_n^{\pm}|^p \ln |u_n^{\pm}| dx \\ &= \int_{\mathbb{R}^3} |u_*^{\pm}|^p \ln |u_*^{\pm}| dx. \end{aligned}$$

Together with (1.9), it shows that

$$\langle I'(u_*), u_*^{\pm}\rangle \le 0.$$

According to Lemma 2.10, there are two positive constants $0 < s_0, t_0 \le 1$ such that $s_0 u_*^+ + t_0 u_*^- \in \mathcal{M}$. Define $\tilde{u} := s_0 u_*^+ + t_0 u_*^-$, it follows from (1.5), (1.8) and weak semi-continuity of norm that

$$\begin{aligned}
m &\le I(\tilde{u}) - \frac{1}{p}\langle I'(\tilde{u}), \tilde{u}\rangle \\
&= a\left(\frac{1}{2}-\frac{1}{p}\right)\|s_0u_*^+ + t_0u_*^-\|^2 + b\left(\frac{1}{4}-\frac{1}{p}\right)\|s_0u_*^+ + t_0u_*^-\|^4 \\
&\quad + \frac{1}{p^2}\int_{\mathbb{R}^3}|s_0u_*^+ + t_0u_*^-|^p dx \\
&= a\left(\frac{1}{2}-\frac{1}{p}\right)(s_0^2\|u_*^+\|^2 + t_0^2\|u_*^-\|^2) \\
&\quad + b\left(\frac{1}{4}-\frac{1}{p}\right)(s_0^4\|u_*^+\|^4 + t_0^4\|u_*^-\|^4 + 2s_0^2t_0^2\|u_*^+\|^2\|u_*^-\|^2) \\
&\quad + \frac{1}{p^2}\left(s_0^p\int_{\mathbb{R}^3}|u_*^+|^p dx + t_0^p\int_{\mathbb{R}^3}|u_*^-|^p dx\right) \\
&\le a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_*\|^2 + b\left(\frac{1}{4}-\frac{1}{p}\right)\|u_*\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_*|^p dx \\
&\le \liminf_{n\to\infty}\left[a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4}-\frac{1}{p}\right)\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_n|^p dx\right] \\
&\le \liminf_{n\to\infty}\left[I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle\right] \\
&= m.
\end{aligned}$$

This means that $s_0 = t_0 = 1$, i.e. $\tilde{u} = u_* \in \mathcal{M}$ and $I(u_*) = m > 0$. By a similar argument as above, we have that $c > 0$ is achieved. □

Lemma 2.12. *The minimizers of* $\inf_{u\in\mathcal{N}} I(u)$ *and* $\inf_{u\in\mathcal{M}} I(u)$ *are critical points of* I.

Proof. According to Lemma 2.11, we have $u_* = u_*^+ + u_*^- \in \mathcal{M}$ and $I(u_*) = m$, Therefore it is only necessary to prove that $I'(u_*) = 0$. Arguing by contradiction, assume that $I'(u_*) \ne 0$. Then, there exist $\delta > 0$ and $\gamma > 0$ such that

$$\|I'(u)\| \ge \gamma, \qquad \forall \|u - u_*\| \le 3\delta \text{ and } u \in H.$$

Let $D := \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$, by Lemma 2.7, one has

$$\tilde{m} := \max_{(s,t)\in\partial D} I(su_*^+ + tu_*^-) < m. \tag{2.18}$$

Set $\varepsilon := \min\{(m-\tilde{m})/3, \delta\gamma/8\}$ and $S_\delta := B(u_*, \delta)$. By applying [19, Lemma 2.3], there exists a deformation $\eta \in ([0,1]\times H, H)$ such that

(i) $\eta(1, u) = u$, if $u \notin I^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I^{m+\varepsilon} \cap S_\delta) \subset I^{m-\varepsilon}$;

(iii) $I(\eta(1,u)) \le I(u)$, $\forall u \in H$.

From (iii) and Lemma 2.7, for each $s, t > 0$ with $|s-1|^2 + |t-1|^2 \geq \delta^2/\|u_*\|^2$, one has

$$I(\eta(1, su_*^+ + tu_*^-)) \leq I(su_*^+ + tu_*^-) < I(u_*) = m. \tag{2.19}$$

By Lemma (2.3), we have $I(su_*^+ + tu_*^-) \leq I(u_*) = m$ for $s, t > 0$. According to (ii), one has

$$I(\eta(1, su_*^+ + tu_*^-)) \leq m - \varepsilon, \qquad \forall s, t > 0, |s-1|^2 + |t-1|^2 < \delta^2/\|u_*\|^2. \tag{2.20}$$

Thus, from (2.19) and (2.20), we get

$$\max_{(s,t)\in\bar{D}} I(\eta(1, su_*^+ + tu_*^-)) < m. \tag{2.21}$$

Let $h(s,t) = su_*^+ + tu_*^-$, we prove that $\eta(1, h(D)) \cap \mathcal{M} \neq \emptyset$, which contradicts the definition of m. Define

$$k(s,t) := \eta(1, h(s,t)),$$
$$\Phi(s,t) := \left(\langle I'(h(s,t)), u_*^+\rangle, \langle I'(h(s,t)), u_*^-\rangle\right) := \left(\Phi_1(s,t), \Phi_2(s,t)\right),$$

and

$$\Psi(s,t) := \left(\frac{1}{s}\langle I'(k(s,t)), (k(s,t))^+\rangle, \frac{1}{t}\langle I'(k(s,t)), (k(s,t))^-\rangle\right),$$

where

$$\begin{aligned}\Phi_1(s,t) &= \frac{1}{s}\langle I'(su_*^+ + tu_*^-), su_*^+\rangle \\ &= a(s - s^{p-1})\|u_*^+\|^2 + b(s^3 - s^{p-1})\|u_*^+\|^4 + b(st^2 - s^{p-1})\|u_*^+\|^2\|u_*^-\|^2 \\ &\quad - s^{p-1}\ln s\int_{\mathbb{R}^3}|u_*^+|^p dx,\end{aligned}$$

and

$$\begin{aligned}\Phi_2(s,t) &= \frac{1}{t}\langle I'(su_*^+ + tu_*^-), tu_*^-\rangle \\ &= a(t - t^{p-1})\|u_*^-\|^2 + b(t^3 - t^{p-1})\|u_*^-\|^4 + b(ts^2 - t^{p-1})\|u_*^+\|^2\|u_*^-\|^2 \\ &\quad - t^{p-1}\ln t\int_{\mathbb{R}^3}|u_*^-|^p dx.\end{aligned}$$

Obviously, Φ is C^1 functions. Moreover, by a direct calculation we have

$$\left.\frac{\partial\Phi_1(s,t)}{\partial s}\right|_{(1,1)} = a(2-p)\|u_*^+\|^2 + b(4-p)\|u_*^+\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3}|u_*^+|^p dx,$$

and

$$\left.\frac{\partial\Phi_1(s,t)}{\partial t}\right|_{(1,1)} = 2b\|u_*^+\|^2\|u_*^-\|^2.$$

Similarly, we obtain

$$\left.\frac{\partial\Phi_2(s,t)}{\partial s}\right|_{(1,1)} = 2b\|u_*^+\|^2\|u_*^-\|^2,$$

and

$$\left.\frac{\partial\Phi_2(s,t)}{\partial t}\right|_{(1,1)} = a(2-p)\|u_*^-\|^2 + b(4-p)\|u_*^-\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3}|u_*^-|^p dx.$$

Let

$$M = \begin{bmatrix} \frac{\partial \Phi_1(s,t)}{\partial s}\big|_{(1,1)} & \frac{\partial \Phi_2(s,t)}{\partial s}\big|_{(1,1)} \\ \frac{\partial \Phi_1(s,t)}{\partial t}\big|_{(1,1)} & \frac{\partial \Phi_2(s,t)}{\partial t}\big|_{(1,1)} \end{bmatrix},$$

then we have that

$$\begin{aligned} \det M &= \frac{\partial \Phi_1(s,t)}{\partial s}\bigg|_{(1,1)} \times \frac{\partial \Phi_2(s,t)}{\partial t}\bigg|_{(1,1)} - \frac{\partial \Phi_1(s,t)}{\partial t}\bigg|_{(1,1)} \times \frac{\partial \Phi_2(s,t)}{\partial s}\bigg|_{(1,1)} \\ &= \left[a(2-p)\|u_*^+\|^2 + b(4-p)\|u_*^+\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3} |u_*^+|^p dx\right] \\ &\quad \times \left[a(2-p)\|u_*^-\|^2 + b(4-p)\|u_*^-\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3} |u_*^-|^p dx\right] \\ &\quad - 4b^2\|u_*^+\|^4\|u_*^-\|^4 \\ &> 0. \end{aligned}$$

Hence, the solution of equation (1.1) is the unique isolated zero point of $\Phi(s,t)$. Then, the topological degree theory [6, 21] implies that $\deg(\Phi, D, 0) = 1$. Combining (2.18) with (i), we have that $h = k$ on ∂D, then we obtain

$$\deg(\Phi, D, 0) = \deg(\Psi, D, 0) = 1.$$

So, $\Psi(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, and

$$\eta(1, h(s_0, t_0)) = k(s_0, t_0) \in \mathcal{M},$$

which is a contradiction with (2.21). So we get that $I'(u_*) = 0$. Similarly, we can prove that any minimizer of $\inf_{u\in\mathcal{N}} I(u)$ are a critical point of $I(u)$. □

3 Proof of theorems

Firstly, we prove the existence of positive ground state solutions for equation (1.1).

Proof of Theorem 1.1. According to Lemma 2.11 and Lemma 2.12, there exists $\bar{u} \in \mathcal{N}$ such that

$$I(\bar{u}) = c, \qquad I'(\bar{u}) = 0.$$

Now, we only need to prove that $\bar{u}$ is a positive solution of equation (1.1). Indeed, replacing $I(u)$ with the functional

$$\begin{aligned} I(\bar{u}) = \frac{a}{2}\int_{\mathbb{R}^3} (|\nabla \bar{u}|^2 + V(x)\bar{u}^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} (|\nabla \bar{u}|^2 + V(x)\bar{u}^2)dx\right) \\ - \frac{1}{p}\int_{\mathbb{R}^3} |\bar{u}^+|^p \ln|\bar{u}^+| dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |\bar{u}^+|^p dx. \end{aligned}$$

In this way we can get a solution $\bar{u}$ such that

$$\left(a + b\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + V(x)\bar{u}^2 dx\right)[-\Delta \bar{u} + V(x)\bar{u}] = |\bar{u}^+|^{p-2}\bar{u}^+ \ln|\bar{u}^+|, \qquad x \in \mathbb{R}^3. \tag{3.1}$$

Multiplying the both sides of (3.1) with u^-, we deduce that

$$a\|\bar{u}^-\|^2 + b\|\bar{u}^-\|^4 + b\|\bar{u}^+\|^2\|\bar{u}^-\|^2 = 0.$$

It follows that $\bar{u}^-(x) = 0$, and then $\bar{u}(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$. The regularity theory of elliptic equation implies that $\bar{u} \in C^2(\mathbb{R}^3)$ is nonnegative classical solution of equation (1.1). Since $p \in (4,6)$, we know that $\lim_{\bar{u}\to 0^+} \bar{u}^{p-1}\ln \bar{u} = 0$. It makes sense to denote $\bar{u}^{p-1}\ln \bar{u} = 0$ for $\bar{u} = 0$. Let $\Omega^+ = \{\bar{u} \in \mathbb{R}^3 : \bar{u}(x) \geq 0\}$. Then $\bar{u}(x)$ is positive solution in $\mathbb{R}^3$ if $\partial\Omega^+ = \emptyset$. In the following, we prove that $\partial\Omega^+ = \emptyset$. Otherwise, let $x_0 \in \partial\Omega^+$ and $B_\rho(x_0) = \{x \in \mathbb{R}^3 : |x - x_0| < \rho\}$ with small $\rho > 0$. Define

$$\alpha = \left(a + b\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + V(x)\bar{u}^2 dx\right) > 0,$$

and

$$c(x) = \left(a + b\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + V(x)\bar{u}^2 dx\right) V(x) - \bar{u}^{p-1}\ln|\bar{u}|.$$

Then, $\bar{u}|_{B_\rho(x_0)}$ is nontrivial solution of the following boundary value problem

$$-\alpha\triangle v + c(x)v = 0, \qquad x \in B_\rho(x_0) \quad \text{and} \quad v(x) = \bar{u}(x) \quad \text{for } x \in \partial B_\rho(x_0).$$

Under the assumptions, we see that $c(x) > 0$ in $B_\rho(x_0)$ for $\rho > 0$ small enough. By the maximum principle [18], we see that $\bar{u}(x) > 0$ for all $x \in B_\rho(x_0)$, which contradicts to that $x_0 \in \partial\Omega^+$. In conclusion, $\bar{u}$ is a positive ground state solution of equation (1.1). Thus the proof of Theorem 1.1 is completed. □

Secondly, we verify that equation (1.1) has a ground state sign-changing solution with precisely two nodal domains.

Proof of Theorem 1.2. In light of Lemma 2.11 and Lemma 2.12, there exists $u_* \in \mathcal{M}$ such that

$$I(u_*) = m, \qquad I'(u_*) = 0. \tag{3.2}$$

Now, we show that u_* has exactly two nodal domains. Set $u_* = u_1 + u_2 + u_3$, where

$$u_1 \geq 0, \qquad u_2 \leq 0, \qquad \Omega_1 \cap \Omega_2 = \emptyset, \qquad u_1|_{\mathbb{R}^3\setminus\Omega_1} = u_2|_{\mathbb{R}^3\setminus\Omega_2} = u_3|_{\Omega_1\cup\Omega_2} = 0, \tag{3.3}$$

$$\Omega_1 := \{x \in \mathbb{R}^3 : u_1 > 0\}, \qquad \Omega_2 := \{x \in \mathbb{R}^3 : u_2 < 0\},$$

and Ω_1, Ω_2 are connected open subsets of $\mathbb{R}^3$. Letting $v = u_1 + u_2$, then $v^+ = u_1$ and $v^- = u_2$, i.e. $v^\pm \neq 0$. Note that $I'(u_*) = 0$, by a straightforward calculation, we can obtain

$$\langle I'(v), v^+\rangle = -b\|v^+\|^2\|u_3\|^2, \tag{3.4}$$

and

$$\langle I'(v), v^-\rangle = -b\|v^-\|^2\|u_3\|^2. \tag{3.5}$$

It follows from (1.5), (1.8), (2.4), (3.2)–(3.5) that

$$\begin{aligned} m = I(u_*) &= I(u_*) - \frac{1}{p}\langle I'(u_*), u_*\rangle \\ &= I(v) + I(u_3) + \frac{b}{2}\|v\|^2\|u_3\|^2 - \frac{1}{p}\left[\langle I'(v), v\rangle + \langle I'(u_3), u_3\rangle + 2b\|v\|^2\|u_3\|^2\right] \end{aligned}$$

$$\begin{aligned}
&\geq \sup_{s,t\geq 0}\left[I(sv^+ + tv^-) + \frac{1-s^p}{p}\langle I'(v), v^+\rangle + \frac{1-t^p}{p}\langle I'(v), v^-\rangle\right] \\
&\quad + I(u_3) - \frac{1}{p}\langle I'(v), v\rangle - \frac{1}{p}\langle I'(u_3), u_3\rangle \\
&\geq \sup_{s,t\geq 0}\left[I(sv^+ + tv^-) + \frac{bs^p}{p}\|v^+\|^2\|u_3\|^2 + \frac{bt^p}{p}\|v^-\|^2\|u_3\|^2\right] \\
&\quad + a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_3\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|u_3\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_3|^p dx \\
&\geq \max_{s,t\geq 0} I(sv^+ + tv^-) + a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_3\|^2 \\
&\geq m + a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_3\|^2,
\end{aligned}$$

which implies that $u_3 = 0$. Therefore, u_* has exactly two nodal domains.

Next, we show that the energy of sign-changing solution of equation (1.1) is strictly larger than twice of the energy of positive ground state solution. By Lemma 2.9, there exist $s_*, t_* > 0$ such that $s_* u_*^+, t_* u_*^- \in \mathcal{N}$. Then it follows from (3.2) and Lemma 2.7 that

$$\begin{aligned}
m = I(u_*) &\geq I(s_* u_*^+ + t_* u_*^-) \\
&= I(s_* u_*^+) + I(t_* u_*^-) + 2bs_*^2 t_*^2 \|u_*^+\|^2 \|u_*^-\|^2 \\
&> I(s_* u_*^+) + I(t_* u_*^-) \geq 2c > 0.
\end{aligned}$$

To sum up, u_* is a ground state sign-changing solution of equation (1.1) with precisely two nodal domains. Besides, $m > 2c$. Then, the proof of Theorem 1.2 is completed. □

Finally, we prove that equation (1.1) has a sequence of solutions to infinity by the following symmetric mountain pass theorem:

Theorem 3.1 ([15, Theorem 9.12])**.** *Let E be an infinite dimensional Banach space, and let $I \in C^1(E, \mathbb{R})$ be even, satisfying (PS) condition and $I(0) = 0$. If $E = V \bigoplus X$, where V is finite dimensional and I satisfies*

- *(i) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$,*
- *(ii) for each finite dimensional subspace $\tilde{E} \subset E$, there exists an $R = R(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \backslash B_{R(\tilde{E})}$,*

then I possesses an unbounded sequence critical values.

Lemma 3.2. *Assume that (V_1)–(V_2) hold, then I satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset H$ be a sequence with $\{I(u_n)\}$ bounded and $I'(u_n) \to 0$. We first claim that $\{u_n\}$ is bounded in H. Indeed,

$$\begin{aligned}
C + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\
&= a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_n|^p dx \\
&\geq a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2.
\end{aligned}$$

This implies that $\{u_n\}$ is bounded in H. Going if necessary up to subsequence, we may assume that there exists $u \in H$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H, \\ u_n \to u, & \text{in } L^q(\mathbb{R}^3),\ 2 \le q < 6, \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{3.6}$$

By $\lim_{n\to\infty} \|\langle I'(u_n), u_n - u\rangle\| \le \lim_{n\to\infty} \|I'(u_n)\|\|u_n - u\| = 0$ and $\|\langle I'(u), u_n - u\rangle\| \le \lim_{n\to\infty} \|I'(u_n)\|\|u_n - u\| = 0$, we deduce that

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \langle I'(u_n), u_n - u\rangle - \langle I'(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty.$$

On the other hand, it follows from (3.6) and Hölder's inequality that

$$\begin{aligned} \lim_{n\to\infty} \langle u, u_n - u\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}^3} \nabla u[\nabla u_n - \nabla u] + V(x)u(u_n - u)dx \\ &\le \int_{\mathbb{R}^3} \nabla u[\nabla u - \nabla u]dx + \left(\int_{\mathbb{R}^3} (V(x)u)^2 dx\right)^{1/2} |u_n - u|_2 \\ &= 0. \end{aligned}$$

Therefore, by some preliminary calculations, one has

$$\begin{aligned} &(a + b\|u_n\|^2)\|u_n - u\|^2 \\ &\quad = (a + b\|u_n\|^2) \int_{\mathbb{R}^3} [\nabla u_n \nabla(u_n - u) + V(x)u_n(u_n - u)]dx \\ &\qquad - (a + b\|u\|^2 + b\|u_n\|^2 - b\|u\|^2) \int_{\mathbb{R}^3} [\nabla u \nabla(u_n - u) + V(x)u(u_n - u)]dx \\ &\quad = \langle I'(u_n), u_n - u\rangle + b(\|u\|^2 - \|u_n\|^2) \int_{\mathbb{R}^3} [\nabla u \nabla(u_n - u) + V(x)u(u_n - u)]dx \\ &\qquad + \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n - u)\ln|u_n|dx - \langle I'(u), u_n - u\rangle - \int_{\mathbb{R}^3} |u|^{p-2}u(u_n - u)\ln|u|dx \\ &\quad = \langle I'(u_n) - I'(u), u_n - u\rangle + b(\|u\|^2 - \|u_n\|^2)\langle u, u_n - u\rangle \\ &\qquad + \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n \ln|u_n| - |u|^{p-2}u\ln|u|)(u_n - u)dx. \end{aligned}$$

We obtain the conclusion if the last term of the above formula tend to zero as $n \to +\infty$. Indeed, in view of (1.7), (3.6) and Hölder's inequality, for any $\varepsilon > 0$ small enough we deduce that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n \ln|u_n| - |u|^{p-2}u\ln|u|)(u_n - u)dx \right| \\ &\qquad \le \int_{\mathbb{R}^3} \left(||u_n|^{p-1}\ln|u_n|| + ||u|^{p-1}\ln|u|| \right) |u_n - u|dx \\ &\qquad \le \int_{\mathbb{R}^3} \left[\varepsilon(|u_n| + |u|) + C_\varepsilon(|u_n|^{q-1} + |u|^{q-1}) \right] |u_n - u|dx \\ &\qquad \le 4\varepsilon\left(|u_n|_2^2 + |u|_2^2\right) + C_\varepsilon \left(|u_n|_q^{q-1} + |u|_q^{q-1} \right) |u_n - u|_q \\ &\qquad \le \varepsilon C + C_\varepsilon |u_n - u|_q. \end{aligned}$$

These estimates show that $u_n \to u$ in H, so I satisfies (PS) condition. □

Proof of Theorem 1.3. In Theorem 3.1, let $E = H$ and the functional I given by (1.5). By Lemma 3.2, the functional I satisfies (PS) condition, so we just need to verify that I satisfies conditions (i) and (ii) of Theorem 3.1. Note that

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \ln|u| dx \\ &\geq \frac{a}{2}\|u\|^2 - \frac{1}{p}\int_{\{x:|u|\geq 1\}} |u|^p \ln|u| dx \\ &\geq \frac{a}{2}\|u\|^2 - C_1\|u\|^{p+\sigma}, \end{aligned}$$

where $0 < \sigma < 6 - p$. Thus, we can choose $\rho > 0$ and $\alpha > 0$ small enough such that $I|_{\partial B_\rho} \geq \alpha > 0$.

We suppose that $\tilde{E}$ is a finite dimensional subspace of H, and for $u \in \tilde{E}\backslash\{0\}$, define $v = u/\|u\|$, then $\|v\| = 1$. one has

$$\begin{aligned} I(u) &= \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} + \frac{b}{4}\|u\|^{4-p} - \frac{1}{p}\int_{\mathbb{R}^3} |v|^p \ln|v| dx + \frac{1}{p^2}\int_{\mathbb{R}^3}|v|^p dx - \frac{1}{p}(\ln\|u\|)\int_{\mathbb{R}^3}|v|^p dx\right) \\ &\leq \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} + \frac{b}{4}\|u\|^{4-p} + C_1\|v\|^2 + C_2\|v\|^q + \frac{S_p^{-1}}{p^2}\|v\|^p - \frac{1}{p}(\ln\|u\|)\int_{\mathbb{R}^3}|v|^p dx\right) \\ &= \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} + \frac{b}{4}\|u\|^{4-p} + C - \frac{1}{p}(\ln\|u\|)\int_{\mathbb{R}^3}|v|^p dx\right). \end{aligned}$$

Thus, there exists an $R = R(\tilde{E})$ large enough such that $I \leq 0$ on $\tilde{E}\backslash B_{R(\tilde{E})}$.

To sum up, all conditions of Theorem 3.1 are satisfied. Therefore, equation (1.1) owns a sequence of solutions $\{u_n\}$ with $I(u_n) \to +\infty$ as $n \to \infty$. This completes the proof of Theorem 1.3. □

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References

[1] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348**(1996), No. 1, 305–330. https://doi.org/10.1090/S0002-9947-96-01532-2; MR1333386; Zbl 0858.35083

[2] S. Bernstein, Surune classe d'équations fonctionnelles aux dérivées partielles, *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **4**(1940), No. 1, 17–26. MR0002699; Zbl 0026.01901

[3] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations* **20**(1995), No. 9–10, 1725–1741. https://doi.org/10.1080/03605309508821149; MR1349229; Zbl 0837.35043

[4] T. Bartsch, T. Weth, Three nodal solutions of singularly perturbed elliptic equations on domains without topology, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **22**(2005), No. 3, 259–281. https://doi.org/10.1016/j.anihpc.2004.07.005; MR2136244; Zbl 1114.35068

[5] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation, *Adv. Differential Equations* **6**(2001), No. 6, 701–730. https://doi.org/10.57262/ade/1357140586; MR1829093; Zbl 1007.35049

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985. https://doi.org/10.1007/978-3-662-00547-7; MR0787404; Zbl 0559.47040

[7] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.* **108**(1992), No. 2, 247–262. https://doi.org/10.1007/BF02100605; MR1161092; Zbl 0785.35067

[8] W. X. Feng, X. H. Tang, L. Y. Zhang, Existence of a positive bound state solution for logarithmic Schrödinger equation, *J. Math. Anal. Appl.* **531**(2024), No. 2, Paper No. 127861, 16 pp. https://doi.org/10.1016/j.jmaa.2023.127861; MR4658509; Zbl 07852472

[9] Y. Gao, Y. S. Jiang, L. S. Liu, N. Wei, Multiple positive solutions for a logarithmic Kirchhoff type problem in $\mathbb{R}^3$, *Appl. Math. Lett.* **139**(2023), 108539. https://doi.org/10.1016/j.aml.2022.108539; MR4524016; Zbl 1506.35088

[10] D. Hu, Q. Gao, Multiple solutions to logarithmic Kirchhoff equations (in Chinese), *Acta Math. Sci. Ser. A (Chinese Ed.).* **42**(2022), No. 2, 401–417. https://doi.org/1003-3998(2022)02-401-17; MR4411988; Zbl 1513.35235

[11] C. Ji, A. Szulkin, A logarithmic Schrödinger equation with asymptotic conditions on the potential, *J. Math. Anal. Appl.* **437**(2016), No. 1, 241–254. https://doi.org/10.1016/j.jmaa.2015.11.071; MR3451965; Zbl 1333.35010

[12] G. Kirchhoff, *Mechanik*, Leipzig, Teubner, 1883.

[13] J. L. Lions, On some questions in boundary value problems of mathematical physics, in: *Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, North-Holland Math. Stud., Vol. 30, North-Holland, Amsterdam–New York, 1978, pp. 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3; MR519648; Zbl 0404.35002

[14] C. Miranda, Un'osservazione su un teorema di Brouwer, *Unione Mat. Ital. (2)* **3**(1940), 5–7. MR0004775; Zbl 0024.02203

[15] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, CBMS Regional Conf. Ser. in Math., Vol. 65, 1986. https://doi.org/10.1090/cbms/065; MR0845785; Zbl 0609.58002

[16] W. Shuai, Multiple solutions for logarithmic Schrödinger equations, *Nonlinearity* **32**(2019), No. 6, 2201–2225. https://doi.org/10.1088/1361-6544/ab08f4; MR3951962; Zbl 1418.35102

[17] W. Shuai, Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains, *J. Differential Equations* **259**(2015), No. 4, 1256–1274. https://doi.org/10.1016/j.jde.2015.02.040; MR3345850; Zbl 1319.35023

[18] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(1984), No. 3, 191–202. https://doi.org/10.1007/BF01449041; MR0768629; Zbl 0561.35003

[19] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[20] L. X. Wen, X. H. Tang, S. T. Chen, Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 47, 1–13. https://doi.org/10.14232/ejqtde.2019.1.47; MR3991096; Zbl 1438.35159

[21] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed point theorems*, Springer-Verlag, New York, 1986. https://doi.org/10.1007/978-1-4612-4838-5; MR0816732; Zbl 0583.47050

[22] K. G. Zloshchastiev, Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences, *Gravit. Cosmol.* **16**(2010), No. 4, 288–297. https://doi.org/10.1134/S0202289310040067; MR2740900; Zbl 1232.83044

[23] Z. T. Zhang, K. Perera, Sign changing solutions of Kirchhoff problems via invariant sets of descent flow, *J. Math. Anal. Appl.* **317**(2006), No. 2, 456–463. https://doi.org/10.1016/j.jmaa.2005.06.102; MR2208932; Zbl 1100.35008

[24] C. X. Zhang, L. Y. Zhang, Qualitative analysis on logarithmic Schrödinger equation with general potential, *J. Fixed Point Theory Appl.* **24**(2022), No. 4, 1–33. https://doi.org/10.1007/s1178-022-00989-y; MR4500268; Zbl 1500.35178