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Global multiplicity of positive solutions for anisotropic (p, q)-Robin boundary value problems with an indefinite potential term

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Abstract. We consider a nonlinear Robin problem driven by the anisotropic (p, q)-Laplacian plus an indefinite potential term. In the reaction, we have the competing effects of a parametric concave (sublinear) term perturbed by a superlinear one (concave-convex problem). We prove the existence and multiplicity result for positive solutions which is global with respect to the parameter. We also show the existence of a minimal positive solution and determine its dependence on the parameter.

Keywords: nonlinear nonhomogeneous differential operator, indefinite potential, positive solutions, truncations and comparisons, nonlinear regularity, (p, q)-Laplacian, Robin boundary.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric anisotropic Robin boundary value problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)(u(z))^{p(z)-1} = \lambda(u(z))^{\tau(z)-1} + f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z)(u(z))^{p(z)-1} = 0 \quad \text{on } \partial\Omega, \ \lambda > 0, \ u > 0, \ 1 < \tau < p. \end{cases} \tag{p_λ}$$

In this problem the variable exponents $p(\cdot)$ and $q(\cdot)$ of the two differential operators are Lipschitz continuous on $\overline{\Omega}$, that is, $p, q \in C^{0,1}(\overline{\Omega})$. Then the two operators are defined by

$$\Delta_{p(z)}u = \operatorname{div}(|Du|^{p(z)-2}Du), \quad \forall u \in W^{1,p(z)}(\Omega),$$

$$\Delta_{q(z)}u = \operatorname{div}(|Du|^{q(z)-2}Du), \quad \forall u \in W^{1,q(z)}(\Omega).$$

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There is also a potential term $\xi(z)(u(z))^{p(z)-1}$ which is in general indefinite since $\xi \in L^\infty(\Omega)$ can be sign-changing (nodal). Therefore the left hand side of (p_λ) is not coercive. In the reaction (right hand side of (p_λ)), we have the combined effects of a parametric "concave" (sublinear) term $\lambda u^{\tau(z)-1}$ with $\tau \in C(\overline{\Omega})$, $\tau_+ = \max_{\overline{\Omega}} \tau < q_- = \min_{\overline{\Omega}} q$ and of a Carathéodory perturbation $f(z,x)$ (that is, for all $x \geq 0$ $z \to f(z,x)$ is measurable and for almost a.a. $z \in \Omega$ $x \to f(z,x)$ is continuous) which is "convex" (($p_+ - 1$)-superlinear with $p_+ = \max_{\bar{\Omega}} p$) but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition.

In the boundary condition, $\frac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative of u, corresponding to the anisotropic (p,q)-Laplacian. If $u \in C^1(\bar{\Omega})$ then

$$\frac{\partial u}{\partial n_{pq}} = \left(|Du|^{p(z)-2} + |Du|^{q(z)-2}\right)\frac{\partial u}{\partial n}$$

with $n(.)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in C^{0,1}(\partial\Omega)$ with $\beta(z) \geq 0$ and either $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Therefore (p_λ) is an anisotropic version of the classical "concave-convex problem", with an indefinite potential term and Robin boundary condition. Concave-convex problems, were first studied by Ambrosetti–Brezis–Cerami [1], for semilinear Dirichlet problems driven by the Laplacian with no potential term (that is, $\xi \equiv 0$) and a reaction of the form $u \to \lambda u^{\tau-1} + u^{r-1}$ with $1 < \tau < 2 < r$. Their work was extended to p-Laplacian equations by García Azorero–Peral Alonso–Manfredi [10] and Guo–Zhang [14]. Further extensions involved more general nonlinear nonhomogeneous differential operators and more general reactions (see Marano–Marino–Papageorgiou [19], Papageorgiou–Rădulescu [29] and the references therein). For anisotropic problems, there are significantly fewer papers. We mention the works of Papageorgiou–Qin–Rădulescu [24] (Dirichlet problems driven by the anisotropic p-Laplacian) and by Deng [6], Liu–Papageorgiou [18] (Robin problems, in [6] the differential operator is the $p(z)$-Laplacian and $\xi \equiv 0$, while in [18] the equation is driven by the anisotropic (p,q)-Laplacian, with $\xi(z) > 0$ for a.a. $z \in \Omega$, the conditions on $f(z,.)$ are stronger near 0^+ and the authors employ a different superlinearity condition). Using variational tools from the critical point theory, together with truncation and comparison techniques, we prove an existence and multiplicity result for positive solutions which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem). Our result here extends all the aforementioned anisotropic works. We also mention the works of [3], [12], [20], [27] and [28] on isotropic Neumann and Robin problems with indefinite potential term.

Anisotropic problems are interesting from a purely mathematical viewpoint since they exhibit challenging nonlinearities that we do not encounter in isotropic problems. The $p(z)$-Laplace differential operator is not homogeneous in contrast to the p-Laplacian. This excludes from consideration techniques which proved to be very effective in the context of isotropic problems. This makes anisotropic problems in principle more difficult to deal with. Anisotropic equations, proved to be the right mathematical tool to describe various phenomena from physics and engineering. Materials with inhomogeneities, such as electrorheological fluids (also known as "smart fluids"), can not be modelled adequately using the formalism of the classical Lebesgue and Sobolev spaces. They require the use of variable such spaces (a particular case of the so-called "Musielak–Orlicz spaces"). The book of Růžička [33] contains mathematical models of such fluids and the phenomena characterizing them (Winslow effect). Another important application of anisotropic problem is in image restoration, where we try to eliminate the effect of noise. Initially, this problem was approached by smoothing the input, which corresponds to minimizing the energy functional:

$$\varphi_1(u) = \int_\Omega (|Du|^2 + |u-i|^2)dz$$

with $i(.)$ being the input which corresponds to shades of grey in $\Omega \subset \mathbb{R}^n$. We assume that noise is additive, that is, $i = t_0 + n$ with t_0 representing the true image and n the noise which is a random variable with zero mean. It turns out that this approach destroys the small details of the image. To remedy this, it was proposed to use the "total variation smoothing", which corresponds to minimizing the energy functional:

$$\varphi_2(u) = \int_\Omega (|Du| + |u-i|^2)dz.$$

This approach does a good job of preserving the edges of the image (an edge gives rise to a large gradient of $u(\cdot)$). But unfortunately, this approach also introduces edges, where they did not exist before. For this reason Chen–Levine–Rao [4], suggested to consider the energy functional:

$$\varphi_3(u) = \int_\Omega (|Du|^{p(z)} + |u-i|^2)dz$$

with $1 \le p(z) \le 2$. This function is close to 1 where there are no edges and close to 2 where there are. Therefore, we have an energy functional which incorporates the positive aspects of both $\varphi_1(.)$ and $\varphi_2(.)$.

More details on the mathematical and physical applications of variable spaces can be found in the books of Cruz Uribe–Fiorenza [5], Diening–Harjulehto–Hästö–Růžička [7], Rădulescu–Repovš [31], Růžička [33].

The Robin boundary condition is a weighted combination of Dirichlet and Neumann boundary conditions and so it is more difficult to handle and for this reason it is less common in the literature. However, it is important from a physical viewpoint since it appears in electromagnetic problems (impedance boundary condition) and in heat transfer problems (convective boundary condition).

2 Mathematical background and hypotheses

In this section, we briefly review some basic facts about variable exponent spaces. A comprehensive presentation of variable exponent Lebesgue and Sobolev spaces can be found in the books of Cruz Uribe–Fiorenza [5], Diening–Harjulehto–Hästö–Růžička [7].

Let $L_1^\infty(\Omega) = \{p \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega p \ge 1\}$. For $p \in L_1^\infty(\Omega)$, we set

$$p_- = \operatorname*{ess\,inf}_\Omega p \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_\Omega p.$$

Also, let $M(\Omega) = \{u : \Omega \to \mathbb{R} : u(\cdot) \text{ is measurable}\}$. As usual, we identify two functions which differ on a set of zero measure.

Given $p \in L_1^\infty(\Omega)$, we define the following variable exponent Lebesgue space

$$L^{p(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{p(z)}\, dz < +\infty \right\}.$$

We equip $L^{p(z)}(\Omega)$ with the following norm (known as the Luxemburg norm)

$$\|u\|_{p(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda} \right)^{p(z)} dz \le 1 \right\}.$$

Also, we introduce the variable exponent Sobolev spaces as follows:

$$W^{1,p(z)}(\Omega) = \left\{u \in L^{p(z)}(\Omega) : |Du| \in L^{p(z)}(\Omega)\right\}.$$

We equip this space with the following norm:

$$\|u\|_{1,p(z)} = \|u\|_{p(z)} + \|Du\|_{p(z)}.$$

An equivalent norm of $W^{1,p(z)}(\Omega)$ is given by:

$$\|u\|_{1,p(z)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\left(\frac{|Du|}{\lambda}\right)^{p(z)} + \left(\frac{|u|}{\lambda}\right)^{p(z)}\right) dz \le 1\right\}.$$

We define $W_0^{1,p(z)}(\Omega)$ as the closure in the $\|\cdot\|_{1,p(z)}$ norm of all compactly supported $W^{1,p(z)}(\Omega)$-functions.

When $p \in L_1^\infty(\Omega)$ and $p_- > 1$, then the spaces $L^{p(z)}(\Omega)$, $W^{1,p(z)}(\Omega)$, and $W_0^{1,p(z)}(\Omega)$ are all separable, reflexive, and uniformly convex.

If $p, p' \in L_1^\infty(\Omega)$ and $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$, then $L^{p(z)}(\Omega)^* = L^{p'(z)}(\Omega)$, and we have the following Hölder-type inequality:

$$\int_\Omega |uv|\, dz \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(z)}\|v\|_{p'(z)}$$

for all $u \in L^{p(z)}(\Omega)$, $v \in L^{p'(z)}(\Omega)$.

We set

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)}, & \text{if } p(z) < N, \\ +\infty, & \text{if } p(z) \ge N. \end{cases}$$

Theorem 2.1. *If $p, q \in C(\bar{\Omega})$, $p_+ < N$ and $1 \le q(z) \le p^*(z)$ (resp. $1 \le q(z) < p^*(z)$) for all $z \in \bar{\Omega}$, then $W^{1,p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$ are embedded continuously (resp. compactly) into $L^{q(z)}(\Omega)$.*

We set

$$p^\partial(z) = \begin{cases} \frac{(N-1)p(z)}{N-p(z)}, & \text{if } p(z) < N, \\ +\infty, & \text{if } p(z) \ge N. \end{cases}$$

Theorem 2.2. *If $p \in C(\bar{\Omega})$, $p_- > 1$ and $q \in C(\partial\Omega)$ satisfies the condition*

$$1 \le q(z) < p^\partial(z) \quad \text{for all } z \in \partial\Omega$$

then $W^{1,p(z)}(\Omega)$ embedded compactly into $L^{q(z)}(\partial\Omega)$. In particular, $W^{1,p(z)}(\Omega)$ embedded compactly into $L^{p(z)}(\partial\Omega)$.

We introduce the following modular functions:

$$\rho(u) = \int_\Omega |u|^{p(z)}\, dz \quad \text{for all } u \in L^{p(z)}(\Omega),$$

$$\hat{\rho}(u) = \int_\Omega \left(|Du|^{p(z)} + |u|^{p(z)}\right) dz \quad \text{for all } u \in W^{1,p(z)}(\Omega).$$

We have the following properties.

Proposition 2.3.

(a) For every $u \in L^{p(z)}(\Omega)$, $u \neq 0$, we have $\|u\|_{p(z)} = \lambda \iff \rho\left(\frac{u}{\lambda}\right) = 1$;

(b) $\|u\|_{p(z)} < 1$ (resp. $= 1, > 1$) $\iff \rho(u) < 1$ (resp. $= 1, > 1$);

(c) $\|u\|_{p(z)} < 1 \Rightarrow \|u\|_{p(z)}^{p_+} \leq \rho(u) \leq \|u\|_{p(z)}^{p_-}$ and $\|u\|_{p(z)} > 1 \Rightarrow \|u\|_{p(z)}^{p_-} \leq \rho(u) \leq \|u\|_{p(z)}^{p_+}$;

(d) $\|u_n\|_{p(z)} \to 0 \iff \rho(u_n) \to 0$;

(e) $\|u_n\|_{p(z)} \to +\infty \iff \rho(u_n) \to +\infty$.

Similarly, we have the following implications when $p \in C^{0,1}(\overline{\Omega})$.

Proposition 2.4.

(a) For every $u \in W^{1,p(z)}(\Omega)$, $u \neq 0$, we have $\|u\|_{1,p(z)} = \lambda \iff \hat{\rho}\left(\frac{u}{\lambda}\right) = 1$;

(b) $\|u\|_{1,p(z)} < 1$ (resp. $= 1, > 1$) $\iff \hat{\rho}(u) < 1$ (resp. $= 1, > 1$);

(c) $\|u\|_{1,p(z)} < 1 \Rightarrow \|u\|_{1,p(z)}^{p_+} \leq \hat{\rho}(u) \leq \|u\|_{1,p(z)}^{p_-}$ and $\|u\|_{1,p(z)} > 1 \Rightarrow \|u\|_{1,p(z)}^{p_-} \leq \hat{\rho}(u) \leq \|u\|_{1,p(z)}^{p_+}$;

(d) $\|u_n\|_{1,p(z)} \to 0 \iff \hat{\rho}(u_n) \to 0$;

(e) $\|u_n\|_{1,p(z)} \to +\infty \iff \hat{\rho}(u_n) \to +\infty$.

Let $\beta \in L^\infty(\partial\Omega)$ with $\beta_- := \inf_{z\in\partial\Omega} \beta(z) > 0$, and for any $u \in W^{1,p(z)}(\Omega)$, define

$$\|u\|_\beta := \inf\left\{\tau > 0 : \int_\Omega \left(\frac{|\nabla u|}{\tau}\right)^{p(z)} dz + \int_{\partial\Omega} \beta(z) \left(\frac{|u|}{\tau}\right)^{p(z)} d\sigma \leq \tau\right\}.$$

Proposition 2.5. *Let* $\rho_\beta(u) = \int_\Omega |\nabla u|^{p(z)}\, dz + \int_{\partial\Omega} \beta(z)|u|^{p(z)}\, d\sigma$ *with* $\beta_- > 0$, *where* $d\sigma$ *is the measure on the boundary of* Ω. *For any* $u, u_k \in W^{1,p(z)}(\Omega)$ $(k = 1, 2, \dots)$, *we have that*

(a) $\|u\|_\beta \leq 1 \Rightarrow \|u\|_\beta^{p_-} \leq \rho_\beta(u) \leq \|u\|_\beta^{p_+}$;

(b) $\|u\|_\beta \geq 1 \Rightarrow \|u\|_\beta^{p_+} \leq \rho_\beta(u) \leq \|u\|_\beta^{p_-}$;

(c) $\|u_k\|_\beta \to 0 \iff \rho_\beta(u_k) \to 0$ (as $k \to \infty$);

(d) $\|u_k\|_\beta \to \infty \iff \rho_\beta(u_k) \to \infty$ (as $k \to \infty$).

Proposition 2.6. *(see [32]) If there is a vector* $l \in \mathbb{R}^n \setminus \{0\}$ *such that for any* $z \in \Omega$ *the function* $f(t) = q(z + tl)$ *is monotone for* $t \in I_z = \{t : \ z + tl \in \Omega\}$, *then*

$$0 < \mu^* = \inf_{u\neq 0} \frac{\int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz}{\int_\Omega \frac{1}{q(z)} |u|^{q(z)} dz}.$$

Theorem 2.7. *For any* $u \in W^{1,p(z)}(\Omega)$, *let*

$$\|u\|_\partial := \|\nabla u\|_{p(z)} + \|u\|_\beta.$$

Then $\|u\|_\partial$ *is a norm on* $W^{1,p(z)}(\Omega)$ *which is equivalent to*

$$\|u\|_{1,p(z)} = \|\nabla u\|_{p(z)} + \|u\|_{p(z)}.$$

The Banach space $C^1(\overline{\Omega})$ is an ordered with a positive (order) cone C_+ which is defined by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Given $u : \Omega \to \mathbb{R}$ is measurable, then we define

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\} \quad \text{for all } z \in \Omega.$$

These are measurable functions and $u = u^+ - u^-$, $|u| = u^+ + u^-$. Moreover, if $u \in W^{1,p(.)}(\Omega)$, then $u^\pm \in W^{1,p(.)}(\Omega)$. Suppose $u, v : \Omega \to \mathbb{R}$ are measurable functions with $u(z) \leq v(z)$ for a.a. $z \in \Omega$. We define

$$[u,v] = \{h \in W^{1,p(.)}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\},$$

$$\operatorname{int}_{C^1(\overline{\Omega})}[u,v] = \text{the interior in } C^1(\overline{\Omega}) \text{ of } [u,v] \cap C^1(\overline{\Omega}),$$

$$[u) = \{h \in W^{1,p(.)}(\Omega) \mid u(z) \leq h(z) \text{ for a.a. } z \in \Omega\}.$$

If $u(.)$ is a measurable function, then we write $0 \prec u$ if for all $K \subseteq \Omega$ compact we have $0 < c_K \leq u(z)$ for a.a. $z \in K$.

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(.)$ satisfies the "C-condition", if it has the following property:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

- $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded,
- $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^*,

admits a strongly convergent subsequence."

This is a compactness-type condition on $\varphi(\cdot)$ which compensates for the fact that the ambient space X need not be locally compact (being in general infinite-dimensional). By K_φ we denote the critical set of $\varphi(.)$, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Now we are ready to state our hypotheses on the data of problem (p_λ):

***H*$_0$**: $p, q \in C^{0,1}(\bar{\Omega})$, $\tau \in C(\bar{\Omega})$ and $1 < \tau(z) \leq \tau_+ < q_- \leq q_+ < p(z) < N$ for all $z \in \bar{\Omega}$. $p_+ < \frac{Np_-}{N-p_-}$, there exists $d \in \mathbb{R}^N$ such that $t \to q(z + td)$ is monotone on $I_z = \{t : z + td \in \Omega\}$, $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

***H*$_1$**: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the following conditions:

(i) $0 \leq f(z,x) \leq a(z)(1 + x^{r(z)-1})$ for almost every $z \in \Omega$ and all $x \geq 0$, where $a \in L^\infty(\Omega)$, $r \in C(\bar{\Omega})$ and $p_+ < r(z) < p(z)^*$ for all $z \in \bar{\Omega}$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^{p_+}} = +\infty$ uniformly for almost every $z \in \Omega$;

(iii) if for every $\lambda > 0$, we define $e(z,x) = f(z,x)x - p_+F(z,x)$ and

$$\beta_\lambda(z,x) = \lambda\left(1 - \frac{p_+}{\tau(z)}\right)x^{\tau(z)} + \xi(z)\left(\frac{p_+}{p(z)} - 1\right)x^{p(z)} + e(z,x),\ x \geq 0,$$

then there exists $\mu \in L^1(\Omega)$ such that

$$\beta_\lambda(z,x) \leq \beta_\lambda(z,y) + \mu(z)$$

for almost all $z \in \Omega$ and all $0 \leq x \leq y$;

(iv) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q(z)-1}} = 0$ uniformly for almost every $z \in \Omega$.

Remark 2.8. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis, we may assume that $f(z,x) = 0$ for almost every $z \in \Omega$, all $x \leq 0$. Hypotheses $H_1(iii)$ is satisfied if there exists $M > 0$ such that for a.a. $z \in \Omega$

$$x \to \frac{\lambda x^{\tau(z)-1} + f(z,x) - \xi(z)x^{p(z)-1}}{x^{p_+-1}}$$

is nondecreasing on $x \geq M$ (see [16]).

The following function satisfies hypotheses H_1 above

$$f(z,x) = \begin{cases} (x^+)^{s(z)-1}, & \text{if } x \leq 1, \\ x^{p_+-1}\ln x + x^{\mu(z)-1}, & \text{if } 1 < x \end{cases}$$

with $s \in C(\bar{\Omega})$, $q(z) < s(z)$ for all $z \in \bar{\Omega}$, $\mu \in C(\bar{\Omega})$, $\mu(z) \leq p_+$ for all $z \in \Omega$. This function fails to satisfy the Ambrosetti–Rabinowitz condition (see [2]).

Let $p \in C^{0,1}(\overline{\Omega})$ and consider the operator $V : W^{1,p(z)}(\Omega) \to (W^{1,p(z)}(\Omega))^*$ defined by

$$\langle V(u), h\rangle = \int_\Omega \left(|\nabla u|^{p(z)-2}(Du, Dh)_{\mathbb{R}} + |\nabla u|^{q(z)-2}(Du, Dh)_{\mathbb{R}}\right)dz, \qquad \forall u, h \in W^{1,p(z)}(\Omega).$$

This operator has the following properties (see [13]).

Proposition 2.9. *The map* $V : W^{1,p(z)}(\Omega) \to (W^{1,p(z)}(\Omega))^*$ *defined above is bounded(that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too) and for type* $(S)_+$*, that is*

$$u_n \rightharpoonup u \ \text{(weakly) in } W^{1,p(z)}(\Omega) \text{ and } \limsup_{n\to\infty}\langle V(u_n), (u_n - u)\rangle \leq 0 \Rightarrow u_n \to u \ \text{ in } W^{1,p(z)}(\Omega).$$

3 Positive solutions

We introduce the following two sets:

$$\mathfrak{L} := \{\lambda > 0 : \text{problem } (p_\lambda) \text{ has a positive solution}\},$$
$$S_\lambda := \text{set of positive solutions of } (p_\lambda).$$

Our first goal is to establish some basic properties of $\mathfrak{L}$. From now on $\|\cdot\| := \|\cdot\|_{1,p(z)}$.

Let $\theta > \|\xi\|_\infty$, $\lambda > 0$, and consider the functional $\hat{\varphi}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(z)}}{p(z)} dz + \int_\Omega \frac{|\nabla u|^{q(z)}}{q(z)} dz + \int_\Omega \frac{\xi(z)}{p(z)} |u|^{p(z)} dz + \int_\Omega \frac{\theta}{p(z)} (u^-)^{p(z)} dz \\ + \int_{\partial\Omega} \frac{\beta(z)}{p(z)} |u|^{p(z)} d\sigma - \int_\Omega \frac{\lambda}{\tau(z)} (u^+)^{\tau(z)} dz - \int_\Omega F(z, u^+) dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

Proposition 3.1. *If hypotheses H_0 and H_1 hold, and $\lambda > 0$, then $\hat{\varphi}_\lambda(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p(z)}(\Omega)$ such that

$$|\hat{\varphi}_\lambda(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{3.1}$$

$$(1 + \|u_n\|)\hat{\varphi}_\lambda'(u_n) \to 0 \quad \text{in } W^{1,p(z)}(\Omega)^* \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$|\langle \hat{\varphi}_\lambda'(u_n), h\rangle| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W^{1,p(z)}(\Omega), \text{ all } n \in \mathbb{N}, \tag{3.3}$$

with $\varepsilon_n \to 0^+$, which implies

$$\Big| \langle V(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p(z)-2}u_n h dz - \int_\Omega \theta(u_n^-)^{p(z)-1} h dz \\ - \lambda \int_\Omega (u_n^+)^{\tau(z)-1} h dz + \int_{\partial\Omega} \beta(z)|u_n|^{p(z)-2}u_n h d\sigma - \int_\Omega f(z, u_n^+) h dz \Big| \leq \frac{\varepsilon_n \|h\|}{(1 + \|u_n\|)} \tag{3.4}$$

for all $h \in W^{1,p(z)}(\Omega), n \in \mathbb{N}$.

In (3.4), we choose $h = -u_n^- \in W^{1,p(z)}(\Omega)$. We have

$$\Big| \int_\Omega |Du_n^-|^{p(z)} dz + \int_\Omega |Du_n^-|^{q(z)} dz + \int_\Omega \xi(z)(u_n^-)^{p(z)} dz + \int_\Omega \theta(u_n^-)^{p(z)} dz \\ + \int_{\partial\Omega} \beta(z)(u_n^-)^{p(z)} d\sigma \Big| \leq \varepsilon_n \tag{3.5}$$

for all $n \in \mathbb{N}$.

Then,

$$\Big| \int_\Omega |Du_n^-|^{p(z)} dz + \int_\Omega (\xi(z) + \theta)(u_n^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_n^-)^{p(z)} d\sigma \Big| \leq \varepsilon_n \tag{3.6}$$

which implies

$$u_n^- \to 0 \quad \text{in } W^{1,p(z)}(\Omega) \quad (\text{recall that } \theta > \|\xi\|_\infty). \tag{3.7}$$

In (3.4), we choose $h = u_n^+ \in W^{1,p(z)}(\Omega)$. We have

$$\Big| \int_\Omega |Du_n^+|^{p(z)} dz + \int_\Omega |Du_n^+|^{q(z)} dz + \int_\Omega \xi(z)(u_n^+)^{p(z)} dz - \lambda \int_\Omega (u_n^+)^{\tau(z)} dz \\ - \int_\Omega f(z, u_n^+) u_n^+ dz + \int_{\partial\Omega} \beta(z)(u_n^+)^{p(z)} d\sigma \Big| \leq \varepsilon_n \tag{3.8}$$

for all $n \in \mathbb{N}$.

On the other hand, from (3.1), (3.7) we have

$$\Big| \int_\Omega \frac{p_+}{p(z)} |Du_n^+|^{p(z)} dz + \int_\Omega \frac{p_+}{q(z)} |Du_n^+|^{q(z)} dz + \int_\Omega \frac{p_+}{p(z)} \xi(z) (u_n^+)^{p(z)} dz \\ - \lambda \int_\Omega \frac{p_+}{\tau(z)} (u_n^+)^{\tau(z)} dz - \int_\Omega p_+ F(z, u_n^+) u_n^+ dz + \int_{\partial\Omega} \frac{p_+}{p(z)} \beta(z) (u_n^+)^{p(z)} d\sigma \Big| \le M_2 \quad (3.9)$$

for some $M_2 > 0$, all $n \in \mathbb{N}$.

From (3.8) and (3.9) it follows that

$$\int_\Omega \left(\frac{p_+}{p(z)} - 1 \right) |Du_n^+|^{p(z)} dz + \int_\Omega \left(\frac{p_+}{q(z)} - 1 \right) |Du_n^+|^{q(z)} dz + \int_\Omega \xi(z) \left(\frac{p_+}{p(z)} - 1 \right) (u_n^+)^{p(z)} dz \\ + \int_{\partial\Omega} \left(\frac{p_+}{p(z)} - 1 \right) \beta(z) (u_n^+)^{p(z)} d\sigma - \lambda \int_\Omega \left(\frac{p^+}{\tau(z)} - 1 \right) (u_n^+)^{\tau(z)} dz + \int_\Omega e(z, u_n^+) dz \le M_3 \quad (3.10)$$

for some $M_3 > 0$, all $n \in \mathbb{N}$.

Recall $\beta_\lambda(z,x) = \lambda(1 - \frac{p_+}{\tau(z)}) x^{\tau(z)} + \xi(z)(\frac{p_+}{p(z)} - 1) x^{p(z)} + e(z,x)$ for all $x \ge 0$. Then from (3.10) we have

$$\int_\Omega \beta_\lambda(z, u_n^+) dz \le M_3 \quad \text{for all } n \in \mathbb{N}. \quad (3.11)$$

Claim. The sequence $\{u_n^+\}_{n \ge 1} \subseteq W^{1,p(z)}(\Omega)$ is bounded.

Our argument proceeds through contradiction. So, suppose that the claim is not true. Then passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \quad (3.12)$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \rightharpoonup y \quad \text{(weakly) in } W^{1,p(z)}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{p(z)}(\Omega),\ y \ge 0. \quad (3.13)$$

Let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$ and $\Omega_0 = \{z \in \Omega : y(z) = 0\}$. Then $\Omega = \Omega_+ \cup \Omega_0$.

First we assume that $|\Omega_+|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). We have $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega_+$ and so on account of hypothesis $H_1(ii)$ we have

$$\int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^{p_+}} dz \to +\infty \quad \text{as } n \to +\infty \quad \text{(see [23])}. \quad (3.14)$$

On account of (3.12), we may assume that $\|u_n^+\| \ge 1$ for all $n \in \mathbb{N}$. Then from (3.1) and (3.5), we have

$$\lambda \int_\Omega \frac{1}{\tau(z)} \frac{(u_n^+)^{\tau(z)}}{\|u_n^+\|^{p_+}} dz + \int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^{p_+}} dz \le \varepsilon_n' + \frac{1}{p_-} \int_\Omega \frac{|Du_n^+|^{p(z)}}{\|u_n^+\|^{p_+}} dz + \frac{1}{q_-} \int_\Omega \frac{|Du_n^+|^{q(z)}}{\|u_n^+\|^{p_+}} dz \\ + \frac{\|\xi\|_\infty}{p_-} \int_\Omega \frac{(u_n^+)^{p(z)}}{\|u_n^+\|^{p_+}} dz + \frac{\|\beta\|_\infty}{p_-} \int_{\partial\Omega} \frac{(u_n^+)^{p(z)}}{\|u_n^+\|^{p_+}} d\sigma \\ \le \varepsilon_n' + \frac{1}{p_-} \int_\Omega |Dy_n|^{p(z)} dz + \frac{1}{q_- \|u_n\|^{p_+ - q_+}} \int_\Omega |Dy_n|^{q(z)} dz + \frac{\|\xi\|_\infty}{p_-} \int_\Omega (y_n)^{p(z)} dz \\ + \frac{\|\beta\|_\infty}{p_-} \int_{\partial\Omega} (y_n)^{p(z)} d\sigma \le M_4 \quad (3.15)$$

for some $M_4 > 0$ with $\varepsilon'_n \to 0$.

Comparing (3.14) and (3.15), we have a contradiction. We can assume that $y \equiv 0$, that is $|\Omega|_N = |\Omega_0|_N$. We define

$$\hat{\varphi}_\lambda(t_n u_n^+) := \max\{\hat{\varphi}_\lambda(t u_n^+) : 0 \le t \le 1\}. \tag{3.16}$$

Let $v_n = \eta^{1/p_-} y_n$ for all $n \in \mathbb{N}$, with $\eta \ge 1$. Then, we have

$$v_n \rightharpoonup 0 \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad \int_\Omega F(z, v_n)\, dz \to 0 \quad \text{as } n \to \infty \quad (\text{see } [23]). \tag{3.17}$$

Also, we have

$$\int_\Omega \frac{1}{\tau(z)} v_n^{\tau(z)}\, dz \to 0. \tag{3.18}$$

Moreover (3.12) implies that we can find $n_0 \in \mathbb{N}$ such that

$$\frac{\eta^{\frac{1}{p_-}}}{\|u_n^+\|} \in (0, 1] \quad \text{for all } n \ge n_0. \tag{3.19}$$

Hence from (3.16) and (3.19)

$$\begin{aligned}
\hat{\varphi}_\lambda(t_n u_n^+) \ge \hat{\varphi}_\lambda(v_n) &= \int_\Omega \frac{1}{p(z)} |Dv_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Dv_n|^{q(z)} dz + \int_\Omega \frac{1}{p(z)} \xi(z) v_n^{p(z)}\, dz \\
&\quad - \lambda \int_\Omega \frac{1}{\tau(z)} v_n^{\tau(z)} dz + \int_{\partial\Omega} \frac{1}{p(z)} \beta(z) v_n^{p(z)}\, d\sigma - \int_\Omega F(z, v_n)\, dz \quad \text{for all } n \ge n_0 \\
&\ge \frac{1}{p_+} \left(\int_\Omega |Dv_n|^{p(z)} dz + \int_\Omega \xi(z) v_n^{p(z)}\, dz + \int_{\partial\Omega} \beta(z) v_n^{p(z)}\, d\sigma \right) - \int_\Omega F(z, v_n)\, dz \\
&\ge \frac{\eta}{2p_+} \quad (\text{see hypotheses } H_0 \text{ and } (3.17))
\end{aligned}$$

for all $n \ge n_1 \ge n_0$. Since $\eta \ge 1$ is an arbitrary number, we can infer that

$$\hat{\varphi}_\lambda(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{3.20}$$

We know that

$$\hat{\varphi}_\lambda(0) = 0 \quad \text{and} \quad \hat{\varphi}_\lambda(u_n^+) \le M_5, \quad \text{all } n \in \mathbb{N}. \tag{3.21}$$

From (3.20) and (3.21) it follows that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0, 1) \quad \text{for all } n \ge n_2. \tag{3.22}$$

Then from (3.16) and (3.22) we infer that

$$t_n \frac{d}{dt} \hat{\varphi}_\lambda(t u_n^+)\Big|_{t=t_n} = 0, \quad \text{then } \langle \hat{\varphi}'_\lambda(t_n u_n^+), t_n u_n^+ \rangle = 0, \quad \forall n \ge n_2,$$

$$\begin{aligned}
\hat{\varphi}_\lambda(t_n u_n^+) &= \hat{\varphi}_\lambda(t_n u_n^+) - \frac{1}{p_+}\langle \hat{\varphi}'_\lambda(t_n u_n^+), t_n u_n^+\rangle \\
&= \int_\Omega t_n^{p(z)}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]|Du_n^+|^{p(z)}dz + \int_\Omega t_n^{q(z)}\left[\frac{1}{q(z)} - \frac{1}{p_+}\right]|Du_n^+|^{q(z)}dz \\
&\quad + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\xi(z)(t_n u_n^+)^{p(z)}dz - \int_\Omega \lambda\left[\frac{1}{\tau(z)} - \frac{1}{p_+}\right](t_n u_n^+)^{\tau(z)}dz \\
&\quad + \int_{\partial\Omega}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\beta(z)(t_n u_n^+)^{p(z)}d\sigma + \frac{1}{p_+}\int_\Omega e(z, t_n u_n^+)dz \\
&\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]|Du_n^+|^{p(z)}dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right]|Du_n^+|^{q(z)}dz + \frac{1}{p_+}\int_\Omega \beta_\lambda(z, t_n u_n^+)dz \\
&\quad + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\xi(z)(t_n u_n^+)^{p(z)}dz + \int_{\partial\Omega}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\beta(z)(t_n u_n^+)^{p(z)}d\sigma \\
&\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]|Du_n^+|^{p(z)}dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right]|Du_n^+|^{q(z)}dz \\
&\quad + \frac{1}{p_+}\int_\Omega \beta_\lambda(z, u_n^+)dz + \frac{1}{p_+}\|\mu\|_1 + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\xi(z)(u_n^+)^{p(z)}dz \\
&\quad + \int_{\partial\Omega}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\beta(z)(u_n^+)^{p(z)}d\sigma \\
&= \hat{\varphi}_\lambda(u_n^+) - \frac{1}{p_+}\langle \hat{\varphi}_\lambda(u_n^+), u_n^+\rangle + \frac{1}{p_+}\|\mu\|_1.
\end{aligned} \tag{3.23}$$

Hence we have,

$$\begin{aligned}
\hat{\varphi}_\lambda(t_n u_n^+) &\leq \hat{\varphi}_\lambda(u_n^+) - \frac{1}{p_+}\langle \hat{\varphi}_\lambda(u_n^+), u_n^+\rangle + \frac{1}{p_+}\|\mu\|_1 \quad \text{for all } n \geq n_2 \text{ (see (3.8))} \\
&\leq \hat{\varphi}_\lambda(u_n^+) + \frac{\varepsilon_n}{p_+} + \frac{1}{p_+}\|\mu\|_1
\end{aligned} \tag{3.24}$$

(3.20) and (3.24) give us that $\hat{\varphi}_\lambda(u_n^+) \to +\infty$, and this contradicts with (3.21).

Therefore $\{u_n^+\} \subset W^{1,p(z)}(\Omega)$ is bounded. Then from (3.7) and the claim it follows that

$$\{u_n\} \subset W^{1,p(z)}(\Omega) \quad \text{is bounded.}$$

We may assume that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{r(z)}(\Omega) \quad \text{as } n \to \infty. \tag{3.25}$$

In (3.4), we choose $h = u_n - u \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.25). Then

$$\lim_{n\to\infty}\langle V(u_n), u_n - u\rangle = 0 \tag{3.26}$$

(3.26) and Proposition (2.9) give us $u_n \to u$ in $W^{1,p(z)}(\Omega)$. So $\hat{\varphi}_\lambda(\cdot)$ satisfies the C-condition. □

Proposition 3.2. *If hypotheses H_0 and H_1 hold, then $\mathfrak{L} \neq \emptyset$ and we have then $S_\lambda \subset \operatorname{int} C_+$ for every $\lambda \in \mathfrak{L}$.*

Proof. On account of hypotheses $H_1(iv)$, we see that given $\varepsilon > 0$, we can find $C_\varepsilon = C(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{\varepsilon}{q(z)}x^{q(z)} + C_\varepsilon x^{r_+} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

For every $u \in W^{1,p(z)}(\Omega)$, we have

$$\begin{aligned}\hat{\varphi}_\lambda(u) \geq & \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma + \int_\Omega \frac{\xi(z)}{p(z)}|u|^{p(z)}dz \\ & + \int_\Omega \frac{\theta}{p(z)}(u^-)^{p(z)}dz - \int_\Omega \frac{\varepsilon}{q(z)}|u|^{q(z)}\,dz - C_\varepsilon \int_\Omega |u|^{r_+}dz - \lambda \int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \\ \geq & \tilde{C}\min\{\|u\|^{p_+}, \|u\|^{p_-}\} + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega \frac{\varepsilon}{q(z)}|u|^{q(z)}\,dz \\ & - C_\varepsilon \int_\Omega |u|^{r_+}dz - \lambda \int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \end{aligned} \tag{3.27}$$

for some $\tilde{C} > 0$ (recall that $\theta > \|\xi\|_\infty$).

Observe next that,

$$\int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz \geq \mu^* \int_\Omega \frac{1}{q(z)}|u|^{q(z)}\,dz \text{ (see Proposition 2.6).} \tag{3.28}$$

$$\int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \leq \frac{1}{\tau_-}\max\{\|u\|^{\tau_+}_{\tau(z)}, \|u\|^{\tau_-}_{\tau(z)}\}. \tag{3.29}$$

We return to (3.27) and use (3.28) and (3.29). Then for $u \in W^{1,p(z)}(\Omega)$ with $\|u\| \leq 1$, $\varepsilon < \mu^*$ we have

$$\begin{aligned}\hat{\varphi}_\lambda(u) &\geq \tilde{C}\|u\|^{p_+} - \lambda C_1\|u\|^{\tau_-} - C_\varepsilon\|u\|^{r_+} \\ &= (\tilde{C} - \lambda C_1\|u\|^{\tau_- - p_+} - C_\varepsilon\|u\|^{r_+ - p_+})\|u\|^{p_+},\ u \in W^{1,p(z)}(\Omega)\end{aligned} \tag{3.30}$$

for some $C_1 > 0$.

Let us set, for any $t > 0$,

$$k_\lambda(t) = \lambda C_1 t^{\tau_- - p_+} - C_\varepsilon t^{r_+ - p_+}.$$

Since $\tau_- < p_+ < r_+$ we have $\lim_{t\to\infty} k_\lambda(t) = \lim_{t\to 0^+} k_\lambda(t) = \infty$.

Then there exists $t_0 > 0$ satisfying $k'_\lambda(t_0) = 0$. One has

$$\lambda C_1(\tau_- - p_+)t_0^{\tau_- - p_+ - 1} = -C_\varepsilon(r_+ - p_+)t_0^{r_+ - p_+ - 1}$$

$$\Rightarrow t_0 = t_0(\lambda) = \left(\frac{\lambda C_1}{C_\varepsilon}\frac{p_+ - \tau_-}{r_+ - p_+}\right)^{\frac{1}{r_+ - \tau_-}}.$$

Then

$$k_\lambda(t_0) = \lambda C_1\left(\frac{\lambda C_1}{C_\varepsilon}\frac{p_+ - \tau_-}{r_+ - p_+}\right)^{\frac{\tau_- - p_+}{r_+ - \tau_-}} + C_\varepsilon\left(\frac{\lambda C_1}{C_\varepsilon}\frac{p_+ - \tau_-}{r_+ - p_+}\right)^{\frac{r_+ - p_+}{r_+ - \tau_-}}$$

and since $p_+ < \tau_+$ we have $\lim_{\lambda\to 0^+} k_\lambda(t_0) = 0$. So we can find $\lambda_0 > 0$ such that

$$k_\lambda(t_0) < \tilde{C} \quad \text{for all } \lambda \in (0, \lambda_0).$$

Then from (3.30) it follows that

$$\hat{\varphi}_\lambda(u) \geq \hat{m}_\lambda > 0 \quad \text{for all } \|u\| = t_0. \tag{3.31}$$

For $u \in \operatorname{int} C_+$, on account of the superlinearity hypothesis $H_1(ii)$, we have

$$\hat{\varphi}_\lambda(tu) \to -\infty \quad \text{as } t \to \infty. \tag{3.32}$$

Then, (3.31), (3.32) and Proposition (3.1) permit the use of mountain pass theorem. Therefore for every $\lambda \in (0, \lambda_0)$ we can find $u_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$u_\lambda \in K_{\hat{\varphi}_\lambda} \quad \text{and} \quad 0 < \hat{m}_\lambda \leq \hat{\varphi}_\lambda(u_\lambda). \tag{3.33}$$

From (3.33) we have $u_\lambda \neq 0$ (recall that $\hat{\varphi}_\lambda(0) = 0$) and

$$\langle \hat{\varphi}'_\lambda(u_\lambda), h \rangle = 0 \quad \text{for all } h \in W^{1,p(z)}(\Omega). \tag{3.34}$$

Choosing $h = -u_\lambda^- \in W^{1,p(z)}(\Omega)$, we obtain

$$\int_\Omega |Du_\lambda^-|^{p(z)} dz + \int_\Omega |Du_\lambda^-|^{q(z)} dz + \int_\Omega (\theta + \xi(z))(u_\lambda^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_\lambda^-)^{p(z)} d\sigma = 0$$

$$\Rightarrow \int_\Omega |Du_\lambda^-|^{p(z)} dz + \int_\Omega (\theta + \xi(z))(u_\lambda^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_\lambda^-)^{p(z)} d\sigma \leq 0$$

$$\Rightarrow u_\lambda \geq 0, \ u_\lambda \neq 0.$$

Then from (3.34) it follows that u_λ is a positive solution (p_λ). From the anisotropic regularity theory (see [8] and [17] for the corresponding isotropic theory) we have

$$u_\lambda \in C_+ \setminus \{0\}.$$

For every $u \in S_\lambda$, we have $u \in C_+ \setminus \{0\}$ and

$$-\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) + \xi(z) u(z)^{p(z)-1} \geq 0 \quad \text{for a.a. } z \in \Omega,$$

$$\Rightarrow \Delta_{p(z)} u(z) + \Delta_{q(z)} u(z) \leq \|\xi\|_\infty u(z)^{p(z)-1} \quad \text{for a.a. } z \in \Omega,$$

$$\Rightarrow u \in \operatorname{int} C_+ \text{ (see [35] and [26], Proposition A2).}$$

So, we have proved that $(0, \lambda_0) \subseteq \mathfrak{L}$ and so $\mathfrak{L} \neq \emptyset$. Moreover, we have $S_\lambda \subseteq \operatorname{int} C_+$ for all $\lambda > 0$. ☐

Next, we show that $\mathfrak{L}$ is an interval.

Proposition 3.3. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathfrak{L}$ and $0 < \mu < \lambda$ then $\mu \in \mathfrak{L}$ and given $u_\lambda \in S_\lambda$, we can find $u_\mu \in S_\mu$ such that $u_\mu \leq u_\lambda$.*

Proof. Let us introduce the Carathéodory function $g_\mu(z, x)$ defined by

$$g_\mu(z,x) = \begin{cases} \mu (x^+)^{\tau(z)-1} + f(z, x^+) + \theta (x^+)^{p(z)-1}, & \text{if } x \leq u_\lambda(z), \\ \mu u_\lambda(z)^{\tau(z)-1} + f(z, u_\lambda(z)) + \theta u_\lambda(z)^{p(z)-1}, & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.35}$$

Here $\theta > \|\xi\|_\infty$.

We set $G_\mu(z,x) = \int_0^x g_\mu(z,s)\, ds$ and consider the C^1-functional $\Psi_\mu : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \Psi_\mu(u) = & \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz \\ & + \int_\Omega \frac{\theta + \xi(z)}{p(z)} |u|^{p(z)}\, dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)} |u|^{p(z)}\, d\sigma - \int_\Omega G_\mu(z,u)\, dz \end{aligned} \tag{3.36}$$

for all $u \in W^{1,p(z)}(\Omega)$. Since $\theta > \|\xi\|_\infty$, it is clear that $\Psi_\mu(\cdot)$ is coercive. Also, using the fact that $W^{1,p(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)$ compactly, we see that $\Psi_\mu(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem, there exists $u_\mu \in W^{1,p(z)}(\Omega)$ such that

$$\Psi_\mu(u_\mu) = \inf\left\{\Psi_\mu(u) : u \in W^{1,p(z)}(\Omega)\right\}. \tag{3.37}$$

Since $\tau_+ < p_-$, we see that

$$\Psi_\mu(u_\mu) < 0 = \Psi_\mu(0) \Rightarrow u_\mu \neq 0.$$

From (3.37) we have

$$\Psi'_\mu(u_\mu) = 0 \Rightarrow$$

$$\langle V(u_\mu), h\rangle + \int_\Omega [\theta + \xi(z)]|u_\mu|^{p(z)-2}u_\mu h\,dz + \int_{\partial\Omega} \beta(z)|u_\mu|^{p(z)-2}u_\mu h\,d\sigma = \int_\Omega g_\mu(z,u_\mu)h\,dz \tag{3.38}$$

for all $h \in W^{1,p(z)}(\Omega)$. In (3.38) first we choose $h = -u_\mu^- \in W^{1,p(z)}(\Omega)$. We obtain

$$\int_\Omega |Du_\mu^-|^{p(z)}dz + \int_\Omega |Du_\mu^-|^{q(z)}dz + \int_\Omega (\theta + \xi(z))(u_\mu^-)^{p(z)}dz + \int_{\partial\Omega} \beta(z)(u_\mu^-)^{p(z)}d\sigma = \int_\Omega g_\mu(z,u_\mu)u_\mu^-\,dz$$

$$\Rightarrow u_\mu \geq 0, u_\mu \neq 0 \quad (\text{see } (3.35)).$$

Next, in (3.38) we choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle V(u_\mu), (u_\mu - u_\lambda)^+\rangle + \int_\Omega [\theta + \xi(z)]u_\mu^{p(z)-1}(u_\mu - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)|u_\mu|^{p(z)-1}u_\mu(u_\mu - u_\lambda)^+\,d\sigma \\
&= \int_\Omega [\mu u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \theta u_\lambda^{p(z)-1}](u_\mu - u_\lambda)^+\,dz \quad (\text{see } (3.35)) \\
&\leq \int_\Omega [\lambda u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \theta u_\lambda^{p(z)-1}](u_\mu - u_\lambda)^+\,dz \quad (\text{since } \mu < \lambda) \\
&= \langle V(u_\lambda), (u_\mu - u_\lambda)^+\rangle + \int_\Omega [\theta + \xi(z)]u_\lambda^{p(z)-1}(u_\mu - u_\lambda)^+\,dz \quad (\text{since } u_\lambda \in S_\lambda).
\end{aligned}$$

The monotonicity of $V(\cdot)$ (see Proposition (2.9)) and the fact that $\theta > \|\xi\|_\infty$ imply that $u_\mu \leq u_\lambda, \Rightarrow u_\mu \in [0, u_\lambda], u_\mu \neq 0, \Rightarrow u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ (see (3.35) and (3.38)). □

So, according to Proposition 3.3, the solution multifunction $\lambda \mapsto S_\lambda$ has a kind of weak monotonicity property. We can improve this monotonicity property by adding one more condition on the perturbation $f(z,\cdot)$.

The new hypotheses on $f(z,x)$ are the following:

***H*$_2$**: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable in $z \in \Omega$, for a.a. $z \in \Omega$ we have $f(z,\cdot) \in C^1(\mathbb{R})$,

(i)–(iv) hypotheses $H_2(i)$–(iv) are the same as the corresponding hypotheses $H_1(i)$–(iv), and

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z,x) + \hat{\xi}_\rho x^{p(z)-1}$ is nondecreasing on $[0,\rho]$.

Remark 3.4. Hypotheses $H_2(v)$ is a one-sided local Hölder condition on $f(z,\cdot)$. It is satisfied for all $z \in \Omega$, $f(z,x)$ is differentiable and for every $\rho > 0$, we can find $C_{\hat{\rho}} > 0$ such that $f'_x(z,x) \geq -C_{\hat{\rho}} x^{p(z)-1}$ for a.a. $z \in \Omega$ and all $0 \leq x \leq \rho$.

Proposition 3.5. *If hypotheses H_0, H_2 hold, $\lambda \in \mathfrak{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $\mu \in (0,\lambda)$, then $\mu \in \mathfrak{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. From Proposition 3.3 we know that $\mu \in \mathfrak{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\lambda - u_\mu \in C_+ \setminus \{0\} \tag{3.39}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_2(v)$. We can always assume that $\hat{\xi}_\rho > \|\xi\|_\infty$. Then we have

$$\begin{aligned} &-\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + [\xi(z) + \hat{\xi}_\rho] u_\mu^{p(z)-1} \\ &\quad = \mu u_\mu^{\tau(z)-1} + f(z,u_\mu) + \hat{\xi}_\rho u_\mu^{p(z)-1} \\ &\quad \leq \mu u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} \quad \text{(see (3.39) and hypothesis } H_2(v)) \\ &\quad \leq \lambda u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} \quad (\text{since } \mu < \lambda) \\ &\quad = -\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + [\xi(z) + \hat{\xi}_\rho] u_\lambda^{p(z)-1}. \end{aligned} \tag{3.40}$$

Note that since $u_\lambda \in \operatorname{int} C_+$ and $\mu < \lambda$, we have

$$0 \prec (\lambda - \mu) u_\lambda^{q(z)-1}. \tag{3.41}$$

Then from (3.40), (3.41) and Proposition 2.4 in [23], we can conclude that

$$u_\lambda - u_\mu \in \operatorname{int} C_+.$$

The proof is now complete. □

Next, we show that for every $\lambda \in \mathfrak{L}$, the solution set S_λ has the smallest element (minimal positive solution). To this end, first, we consider the following auxiliary problem:

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) + |\xi(z)||u(z)|^{p(z)-2} u(z) = \lambda |u(z)|^{\tau(z)-2} u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z)|u(z)|^{p(z)-1} = 0 \text{ on } \partial\Omega, \ \lambda > 0, \ u > 0. \end{cases} \tag{3.42}$$

Proposition 3.6. *If hypotheses H_0 hold and $\lambda > 0$, then problem (3.42) admits a unique positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$.*

Proof. We consider the C^1-functional $\gamma_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \gamma_\lambda(u) = &\int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, dz + \int_\Omega |\xi(z)||u|^{p(z)} \, dz \\ &- \lambda \int_\Omega \frac{1}{\tau(z)} (u^+)^{\tau(z)} \, dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)} |u|^{p(z)} d\sigma \end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. Evidently, $\gamma_\lambda(\cdot)$ is coercive (since $\tau_+ < p_-$, $q_+ < p_-$.) and sequentially weakly lower semicontinuous. So, we can find $\bar{u}_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$\gamma_\lambda(\bar{u}_\lambda) = \min\{\gamma_\lambda(u) : u \in W^{1,p(z)}(\Omega)\} < 0 = \gamma_\lambda(0) \quad (\text{since } \tau_+ < p_-),$$

which implies $\bar{u}_\lambda \neq 0$. We have

$$\gamma'_\lambda(\bar{u}_\lambda) = 0,$$

which implies

$$\langle V(\bar{u}_\lambda), h\rangle + \int_\Omega |\xi(z)||\bar{u}_\lambda|^{p(z)-2}\bar{u}_\lambda h\,dz - \lambda\int_\Omega (\bar{u}_\lambda^+)^{\tau(z)-1}h\,dz + \int_{\partial\Omega}\beta(z)|\bar{u}_\lambda|^{p(z)-2}\bar{u}_\lambda d\sigma = 0 \tag{3.43}$$

for all $h \in W^{1,p(z)}(\Omega)$.

In (3.43) we choose $h = -\bar{u}_\lambda^- \in W^{1,p(z)}(\Omega)$. Then

$$\int_\Omega |D\bar{u}_\lambda^-|^{p(z)}dz + \int_\Omega |D\bar{u}_\lambda^-|^{q(z)}dz + \int_\Omega |\xi(z)|(\bar{u}_\lambda^-)^{p(z)}dz + \int_{\partial\Omega}\beta(z)(\bar{u}_\lambda^-)^{p(z)}d\sigma = 0$$

$$\Rightarrow \int_\Omega |D\bar{u}_\lambda^-|^{p(z)}dz + \int_\Omega |\xi(z)|(\bar{u}_\lambda^-)^{p(z)}dz + \int_{\partial\Omega}\beta(z)(\bar{u}_\lambda^-)^{p(z)}d\sigma \le 0$$

which implies $\bar{u}_\lambda \ge 0$, $\bar{u}_\lambda \neq 0$, hence $\bar{u}_\lambda$ is a positive solution of (3.42) (see (3.43)), therefore $\bar{u}_\lambda \in C_+ \setminus \{0\}$ (anisotropic regularity theory).

Therefore

$$\Delta_{p(z)}\bar{u}_\lambda(z) + \Delta_{q(z)}\bar{u}_\lambda(z) \le \|\xi\|_\infty(\bar{u}_\lambda(z))^{p(z)-1} \quad \text{for a.a. } z \in \Omega,$$

which implies $\bar{u}_\lambda \in \operatorname{int} C_+$ (see Zhang [35]).

Next, we show that this positive solution of (3.42) is unique. Suppose that $\bar{v}_\lambda$ is another positive solution of (3.42). Again we have $\bar{v}_\lambda \in \operatorname{int} C_+$. On account of Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovs [22], p. 274, we have $\frac{\bar{u}_\lambda}{\bar{v}_\lambda}, \frac{\bar{v}_\lambda}{\bar{u}_\lambda} \in L^\infty(\Omega)$. So, we can apply Theorem 2.5 of Takac and Giacomoni [34] and get

$$\begin{aligned}
0 &\le \int_\Omega \left[\frac{-\Delta_{p(z)}\bar{u}_\lambda - \Delta_{q(z)}\bar{u}_\lambda}{(\bar{u}_\lambda)^{q_--1}} + \frac{-\Delta_{p(z)}\bar{v}_\lambda - \Delta_{q(z)}\bar{v}_\lambda}{(\bar{v}_\lambda)^{q_--1}}\right]((\bar{u}_\lambda)^{q_-} - (\bar{v}_\lambda)^{q_-})\,dz \\
&= \int_\Omega \Bigg[\lambda\left(\frac{1}{(\bar{u}_\lambda)^{q_--\tau(z)}} - \frac{1}{(\bar{v}_\lambda)^{q_--\tau(z)}}\right) \\
&\qquad -|\xi(z)|\left((\bar{u}_\lambda)^{p(z)-q_-} - (\bar{v}_\lambda)^{p(z)-q_-}\right)\Bigg]((\bar{u}_\lambda)^{q_-} - (\bar{v}_\lambda)^{q_-})\,dz,
\end{aligned}$$

which implies $\bar{u}_\lambda = \bar{v}_\lambda$ (since $\tau_+ < p_- \le p(z)$).

Therefore the positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$ of problem (3.42) is unique. □

This solution $\bar{u}_\lambda \in \operatorname{int} C_+$ provides a lower bound for the solution set S_λ.

Proposition 3.7. *If hypotheses H_0, H_1 hold and $\lambda \in \mathfrak{L}$, then $\bar{u}_\lambda \le u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda \subset \operatorname{int} C_+$ and consider the Carathéodory function $\beta_\lambda(z, x)$ defined by

$$\hat{\beta}_\lambda(z,x) = \begin{cases} \lambda(x^+)^{\tau(z)-1}, & \text{if } x \le u(z), \\ \lambda u(z)^{\tau(z)-1}, & \text{if } u(z) < x. \end{cases} \tag{3.44}$$

We set $\hat{B}_\lambda(z,x) = \int_0^x \hat{\beta}_\lambda(z,s)ds$ and consider the C^1-functional $\tau_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
\tau_\lambda(u) = &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{|\xi(z)|}{p(z)}|u|^{p(z)}dz \\
&+ \int_{\partial\Omega}\frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma - \int_\Omega \hat{B}_\lambda(z,u)dz
\end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$.

From (3.44) we see that $\tau_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$\tau_\lambda(\tilde{u}_\lambda) = \min\left\{\tau_\lambda(u) : u \in W^{1,p(z)}(\Omega)\right\} < 0 = \tau_\lambda(0) \quad (\text{since } \tau_+ < p_-),$$

which implies $\tilde{u}_\lambda \neq 0$.

We have

$$\tau'_\lambda(\tilde{u}_\lambda) = 0,$$

$$\langle V(\tilde{u}_\lambda), h\rangle + \int_\Omega |\xi(z)||\tilde{u}_\lambda|^{p(z)-2}\tilde{u}_\lambda h dz + \int_{\partial\Omega} \beta(z)|\tilde{u}_\lambda|^{p(z)-2}\tilde{u}_\lambda h d\sigma = \int_\Omega \hat{\beta}_\lambda(z, \tilde{u}_\lambda) h dz \tag{3.45}$$

for all $h \in W^{1,p(z)}(\Omega)$. In (3.45) we first choose $h = -\tilde{u}_\lambda^- \in W^{1,p(z)}(\Omega)$ and infer that

$$\tilde{u}_\lambda \geq 0, \ \tilde{u}_\lambda \neq 0.$$

Next, in (3.45) we choose $h = (\tilde{u}_\lambda - u)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle V(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega |\xi(z)|(\tilde{u}_\lambda)^{p(z)-1}(\tilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \beta(z)(\tilde{u}_\lambda)^{p(z)-1}(\tilde{u}_\lambda - u)^+ d\sigma \\
&= \int_\Omega \lambda u^{\tau(z)-1}(\tilde{u}_\lambda - u)^+ dz \quad (\text{see } (3.44)) \\
&\leq \int_\Omega [\lambda u^{\tau(z)-1} + f(z, u)](\tilde{u}_\lambda - u)^+ dz \quad (\text{since } f \geq 0) \\
&\leq \langle V(u), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega |\xi(z)| u^{p(z)-1}(\tilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \beta(z) u^{p(z)-1}(\tilde{u}_\lambda - u)^+ d\sigma
\end{aligned}$$

(since $u \in S_\lambda$)

$$\Rightarrow \tilde{u}_\lambda \leq u.$$

So, we have proved that

$$\tilde{u}_\lambda \in [0, u] \setminus \{0\}. \tag{3.46}$$

Then it follows from (3.43), (3.44), (3.46) that

$$\begin{aligned}
&\tilde{u}_\lambda \text{ is a positive solution of (3.42)}, \\
&\Rightarrow \tilde{u}_\lambda = \bar{u}_\lambda \in \operatorname{int} C_+ \quad (\text{see Proposition 3.5}), \\
&\Rightarrow \bar{u}_\lambda \leq u \text{ for all } u \in S_\lambda.
\end{aligned}$$

The proof is now complete. □

Remark 3.8. Reasoning as in the above proof, we show that $\lambda \mapsto \bar{u}_\lambda$ is increasing, that is, if $0 < \mu < \lambda$, then $\bar{u}_\lambda - \bar{u}_\mu \in C_+ \setminus 0$.

We know that S_λ is downward directed (see Filippakis and Papageorgiou [9] and Papageorgiou, Rădulescu and Repovš [21], and recall that $V(\cdot)$ is monotone (see Proposition 2.9)).

Proposition 3.9. *If hypotheses H_0, H_1 hold and $\lambda \in \mathfrak{L}$, then there exists $u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+$ such that*

$$u_\lambda^* \leq u \quad \text{for all } u \in S_\lambda \text{ (minimal positive solution of } (p_\lambda)).$$

Proof. By Lemma 3.10 of Hu and Papageorgiou [15] (p. 178), we know that we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda \subseteq \operatorname{int} C_+$ decreasing (recall that S_λ is downward directed) such that

$$\inf_{n\geq 1} u_n = \inf S_\lambda.$$

Since $u_\lambda \leq u_n \leq u_1$ for all $n \in \mathbb{N}$ (see Proposition 3.6), from hypothesis $H_1(i)$ it follows that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightharpoonup u_\lambda^* \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \quad \text{in } L^{r(z)}(\Omega) \quad \text{as } n \to \infty. \tag{3.47}$$

We have

$$\begin{aligned}\langle V(u_n), u_n - u_\lambda^* \rangle &+ \int_\Omega \xi(z) u_n^{p(z)-1}(u_n - u_\lambda^*)\, dz + \int_{\partial\Omega} \beta(z) u_n^{p(z)-1}(u_n - u_\lambda^*) d\sigma \\ &= \lambda \int_\Omega u_n^{\tau(z)-1}(u_n - u_\lambda^*)\, dz + \int_\Omega f(z, u_n)(u_n - u_\lambda^*)\, dz,\end{aligned}$$

which implies

$$\lim_{n\to\infty} \langle V(u_n), u_n - u_\lambda^* \rangle = 0,$$

and thus

$$u_n \to u_\lambda^* \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{(see Proposition 2.9)}. \tag{3.48}$$

Note that $\bar{u}_\lambda \leq u_\lambda^*$ and so $u_\lambda^* \neq 0$,

$$\begin{aligned}\langle V(u_\lambda^*), h \rangle &+ \int_\Omega \xi(z)(u_\lambda^*)^{p(z)-1} h\, dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p(z)-1} h d\sigma \\ &= \lambda \int_\Omega (u_\lambda^*)^{\tau(z)-1} h\, dz + \int_\Omega f(z, u_\lambda^*) h\, dz\end{aligned}$$

for all $h \in W^{1,p(z)}(\Omega)$ (see (3.48)).

It follows that

$$u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+ \quad \text{and} \quad u_\lambda^* = \inf S_\lambda.$$

The proof is now complete. □

We set $\lambda^* = \sup \mathfrak{L}$.

Proposition 3.10. *If hypotheses H_0, H_2 hold, then $\lambda^* < \infty$.*

Proof. On account of hypotheses $H_0, H_2(iv)$ and since $\tau^+ < p^-$, we see that we can find $\lambda > \lambda^*$ such that

$$\lambda x^{\tau(z)-1} + f(z, x) - \xi(z) x^{p(z)-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.49}$$

Let $\lambda > \hat{\lambda}$ and suppose that $\lambda \in \mathfrak{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. Let $\Omega_0 \subset\subset \Omega$ (that is, $\Omega_0 \subseteq \overline{\Omega}_0 \subseteq \Omega$) and assume that $\partial\Omega_0$ is a C^2-manifold. We set $m_0 = \min_{\Omega_0} u_\lambda > 0$ (recall that $u_\lambda \in \operatorname{int} C_+$). Also, let $\hat{\xi}_\rho > \|\xi\|_\infty$. Let $m_0^\delta = m_0 + \delta$ for $\delta > 0$ small enough. We have

$$\begin{aligned}&-\Delta_{p(z)} m_0^\delta - \Delta_{q(z)} m_0^\delta + [\xi(z) + \hat{\xi}_\rho](m_0^\delta)^{p(z)-1} \\ &\quad\leq [\xi(z) + \hat{\xi}_\rho](m_0^\delta)^{p(z)-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\ &\quad\leq \hat{\lambda} m_0^{\tau(z)-1} + f(z, m_0) + \hat{\xi}_\rho m_0^{p(z)-1} + \chi(\delta) \quad \text{(see (3.49))} \\ &\quad\leq \hat{\lambda} u_\lambda^{\tau(z)-1} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} + \chi(\delta) \quad \text{(see hypothesis } H_2(v)) \\ &\quad\leq \hat{\lambda} u_\lambda^{\tau(z)-1} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} - [\lambda - \hat{\lambda}] m_0^{p(z)-1} + \chi(\delta) \\ &\quad\leq -\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + [\xi(z) + \hat{\xi}_\rho] u_\lambda^{p(z)-1} \quad \text{in } \Omega_0, \text{ for } 0 < \delta < 1 \text{ small.}\end{aligned} \tag{3.50}$$

Note that $\delta \in (0,1)$ small enough, we will have

$$[\lambda - \hat{\lambda}] m_0^{p(z)-1} - \chi(\delta) \geq \eta > 0.$$

So, from (3.50) and Papageorgiou–Qin–Rădulescu [24], Proposition 5 (see also [25], Proposition 6) we have

$$u_\lambda(z) \geq m_{\delta_0} \quad \text{for all } z \in \Omega, \text{ all } 0 < \delta < 1 \text{ small enough}$$

which is a contradiction. Therefore $0 < \lambda^* \leq \hat{\lambda} < \infty$. □

According to this proposition, we have

$$(0, \lambda^*) \subseteq \mathfrak{L} \subseteq (0, \lambda^*]. \tag{3.51}$$

We will show that for all $\lambda \in (0, \lambda^*)$, we have at least two positive smooth solutions for problem (p_λ). To do this we need to strengthen a little the hypotheses on $f(z, \cdot)$. The new conditions on $f(z, x)$ are the following:

***H*$_3$**: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, hypotheses $H_3(i)$–(v) are the same as the corresponding hypotheses $H_2(i)$–(v) = $H_1(i)$–(v) and

(vi) for every $m > 0$, there exists $\eta_m > 0$ such that

$$f(z, x) \geq \eta_m > 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq m.$$

Proposition 3.11. *If hypotheses H_0, H_3 hold and $\lambda \in (0, \lambda^*)$, then problem (p_λ) admits at least two positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \neq \hat{u}$.*

Proof. Let $\eta \in (\lambda, \lambda^*)$. We have $\eta \in \mathfrak{L}$ (see (3.51)) and so we can find $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$. Then according to Proposition 3.5, we can find $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ such that

$$u_\eta - u_0 \in \operatorname{int} C_+. \tag{3.52}$$

Recall that $\bar{u}_\lambda \leq u_0$ (see Proposition 3.7).

Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_3(v)(H_2(v))$. We can assume that $\hat{\xi}_\rho > \|\xi\|_\infty$. Then we have

$$\begin{aligned}
&-\Delta_{p(z)} \bar{u}_\lambda - \Delta_{q(z)} \bar{u}_\lambda + [\xi(z) + \hat{\xi}_\rho] \bar{u}_\lambda^{p(z)-1} \\
&\quad \leq -\Delta_{p(z)} \bar{u}_\lambda - \Delta_{q(z)} \bar{u}_\lambda + [|\xi(z)| + \hat{\xi}_\rho] \bar{u}_\lambda^{p(z)-1} \\
&\quad = \lambda \bar{u}_\lambda^{\tau(z)-1} + \hat{\xi}_\rho \bar{u}_\lambda^{p(z)-1} \quad \text{(see Proposition 3.6)} \\
&\quad \leq \lambda u_0^{\tau(z)-1} + f(z, \bar{u}_\lambda) + \hat{\xi}_\rho u_0^{p(z)-1} \quad \text{(recall that } f \geq 0) \\
&\quad \leq \lambda u_0^{\tau(z)-1} + f(z, u_0) + \hat{\xi}_\rho u_0^{p(z)-1} \quad \text{(see Proposition 3.7 and hypothesis } H_3(v) = H_2(v)) \\
&\quad = -\Delta_{p(z)} u_0 - \Delta_{q(z)} u_0 + [\xi(z) + \hat{\xi}_\rho] u_0^{p(z)-1} \quad \text{(since } u_0 \in S_\lambda).
\end{aligned} \tag{3.53}$$

On account of hypothesis H_3(vi) and since $u_\lambda \in \operatorname{int} C_+$, we see that

$$0 \prec f(\cdot, \bar{u}_\lambda(\cdot)).$$

Then from (3.53) and Proposition 2.4 in [23] (see also [25], Proposition 7), we can conclude that

$$u_0 - \bar{u}_\lambda \in \text{int } C_+. \tag{3.54}$$

It follows from (3.52) and (3.54) that

$$u_0 \in \text{int}_{C^1(\bar{\Omega})}[\bar{u}_\lambda, u_\eta]. \tag{3.55}$$

As before, let $\theta > \|\xi\|_\infty$ and consider the Carathéodory function $k_\lambda(z,x)$ defined by

$$k_\lambda(z,x) = \begin{cases} \lambda\bar{u}_\lambda(z)^{\tau(z)-1} + f(z,\bar{u}_\lambda(z)) + \theta\bar{u}_\lambda(z)^{p(z)-1}, & \text{if } x < \bar{u}_\lambda(z) \\ \lambda x^{\tau(z)-1} + f(z,x) + \theta x^{p(z)-1}, & \text{if } \bar{u}_\lambda(z) \le x \le u_\eta(z) \\ \lambda u_\eta(z)^{\tau(z)-1} + f(z,u_\eta(z)) + \vartheta u_\eta(z)^{p(z)-1}, & \text{if } u_\eta(z) < x. \end{cases} \tag{3.56}$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)ds$ and consider the C^1-functional $\tau_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\tau_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{1}{p(z)}(\theta + \xi(z))|u|^{p(z)}dz$$
$$- \int_\Omega K_\lambda(z,u)dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma$$

for all $u \in W^{1,p(z)}(\Omega)$.

From (3.56) and since $\theta > \|\xi\|_\infty$, we infer that $\tau_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} \tau_\lambda(\tilde{u}_0) &= \min\{\tau_\lambda(u) : u \in W^{1,p(z)}(\Omega)\}, \\ &\Rightarrow \tau_\lambda'(\tilde{u}_0) = 0, \\ &\Rightarrow \langle \tau_\lambda'(\tilde{u}_0), h\rangle = 0 \quad \text{for all } h \in W^{1,p(z)}(\Omega). \end{aligned}$$

Choosing $h = (\bar{u}_\lambda - \tilde{u}_0)^+$ and $h = (\tilde{u}_0 - u_\eta)^+$ and using (3.56), we show as before that

$$\tilde{u}_0 \in [\bar{u}_\lambda, u_\eta] \cap \text{int}\, C_+.$$

Therefore, we may assume that $\tilde{u}_0 = u_0$ or otherwise, we already have a second positive smooth solution and so, we are done.

Next, we consider the Carathéodory function

$$\hat{k}_\lambda(z,x) = \begin{cases} \lambda\bar{u}_\lambda(z)^{\tau(z)-1} + f(z,\bar{u}_\lambda(z)) + \theta\bar{u}_\lambda(z)^{p(z)-1}, & \text{if } x \le \bar{u}_\lambda(z) \\ \lambda x^{\tau(z)-1} + f(z,x) + \vartheta x^{p(z)-1}, & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \tag{3.57}$$

We define $\hat{K}_\lambda(z,x) = \int_0^x \hat{k}_\lambda(z,s)ds$ and introduce the C^1-functional $\hat{\tau}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\tau}_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{1}{p(z)}(\theta + \xi(z))|u|^{p(z)}dz$$
$$- \int_\Omega \hat{K}_\lambda(z,u)dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma$$

for all $u \in W^{1,p(z)}(\Omega)$.

From (3.56) and (3.57), it is clear that

$$\tau_\lambda\big|_{[\bar{u}_\lambda,u_\eta]} = \hat{\tau}_\lambda\big|_{[\bar{u}_\lambda,u_\eta]}.$$

On account of (3.55), we have that u_0 is a local $C^1(\Omega)$-minimizer of $\hat{\tau}_\lambda$,

$$\Rightarrow u_0 \text{ is a local } W^{1,p(z)}(\Omega) - \text{minimizer of } \hat{\tau}_\lambda.$$

(see Gasiński and Papageorgiou [[13], Proposition 3.3])

Using (3.57), we can easily see that

$$K_{\hat{\tau}_\lambda} \subset [\bar{u}_\lambda) \cap \operatorname{int} C_+. \tag{3.58}$$

Then from (3.57) and the above, we can infer that we may assume that $K_{\hat{\tau}_\lambda}$ is finite or otherwise, we already have an infinity of positive smooth solutions all distinct from u_0 and so, we are done. According to Theorem 5.7.6 of Papageorgiou, Rădulescu, and Repovš [[22], p. 449], we can find $\rho \in (0,1)$ small such that

$$\hat{\tau}_\lambda(u_0) < \inf\{\hat{\tau}_\lambda(u) : \|u-u_0\| = \rho\} = \hat{m}_\rho. \tag{3.59}$$

On account of hypothesis H_3(ii) for $u \in \operatorname{int} C_+$, we have

$$\hat{\tau}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.60}$$

Finally, from (3.57), it follows that

$$\varphi_{\hat{\lambda}}\big|_{[\bar{u}_\lambda)} = \hat{\tau}_\lambda\big|_{[\bar{u}_\lambda)} + \hat{\eta} \quad \text{with } \hat{\eta} \in \mathbb{R},$$

$$\Rightarrow \hat{\tau}_\lambda(\cdot) \text{ satisfies the } C\text{-condition} \quad \text{(see Proposition 3.1)}. \tag{3.61}$$

Then (3.59), (3.60), (3.61) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W^{1,p(z)}(\Omega)$ such that

$$\hat{u} \in K_{\hat{\tau}_\lambda} \subset [\bar{u}_\lambda) \cap \operatorname{int} C_+ \quad \text{and} \quad \hat{m}_\rho \le \hat{\tau}_\lambda(\hat{u}). \tag{3.62}$$

From (3.62) and (3.57), we see that $\hat{u} \in S_\lambda \subseteq \operatorname{int} C_+$, while from (3.62) and (3.59), we have that $\hat{u} \neq u_0$. □

Finally, we show that the critical parameter value λ^* is admissible, that is, λ^*

Proposition 3.12. *If hypotheses H_0, H_1 hold, then $\lambda^* \in \mathfrak{L}$.*

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq \mathfrak{L}$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$. From the proof of Proposition 3.3, we know that we can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ such that $\varphi_{\hat{\lambda}_n}(u_n) < 0$ for all $n \in \mathbb{N}$.

Also, we have $\varphi'_{\hat{\lambda}_n}(u_n) = 0$, for all $n \in \mathbb{N}$. Then as in the proof of Proposition 3.1, we show that $\{u_n\}_{n\ge1} \subseteq W^{1,p(z)}(\Omega)$ is bounded.

We may assume that

$$u_n \rightharpoonup u^* \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u^* \quad \text{in } L^{r(z)}(\Omega) \quad \text{as } n \to \infty. \tag{3.63}$$

We have

$$\langle V(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p(z)-1} h\, dz + \int_{\partial\Omega} \beta(z) u_n^{p(z)-1} h d\sigma = \lambda_n \int_\Omega u_n^{\tau(z)-1} h\, dz + \int_\Omega f(z,u_n) h\, dz$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$.

Choosing $h = u_n - u^*$, passing to the limit as $n \to \infty$ and using (3.63), we obtain

$$u_n \to u^* \quad \text{in } W^{1,p(z)}(\Omega).$$

So, in the limit as $n \to \infty$, we have

$$\begin{aligned}\langle V(u^*), h\rangle + \int_\Omega \xi(z)(u^*)^{p(z)-1} h\, dz + \int_{\partial\Omega} \beta(z)(u^*)^{p(z)-1} h d\sigma \\ = \lambda^* \int_\Omega (u^*)^{q(z)-1} h\, dz + \int_\Omega f(z, u^*) h\, dz\end{aligned}$$

for all $h \in W^{1,p(z)}(\Omega)$.

We have $\bar{u}_{\lambda_1} \leq u_n$ for all $n \in \mathbb{N}$ (see the Remark 3.8),

$$\begin{aligned}&\Rightarrow \bar{u}_{\lambda_1} \leq u^*,\\ &\Rightarrow u^* \in S_{\lambda^*} \subseteq \operatorname{int} C_+ \quad \text{and so } \lambda^* \in \mathfrak{L}.\end{aligned}$$

The proof is now complete. □

Summarizing, we can state the following existence and multiplicity theorem for the problem (p_λ), which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem).

Theorem 3.13. *If hypotheses H_0, H_3, hold, then there exists $\lambda^* > 0$ such that*

(a) for all $\lambda \in (0, \lambda^*)$, problem ($p_\lambda$) has at least two positive solutions

$$u_0, \hat{u} \in \operatorname{int} C_+;$$

(b) for $\lambda = \lambda^*$, problem (p_λ) has at least one positive solution

$$u^* \in \operatorname{int} C_+;$$

(c) for all $\lambda > \lambda^*$, problem (p_λ) has no positive solutions;

(d) for every $\lambda \in \mathfrak{L} = (0, \lambda^*]$, problem ($p_\lambda$) has a smallest positive solution $u^*_\lambda \in \operatorname{int} C_+$ and the map $\lambda \mapsto u^*_\lambda$ from $\mathfrak{L} = (0, \lambda^*]$ into $C_+ \setminus \{0\}$ is increasing, that is,

$$0 < \mu \leq \lambda \in \mathfrak{L} \Rightarrow u^*_\lambda - u^*_\mu \in C_+ \setminus \{0\}.$$

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