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# Multiple normalized solutions of a nonlinear Schrödinger–Poisson system with $L^2$-subcritical growth

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**Abstract.** In this paper, we study the existence of multiple normalized solutions to the following Schrödinger–Poisson system with general nonlinearities:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = \varepsilon^3 a^2, \end{cases}$$

where $\varepsilon$, $a > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $V(x) : \mathbb{R}^3 \to [0, \infty)$ is a continuous function, and $f$ is a differentiable function satisfying $L^2$-subcritical growth. Through using the minimization techniques and the Lusternik–Schnirelmann category, we prove that the numbers of normalized solutions are related to the topology of the set where the potential $V(x)$ attains its minimum value.

**Keywords:** Schrödinger–Poisson system, normalized solutions, Lusternik–Schnirelmann category, variational methods.

**2020 Mathematics Subject Classification:** 35B09, 35J61.

## 1 Introduction

In this paper, we are concerned with the existence of multiple normalized solutions to the following Schrödinger–Poisson system with general nonlinearities:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = \varepsilon^3 a^2, \end{cases} \tag{1.1}$$

where $\varepsilon$, $a > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier.

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Problem (1.1) arises in the study of the coupled Schrödinger–Poisson system:

$$\begin{cases} i\psi_t - \Delta\psi + V(x)\psi + \phi\psi = g(|\psi|^2)\psi & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |\psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\psi(x,t) : \mathbb{R}^3 \times [0,T]$ is the wave function. Equation (1.2) arises from approximation of the Hartree–Fock equation which describes a quantum mechanical of many particles, see [11,12,24]. Set $\psi(x,t) = e^{i\lambda t}u(x)$ and $u : \mathbb{R}^3 \to \mathbb{R}$, one is led to the equation

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $f(u) = g(|u|^2)u$, $\lambda \in \mathbb{R}$. The system (1.1) was firstly introduced by Benci and Fortunato in [9]. System (1.1) also arises in various fields of physics, for instance, in semiconductor theory (see [10,25,26]), for more details on the physical aspects, we refer the reader to [9] and references therein.

When $\lambda \in \mathbb{R}$ is a fixed parameter, we call (1.1) the fixed frequency problem. In the last decades, the existence, concentration and multiplicity of solutions for the fixed frequency problem (1.1) has been studied by many scholars, for example [2–4, 13, 15, 27, 29] and the references therein.

Recently, the existence and multiplicity of normalized solution are attracted many people's interests. Such solutions have a prescribed $L^2$-norm, that is, solutions which satisfy $\|u\|_2 = a$ for a priori given $a > 0$. In this case, the parameter $\lambda \in \mathbb{R}$ cannot be fixed but instead appears as a Lagrange multiplier.

When $\varepsilon = 1$, $V(x) = 0$ and $f(u) = |u|^{p-2}u$, normalized solutions of (1.1) can be obtained by considering the critical points of the following functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dy dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$$

on the constraint

$$S(a) = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\}.$$

As far as we know, the first work for normalized solutions to Schrödinger–Poisson system is due to Sánchez and Soler [28], they proved that all the minimizing sequence for $\sigma_a$ are compact provided that $a \in (0, a_0)$ for a suitable $a_0 > 0$ small enough and $p = \frac{8}{3}$, where $\sigma_a$ is defined by

$$\sigma_a = \inf_{u \in S(a)} J(u). \tag{1.3}$$

Bellazzini and Siciliano in [5] and [6] proved that $\sigma_a$ is achieved when $a > 0$ is small and $p \in (2,3)$ and when $a > 0$ is large and $p \in (3, \frac{10}{3})$, respectively. Subsequently, Jeanjean and Luo in [22] sharpened the conclusion of [6] by showing that (1.3) has a minimizer if and only if

$$a \geq a_1 = \inf\{a > 0 : \sigma_a < 0\}.$$

Moreover, for the case of $p = 3$ or $p = \frac{10}{3}$, they proved $\sigma_a$ has no minimizer for any $a > 0$.

For the $L^2$-supercritical case, that is, $p \in (\frac{10}{3}, 6)$, the functional $J(u)$ is no more bounded from below on $S(a)$. Bellazzini, Jeanjean and Luo in [7] found critical points of $J(u)$ on $S(a)$ by looking at the mountain-pass level for $a > 0$ sufficiently small. In 2021, Jeanjean and Le in

[21] obtained the existence of two positive solutions for (1.1) with $f(u) = |u|^{p-2}u$ which can be characterized respectively as a local minima and as a mountain pass critical point when $p \in (\frac{10}{3}, 6]$. For the general nonlinearity, Chen, Tang and Yuan in [16] studied the existence of normalized solutions for system (1.1), where $f \in C(\mathbb{R}, \mathbb{R})$ covers the case $f(u) = |u|^{p-2}u$ with $q \in (2, \frac{10}{3}) \cup (\frac{10}{3}, 6)$. When considering more general $L^2$-supercritical conditions without imposing the monotonicity property on $f$, Hu, Tang and Jin [20] obtained the existence of normalized solutions for problem (1.1) under suitable assumptions on $f$.

For the case combining nonlinearity, Kang, Li and Tang in [23] considered system (1.1) with $f(u) = \mu|u|^{q-2}u + |u|^{p-2}u$, where $\mu \in \mathbb{R}$, $2 < q \leq \frac{10}{3} \leq p < 6$ with $q \neq p$. Under some suitable assumptions on $s$ and $\mu$, they proved some existence, nonexistence and multiplicity of normalized solutions.

When $\varepsilon = 1$ and $V(x) \neq 0$, it is more complicated to deal with the existence of normalized solutions. Zeng and Zhang in [32] considered system (1.1) with $f(u) = |u|^p u$ $(0 < p < \frac{4}{3})$ and unbounded potential, where the potential function $V(x)$ satisfies the following conditions

$$V \in C(\mathbb{R}^N, \mathbb{R}^+), \quad \inf_{x \in \mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x| \to +\infty} V(x) = \infty,$$

with the help of the compactness of Sobolev embedding in the working space, they obtained the existence of normalized solutions.

To our best of knowledge, there is few results for the existence of multiple normalized solutions to Schrödinger–Poisson system (1.1). Motivated by [1], the main purpose of this paper is to study the existence of multiple normalized solutions to (1.1) by using the Lusternik–Schnirelmann category when $V(x)$ satisfies the global conditions:

$$(V) \qquad V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N), \qquad V(0) = 0, \qquad 0 = \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to +\infty} V(x) = V_\infty.$$

By change of variable $x \to \varepsilon x$, problem (1.1) reduces to the following system

$$\begin{cases} -\Delta u + V(\varepsilon x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = a^2. \end{cases} \tag{1.4}$$

We assume that $V(x)$ satisfies $(V)$ and $f$ satisfies the following assumptions:

$(f_1)$ $f$ is odd and there exist $q \in (3, \frac{10}{3})$ and $\alpha \in (0, +\infty)$ such that $\lim_{s \to 0} \frac{|f(s)|}{|s|^{q-1}} = \alpha$.

$(f_2)$ There exist constants $c_1, c_2, c_3, c_4 > 0$ and $p \in (3, \frac{10}{3})$ such that

$$|f(s)| \leq c_1 + c_2|s|^{p-1} \quad \forall s \in \mathbb{R} \quad \text{and} \quad |f'(s)| \leq c_3 + c_4|s|^{p-2} \quad \forall s \in \mathbb{R}.$$

$(f_3)$ There exists $q_1 \in (3, \frac{10}{3})$ and $q > q_1$ such that $f(s)/s^{q_1-1}$ is an increasing function of $s$ on $(0, +\infty)$.

**Remark 1.1.** The conditions $(f_1)$ and $(f_3)$ imply that $F(t) \geq 0$ for all $t \in \mathbb{R}$. Indeed,

$$\frac{f(s)}{s^{q_1-1}} \geq \lim_{s \to 0^+} \frac{f(s)}{s^{q-1}} s^{q-q_1} = 0, \qquad s > 0,$$

that is, $f(s) \geq 0$. Hence, $F(t) \geq 0$.

An example of a function $f$ that satisfies the above assumption is

$$f(s) = |s|^{q-2}s + |s|^{r-2}s\ln(1 + |s|) \qquad \forall s \in \mathbb{R},$$

for some $r, q \in (3, \frac{10}{3})$ and $r > q$, here $(f_2)$ and $(f_3)$ hold with $p \in (r, \frac{10}{3})$.

A solution $u$ to the problem (1.4) with $\int_{\mathbb{R}^3} |u|^2 = a^2$ can be obtained by looking for critical points of the following functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x, \qquad u \in H^1(\mathbb{R}^3),$$

restricted to sphere

$$S(a) = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\},$$

where $\|\cdot\|_p$ denotes the usual norm in $L^p(\mathbb{R}^3)$ for $p \in [1, +\infty)$.

Moreover, it is easy to see that $J_\varepsilon \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and

$$J'_\varepsilon(u)v = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(\varepsilon x)uv)\mathrm{d}x + \int_{\mathbb{R}^3} \phi_u uv \mathrm{d}x - \int_{\mathbb{R}^3} f(u)v\mathrm{d}x, \qquad \forall v \in H^1(\mathbb{R}^3).$$

When we study the multiplicity of solutions in the nonautonomous case, we need to use the following sets:

$$M = \{x \in \mathbb{R}^3 : V(x) = 0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \le \delta\}, \qquad \delta > 0.$$

Now, we state our main result as follows.

**Theorem 1.2.** *Suppose that $f$ satisfies the conditions $(f_1)$–$(f_3)$ and that $V$ satisfies $(V)$. Then for each $\delta > 0$, there exist $\varepsilon_0$, $\mu_* > 0$ and $a_* > 0$ such that (1.1) admits at least $\mathrm{cat}_{M_\delta}(M)$ couples $(u_j, \lambda_j) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions for $0 < \varepsilon < \varepsilon_0$, $|V|_\infty < \mu_*$ and $a > a_*$ with $\int_{\mathbb{R}^3} |u_j|^2 \mathrm{d}x = a^2$ and $J_\varepsilon(u_j) < 0$.*

**Remark 1.3.** For $V(0) =: V_0 \neq 0$, $V_0 < V_\infty$, we can also obtain Theorem 1.2.

We recall that, if $Y$ is a closed subset of a topological space $X$, the Lusternik–Schnirelmann category $cat_X(Y)$ is the least number of closed and contractible sets in $X$ which cover $Y$. If $X = Y$, we use the notation $\mathrm{cat}(X)$. For more details about this subject, we cite [30].

The organization of this paper is as follows. In Section 2, we study the autonomous problem. In Section 3, we study the nonautonomous case. In this section, we also study the Palais–Smale condition on the sphere $S(a)$ for the energy functional and provide some crucial tools to establish a multiplicity result. In Section 4, we prove the multiplicity and concentration of solutions to problem (1.1).

## 2 The autonomous case

The following classical Gagliardo–Nirenberg inequality is so crucial in this paper, which can be found in [31]. Precisely, let $l \in [2, 6)$, then

$$|u|_l^l \le C|u|_2^{(1-\beta_l)l} |\nabla u|_2^{\beta_l l} \quad \text{in } \mathbb{R}^3, \qquad \beta_l = \frac{3(l-2)}{2l}, \tag{2.1}$$

for some positive constant $C = C(3, l) > 0$.

In this section, we list some preliminary lemmas which used later involving the existence of normalized solution for the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \mu u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = a^2, \end{cases} \tag{2.2}$$

where $a > 0$, $\mu \geq 0$, and $\lambda \in \mathbb{R}$ is unknown parameter that appears as a Lagrange multiplier and $f$ is a continuous function satisfying $(f_1)$–$(f_3)$.

A solution $u$ to the problem (2.2) corresponds to a critical point of the $C^1$ functional

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + \mu u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\mathrm{d}x, \qquad u \in H^1(\mathbb{R}^3),$$

on the constraint $S(a)$ given by

$$S(a) = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\}.$$

Our main result in this section is stated as follows.

**Theorem 2.1.** *Suppose that $f$ satisfies the conditions $(f_1)$–$(f_3)$. Then, there exists $\mu_* > 0$ and $a_* > 0$ such that problem (2.2) has a couple $(u, \lambda)$ solution when $0 \leq \mu < \mu_*$ and $a > a_*$, where $u$ is positive.*

The proof of the theorem above will be divided into several lemmas.

Now, we recall some properties of the functions $\phi_u$ in the following lemma (for a proof see [27], [18] and [17]).

**Lemma 2.2.** *The following results hold:*

*(1)* $\phi_u \geq 0$;

*(2) there exist some constants* $C_1, C_2 > 0$ *such that* $\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x \leq C_1|u|^4_{\frac{12}{5}} \leq C_2\|u\|^4$;

*(3) if* $u_n \to u$ *in* $L^t(\mathbb{R}^3)$, $\forall t \in [2, 6)$, *then* $\int_{\mathbb{R}^3}\phi_{u_n}u_n{}^2\mathrm{d}x \to \int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x$,

*where* $\|\cdot\|$ *denotes the usual norm in* $H^1(\mathbb{R}^3)$.

Define $N$: $H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x.$$

**Lemma 2.3** ([33, Lemma 2.2])**.** *Let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then as $n \to \infty$,*

*(1)* $N(u_n - u) = N(u_n) - N(u) + o(1)$;

*(2)* $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $(H^1(\mathbb{R}^3))'$.

**Lemma 2.4.** *The functional $I_\mu$ is coercive and bounded from below in $S(a)$.*

*Proof.* According to $(f_1)$–$(f_2)$, there is $C_1$, $C_2 > 0$ such that

$$|F(t)| \leq C_1|t|^q + C_2|t|^p \qquad \forall t \in \mathbb{R}.$$

Then it follows from (2.1) that

$$I_\mu(u) \geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x - CC_1 a^{(1-\beta_q)q}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^{\frac{\beta_q q}{2}}$$
$$- CC_2 a^{(1-\beta_p)p}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^{\frac{\beta_p p}{2}}.$$

Since $q, p \in (2, \frac{10}{3})$, by simple calculation, we get $\beta_q q$, $\beta_p p < 2$, which ensures the coercivity and boundedness of $I_\mu$ from below. □

Lemma 2.4 guarantees that the minimization problem

$$I_{\mu,a} = \inf_{u \in S(a)} I_\mu(u)$$

is well defined. In what follows, we are going to establish some properties of $I_\mu$ related to the parameter $\mu \geq 0$.

**Lemma 2.5.** *There exists $\mu_* > 0$, $a_* > 0$ such that $\mathcal{I}_{\mu,a} < 0$ for $0 \leq \mu < \mu_*$ and $a > a_*$.*

*Proof.* By assumption $(f_3)$, we have

$$f'(t)t - (q_1 - 1)f(t) \geq 0 \qquad \forall t > 0. \tag{2.3}$$

In order to show $t \mapsto \frac{F(t)}{t^{q_1}}$ is increasing on $(0, +\infty)$, we need to prove

$$\frac{d}{dt}\frac{F(t)}{t^{q_1}} = \frac{f(t)t^{q_1} - q_1 F(t)t^{q_1-1}}{t^{2q_1}} = \frac{f(t)t - q_1 F(t)}{t^{q_1+1}} \qquad \forall t > 0.$$

Define $h(t) = f(t)t - q_1 F(t)$, clearly, $h(0) = 0$ and (2.3) yields that

$$h'(t) = f'(t)t - (q_1 - 1)f(t) \geq 0 \qquad \forall t > 0,$$

which implies that

$$h(t) = f(t)t - q_1 F(t) \geq 0. \tag{2.4}$$

This leads to $\frac{\mathrm{d}}{\mathrm{dt}}\frac{F(t)}{t^{q_1}} \geq 0$, that is, the function $t \mapsto \frac{F(t)}{t^{q_1}}$ is increasing on $(0, +\infty)$, thus, we have that

$$\frac{F(ts)}{(ts)^{q_1}} \geq \frac{F(s)}{s^{q_1}} \qquad \forall s > 0 \quad \text{and} \quad t \geq 1,$$

which yields that

$$F(ts) \geq t^{q_1} F(s) \qquad \forall s > 0 \quad \text{and} \quad t \geq 1. \tag{2.5}$$

Given $u_0(x) \in S(a) \cap L^\infty(\mathbb{R}^3)$ a nonnegative function, let

$$u_0^\eta(x) = \eta^2 u_0(\eta x) \quad \text{for all } x \in \mathbb{R}^3 \text{ and all } \eta \in \mathbb{R}. \tag{2.6}$$

By simple computation, we have

$$\int_{\mathbb{R}^3} |u_0^\eta(x)|^2 \mathrm{d}x = \eta a^2,$$

that is, $u_0^\eta(x) \in S(\eta^{\frac{1}{2}} a)$. Therefore,

$$I_\mu(u_0^\eta(x)) \leq \frac{\eta^3}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x + \frac{\mu\eta a^2}{2} + \frac{\eta^3}{4}\int_{\mathbb{R}^3} \phi_{u_0} u_0^2 \mathrm{d}x - \eta^{2q_1-3}\int_{\mathbb{R}^3} F(u_0(x))\mathrm{d}x.$$

When $q_1 \in (3, \frac{10}{3})$ and $\eta > 0$, $2q_1 - 3 > 3$, for $|\eta|$ large, we deduce that

$$\frac{\eta^3}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x + \frac{\eta^3}{4}\int_{\mathbb{R}^3} \phi_{u_0} u_0^2 \mathrm{d}x - \eta^{2q_1-3}\int_{\mathbb{R}^3} F(u_0(x))\mathrm{d}x = A_\eta < 0,$$

thus, we obtain that

$$I_\mu(u_0^\eta(x)) \leq A_\eta + \frac{\mu a^2}{2}.$$

Hence, we fix $\mu_* > 0$ such that

$$I_\mu(u_0^\eta(x)) < 0, \qquad \forall \mu \in [0, \mu_*),$$

showing that $\mathcal{I}_{\mu, t^{\frac{1}{2}}a} < 0$. Thus, for $a$ large enough, $\mathcal{I}_{\mu,a} < 0$. □

**Lemma 2.6.** *Fix $\mu \in [0, \mu_*)$ and let $a_* < a_1 < a_2$. There holds $\frac{a_1^6}{a_2^6}\mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1} < 0$.*

*Proof.* Let $\xi > 1$ such that $a_2 = \xi a_1$ and $\{u_n\} \subset S(a_1)$ be a nonnegative minimizing sequence with respect to the $\mathcal{I}_{\mu,a_1}$ (because $I_\mu(u) = I_\mu(|u|)$ for all $u \in H^1(\mathbb{R}^3)$), that is,

$$I_\mu(u_n) \to \mathcal{I}_{\mu,a_1} \quad \text{as } n \to +\infty.$$

Set $v_n = \xi^4 u_n(\xi^2 x)$. Obviously $v_n \in S(a_2)$ and

$$\begin{aligned}\mathcal{I}_{\mu,a_2} &\le I_\mu(v_n)\\ &= \frac{\xi^6}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x + \frac{\xi^2\mu^2}{2}\int_{\mathbb{R}^3}u_n^2\mathrm{d}x + \frac{\xi^6}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x - \xi^{-6}\int_{\mathbb{R}^3}F(\xi^4 u_n(x))\mathrm{d}x\\ &\le \xi^6\left[\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x + \frac{\mu^2}{2}\int_{\mathbb{R}^3}u_n^2\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x\right] - \xi^{-6}\int_{\mathbb{R}^3}F(\xi^4 u_n(x))\mathrm{d}x.\end{aligned}$$

By (2.5), we deduce that

$$\mathcal{I}_{\mu,a_2} \le I_\mu(v_n) = \xi^6 I_\mu(u_n) + (\xi^6 - \xi^{4q_1-6})\int_{\mathbb{R}^3}F(u_n(x))\mathrm{d}x.$$

**Claim 2.7.** There exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x \ge C \quad \text{for } n \ge n_0.$$

Arguing by contradiction that there exists a subsequence of $\{u_n\}$, still denoted by itself, such that

$$\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

Thus, we have

$$0 > \mathcal{I}_{\mu,a} + o_n(1) = I_\mu(u_n) \ge -\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x,$$

which is absurd. Thus, Claim 2.7 holds. It is easy to verify that $\xi^6 - \xi^{4q_1-6} < 0$. Hence, we have

$$\mathcal{I}_{\mu,a_2} \le \xi^6 I_\mu(u_n) + (\xi^6 - \xi^{4q_1-6})C.$$

As $n \to +\infty$, we get

$$\mathcal{I}_{\mu,a_2} < \xi^6\mathcal{I}_{\mu,a_1} + (\xi^6 - \xi^{4q_1-6})C < \xi^6\mathcal{I}_{\mu,a_1},$$

that is,

$$\frac{a_1^6}{a_2^6}\mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1}.$$

□

The following theorem is a compactness theorem on $S(a)$, which is crucial to study the autonomous case and the nonautonomous case.

**Theorem 2.8.** *Let $\mu \in [0, \mu_*)$, $a > a_*$ and $\{u_n\} \subset S(a)$ be a minimizing sequence with respect to $I_\mu$. Then, for some subsequence either*

*(i) $\{u_n\}$ is strongly convergent in $H^1(\mathbb{R}^3)$;*

*or*

*(ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \to +\infty$ such that $v_n(x) = u_n(x + y_n) \to v$ in $H^1(\mathbb{R}^3)$, where $v \in S(a)$ and $I_\mu(v) = \mathcal{I}_{\mu,a}$.*

*Proof.* Since $\mathcal{I}_\mu$ is coercive on $S(a)$, the sequence $\{u_n\}$ is bounded. Hence, up to a subsequence, still denoted by $u_n$, we may assume that there exists some $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

If $u \neq 0$ and $|u|_2 = b \neq a$, we have that $b \in (0, a)$. It follows from Brézis–Lieb lemma (see [30]) that:

$$|u_n|_2^2 = |u_n - u|_2^2 + |u|_2^2 + o_n(1).$$

Moreover, setting $v_n = u_n - u$, $d_n = |v_n|_2$, $t \in (0,1)$, by using mean value theorem, $(f_1)$, $(f_2)$, and Young's inequality, we get

$$\begin{aligned} |F(v_n + u) - F(v_n) - F(u)| &\leq |F(v_n + u) - F(v_n)| + |F(u)| \\ &\leq |f(v_n + tu)||u| + |F(u)| \\ &\leq [C_1|v_n + tu|^{q-1} + C_2|v_n + tu|^{p-1}]|u| + C_1|u|^{q-1} + C_2|u|^{p-1} \\ &\leq C(|v_n|^{q-1} + |u|^{q-1} + |v_n|^{p-1} + |u|^{p-1})|u| + C_1|u|^{q-1} + C_2|u|^{p-1} \\ &\leq C\varepsilon(|v_n|^q + |v_n|^p) + (C\varepsilon^{-(q-1)} + C_1)|u|^q + (C\varepsilon^{-(p-1)} + C_2)|u|^p. \end{aligned}$$

Since $\lim_{n\to+\infty} |F(v_n + u) - F(v_n) - F(u)| = 0$ a.e. in $\mathbb{R}^3$, by Lebesgue dominated convergence Theorem, it is easy to get that

$$\int_{\mathbb{R}^3} F(v_n + u)\mathrm{d}x = \int_{\mathbb{R}^3} F(v_n)\mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x + o_n(1),$$

that is,

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(u_n - u)\mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x + o_n(1). \tag{2.7}$$

Suppose that $|v_n|_2 \to d$, then $a^2 = b^2 + d^2$ and $d_n \in (0, a)$ for $n$ large enough. Thus, by Lemma 2.3 and (2.7), we have that

$$\mathcal{I}_{\mu,a} + o_n(1) = I_\mu(u_n) = I_\mu(v_n) + I_\mu(u) + o_n(1) \geq \mathcal{I}_{\mu,d_n} + \mathcal{I}_{\mu,b} + o_n(1).$$

From Lemma 2.6, it follows that

$$\mathcal{I}_{\mu,a} + o_n(1) \geq \frac{d_n^6}{a^6}\mathcal{I}_{\mu,a} + \mathcal{I}_{\mu,b} + o_n(1).$$

As $n \to +\infty$, we arrive at the inequality

$$\mathcal{I}_{\mu,a} \geq \frac{d^6}{a^6}\mathcal{I}_{\mu,a} + \mathcal{I}_{\mu,b}. \tag{2.8}$$

Since $b \in (0, a)$, by Lemma 2.6 and (2.8), we obtain

$$0 > \mathcal{I}_{\mu,a} > \frac{d^6}{a^6}\mathcal{I}_{\mu,a} + \frac{b^6}{a^6}\mathcal{I}_{\mu,a} = \frac{b^6 + d^6}{a^6}\mathcal{I}_{\mu,a},$$

which yields that

$$\frac{b^6 + d^6}{a^6} > 1.$$

By using $a^2 = b^2 + d^2$, we deduce that

$$b^6 + d^6 > a^6 = (b^2 + d^2)^3 = b^6 + d^6 + 3b^2d^4 + 3b^4d^2,$$

which is absurd. Hence, we infer that $|u|_2 = a$, that is, $u \in S(a)$.

As $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$, it is easy to verify that

$$u_n \to u \quad \text{in } L^2(\mathbb{R}^3). \tag{2.9}$$

By (2.9) and interpolation theorem in the Lebesgue spaces, one infers that

$$u_n \to u \quad \text{in } L^t(\mathbb{R}^3), \qquad \forall t \in [2,6),$$

which combines with $(f_1)$–$(f_2)$, we can deduce that

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^3} F(u)\mathrm{d}x. \tag{2.10}$$

Thus, by Lemma 2.2-(3) and $\mathcal{I}_{\mu,a} = \lim_{n\to\infty} I_\mu(u_n)$, we have that $\mathcal{I}_{\mu,a} \geq I_\mu(u)$. Since $u \in S(a)$, it follows that $\mathcal{I}_{\mu,a} = I_\mu(u)$, and then $\lim_{n\to\infty} I_\mu(u_n) = \mathcal{I}_\mu(u)$, which combines with (2.9), (2.10) and Lemma 2.2-(3), we have that $u_n \to u$ in $D^{1,2}(\mathbb{R}^3)$. From (2.9), it follows that $\|u_n\|^2 \to \|u\|^2$, that is, $u_n \to u$ in $H^1(\mathbb{R}^3)$.

If $u = 0$, then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Similar to Claim 2.7, we prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \geq C \quad \text{for } n \in \mathbb{N} \text{ large}. \tag{2.11}$$

Next, we prove that there exist $R$, $\beta > 0$ and $y_n \in \mathbb{R}^3$ such that

$$\int_{B_R(y_n)} |u_n|^2\mathrm{d}x \geq \beta \qquad \forall n \in \mathbb{N}. \tag{2.12}$$

Suppose on the contrary, by Lions' vanishing lemma, we get that $u_n \to 0$ in $L^t(\mathbb{R}^3)$ for all $t \in (2, 2^*)$. Hence, it is easy to check that $F(u_n) \to 0$ in $L^1(\mathbb{R}^3)$, which is contradict with (2.11).

Since $u = 0$, we claim that $\{y_n\}$ is unbounded. Arguing by contradiction that $\{y_n\}$ is bounded, there exists $R_0 > 0$, such that $|y_n| < R_0$. Hence, $B_R(y_n) \subset B_{R+R_0}(0)$. Thus, we have that

$$\int_{B_R(y_n)} |u_n|^2\mathrm{d}x \leq \int_{B_{R+R_0}(0)} |u_n|^2\mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

which is contradiction with (2.12). The claim follows.

Setting $\widetilde{u}_n(x) = u(x + y_n)$, clearly $\{\widetilde{u}_n\} \subset S(a)$ and it is also a minimizing sequence with respect to $\mathcal{I}_{\mu,a}$, up to a subsequence, we may assume that there exists $\widetilde{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in } H^1(\mathbb{R}^3) \quad \text{and} \quad \widetilde{u}_n(x) \to \widetilde{u}(x) \quad \text{a.e. in } \mathbb{R}^3.$$

Similarly arguing as the above proof, we can deduce that $\widetilde{u}_n \to \widetilde{u}$ in $H^1(\mathbb{R}^3)$. This completes the proof. □

## 2.1 Proof of Theorem 2.1

From Lemma 2.4, there exists a bounded minimizing sequence $\{u_n\} \subset S(a)$ with respect to $\mathcal{I}_{\mu,a}$, that is, $I_\mu(u_n) \to \mathcal{I}_{\mu,a}$. By Theorem 2.8, there exists $u \in S(a)$ with $I_\mu(u) = \mathcal{I}_{\mu,a}$. Hence, by the Lagrange multiplier, there exists $\lambda_a \in \mathbb{R}$ such that

$${I_\mu}'(u) = \lambda_a \Psi'(u) \quad \text{in } (H^1(\mathbb{R}^3))', \tag{2.13}$$

where $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ is given by

$$\Psi(u) = \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x, \qquad u \in H^1(\mathbb{R}^3).$$

By (2.13), we have

$$-\Delta u + \mu u + \phi u = \lambda_a u + f(u) \quad \text{in } \mathbb{R}^3. \tag{2.14}$$

Next, by simple calculation, it is easy to see that $I_\mu(|u|) = I_\mu(u)$. Besides, since $u \in S(a)$ implies that $|u| \in S(a)$, then the following equality holds:

$$\mathcal{I}_{\mu,a} = I_\mu(u) = I_\mu(|u|) \geq \mathcal{I}_{\mu,a},$$

thus, $I_\mu(|u|) = \mathcal{I}_{\mu,a}$. Then we can replace $u$ by $|u|$, thus we may assume that $u \geq 0$, by standard argument, we can prove that $u(x) > 0$ in $\mathbb{R}^3$.

By Theorem 2.1, it is easy to conclude the following corollary.

**Corollary 2.9.** *Fix $a > a^*$ and let $0 \leq \mu_1 < \mu_2 \leq \mu_*$. There holds $\mathcal{I}_{\mu_1,a} < \mathcal{I}_{\mu_2,a} < 0$.*

*Proof.* Let $u_{\mu_2,a} \in S(a)$ satisfying $I_{\mu_2}(u_{\mu_2,a}) = \mathcal{I}_{\mu_2,a}$. It is easy to infer that

$$\mathcal{I}_{\mu_1,a} \leq I_{\mu_1}(u_{\mu_2,a}) < I_{\mu_2}(u_{\mu_2,a}) = \mathcal{I}_{\mu_2,a}.$$

□

## 3 The nonautonomous case

In this section, we will study the nonautonomous case of the Schrödinger–Poisson system (1.4). Hereafter, we will suppose that $|V|_\infty < \mu_*$ and $a > a_*$, where $\mu_*$ and $a_*$ was given in section 2. In order to prove some properties of the functional $J_\varepsilon$, we give several useful definitions. We define $J_0$, $J_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ by the following functionals:

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x$$

and

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty |u|^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x.$$

Furthermore, we denote $\Upsilon_{\varepsilon,a}$, $\Upsilon_{0,a}$ and $\Upsilon_{\infty,a}$:

$$\Upsilon_{\varepsilon,a} = \inf_{u \in S(a)} J_\varepsilon(u), \qquad \Upsilon_{0,a} = \inf_{u \in S(a)} J_0(u), \qquad \Upsilon_{\infty,a} = \inf_{u \in S(a)} J_\infty(u).$$

Since $0 < V_\infty < +\infty$, we deduce from Corollary 2.9 that

$$\Upsilon_{0,a} < \Upsilon_{\infty,a} < 0. \tag{3.1}$$

In the following, we set $0 < \rho_1 = \frac{1}{2}(\Upsilon_{\infty,a} - \Upsilon_{0,a})$.

The following lemma establishes some essential relations involving the levels $\Upsilon_{\varepsilon,a}$, $\Upsilon_{0,a}$ and $\Upsilon_{\infty,a}$.

**Lemma 3.1.** $\limsup_{\varepsilon \to 0^+} \Upsilon_{\varepsilon,a} \leq \Upsilon_{0,a}$ *and there exists $\varepsilon_0 > 0$ such that $\Upsilon_{\varepsilon,a} < \Upsilon_{\infty,a}$ for all $\varepsilon \in (0, \varepsilon_0)$.*

*Proof.* Let $u_0 \in S(a)$ with $J_0(u_0) = \Upsilon_{0,a}$, we have that

$$\Upsilon_{\varepsilon,a} \le J_\varepsilon(u_0) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_0|^2 + V(\varepsilon x)|u_0|^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_0}u_0^2\mathrm{d}x - \int_{\mathbb{R}^3}F(u_0)\mathrm{d}x.$$

As $\varepsilon \to 0^+$, we arrive at the inequality

$$\limsup_{\varepsilon\to 0^+}\Upsilon_{\varepsilon,a} \le \lim_{\varepsilon\to 0^+} J_\varepsilon(u_0) = J_0(u_0) = \Upsilon_{0,a}.$$

By (3.1) and the above inequality, we can obtain that $\Upsilon_{\varepsilon,a} < \Upsilon_{\infty,a}$ for $\varepsilon$ small enough. □

**Lemma 3.2.** *Fix $\varepsilon \in (0,\varepsilon_0)$ and let $\{u_n\} \subset S(a)$ such that $J_\varepsilon(u_n) \to c$ with $c < \Upsilon_{0,a} + \rho_1 < 0$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $u \neq 0$.*

*Proof.* We argue by contradiction that $u = 0$. From the definition of $J_\varepsilon(u_n)$ and $J_\infty(u_n)$, it follows that

$$\Upsilon_{0,a} + \rho_1 + o_n(1) > c + o_n(1) = J_\varepsilon(u_n) = J_\infty(u_n) + \frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon x) - V_\infty)|u_n|^2\mathrm{d}x.$$

From $(V)$, for any given $\zeta > 0$, there exists $R > 0$ such that

$$V(x) \ge V_\infty - \zeta, \qquad \forall\, |x| \ge R.$$

Thus, there holds

$$\Upsilon_{0,a} + \rho_1 + o_n(1) > J_\varepsilon(u_n) \ge J_\infty(u_n) + \frac{1}{2}\int_{B_{R/\varepsilon}(0)}(V(\varepsilon x) - V_\infty)|u_n|^2\mathrm{d}x - \frac{\zeta}{2}\int_{B^c_{R/\varepsilon}(0)}|u_n|^2\mathrm{d}x.$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \to 0$ in $L^l(B_{R/\varepsilon}(0))$ for all $l \in [1, 2^*)$, we obtain

$$\Upsilon_{0,a} + \rho_1 + o_n(1) \ge J_\infty(u_n) - \zeta C \ge \Upsilon_{\infty,a} - \zeta C$$

for some $C > 0$. Because $\zeta > 0$ is arbitrary, it follows that

$$\Upsilon_{0,a} + \rho_1 \ge \Upsilon_{\infty,a},$$

which is contradict with the definition of $\rho_1$. The proof is completed. □

**Lemma 3.3.** *Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence for $J_\varepsilon$ restricted to $S(a)$ with $c < \Upsilon_{0,a} + \rho_1 < 0$ and $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, that is,*

$$J_\varepsilon(u_n) \to c \quad \text{as } n \to +\infty \quad \text{and} \quad \|J_\varepsilon|'_{S(a)}(u_n)\| \to 0 \quad \text{as } n \to +\infty.$$

*If $v_n = u_n - u_\varepsilon \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, then there exists $\beta > 0$, such that*

$$\liminf_{n\to+\infty}|u_n - u_\varepsilon|_2^2 \ge \beta.$$

*Proof.* Let the functional $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ be given by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x,$$

we have that $S(a) = \Psi^{-1}(\{\frac{a^2}{2}\})$. Then, by Proposition 5.12 in [30], we see that

$$\|J_\varepsilon|'_{S(a)}(u_n)\| = \min_{\lambda\in\mathbb{R}}\|J_\varepsilon{}'(u_n) - \lambda\Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'},$$

thus, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty.$$

Since

$$\|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} = \sup_{v \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\langle J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n), v\rangle}{\|v\|} \to 0 \quad \text{as } n \to +\infty.$$

In view of the boundedness of $\{u_n\}$, we can deduce that

$$\frac{\langle J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n), u_n\rangle}{\|u_n\|} \leq \|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty,$$

which leads to

$$\lambda_n a^2 = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon x) u_n^2) \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x - \int_{\mathbb{R}^3} f(u_n) u_n \mathrm{d}x + o_n(1). \tag{3.2}$$

From the boundedness of $\{u_n\} \in H^1(\mathbb{R}^3)$, it follows that $\{\lambda_n\}$ is also a bounded sequence, up to a subsequence, we may assume that $\lambda_n \to \lambda_\varepsilon$ as $n \to +\infty$. Hence, we have that

$$\|J_\varepsilon{}'(u_n) - \lambda_\varepsilon \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \leq \|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} + |\lambda_n - \lambda_\varepsilon| \|\Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'},$$

which combing with $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, we can deduce that

$$J_\varepsilon{}'(u_\varepsilon) - \lambda_\varepsilon \Psi'(u_\varepsilon) = 0 \quad \text{in } (H^1(\mathbb{R}^3))'.$$

By using Lemma 2.3, we can prove that

$$J_\varepsilon{}'(u_n) = J_\varepsilon{}'(u_\varepsilon) + J_\varepsilon{}'(v_n) + o_n(1),$$

and

$$\Psi'(u_n) = \Psi'(u_\varepsilon) + \Psi'(v_n) + o_n(1).$$

Hence, we have

$$J_\varepsilon{}'(u_n) - \lambda_\varepsilon \Psi'(u_n) = J_\varepsilon{}'(v_n) - \lambda_\varepsilon \Psi'(v_n) + o_n(1),$$

and so

$$\|J_\varepsilon{}'(v_n) - \lambda_\varepsilon \Psi'(v_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty,$$

which implies that

$$\int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(\varepsilon x)|v_n|^2) \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{v_n} {v_n}^2 \mathrm{d}x - \lambda_\varepsilon \int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(v_n) v_n \mathrm{d}x + o_n(1).$$

Suppose on the contrary that $|v_n|_2 \to 0$, by interpolation inequality, one infers that

$$v_n \to 0 \quad \text{in } L^t(\mathbb{R}^3), \qquad \forall t \in [2, 6). \tag{3.3}$$

By $(f_1)$, $(f_2)$ and (3.3), we deduce that

$$\int_{\mathbb{R}^3} f(v_n) v_n \mathrm{d}x \leq \int_{\mathbb{R}^3} C_1 |v_n|^p + C_2 |v_n|^q \mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

and

$$\int_{\mathbb{R}^3} \phi_{v_n} {v_n}^2 \mathrm{d}x \leq |v_n|^4_{\frac{12}{5}} \to 0 \quad \text{as } n \to +\infty,$$

and

$$\int_{\mathbb{R}^3} V(\varepsilon x)|v_n|^2 dx \le \int_{\mathbb{R}^3} \mu_* |v_n|^2 dx \to 0 \quad \text{as } n \to +\infty.$$

Hence, we have that

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to 0 \quad \text{as } n \to +\infty,$$

which leads to $\|v_n\|_{H^1(\mathbb{R}^3)} \to 0$, which gives a contradiction by $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^3)$. Therefore, there exists $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\liminf_{n\to+\infty} |u_n - u_\varepsilon|_2^2 \ge \beta. \qquad \square$$

In what follows, we set

$$0 < \rho < \min\left\{\frac{1}{2}, \frac{\beta^3}{a^6}\right\}(\Upsilon_{\infty,a} - \Upsilon_{0,a}) \le \rho_1. \tag{3.4}$$

**Lemma 3.4.** *For each $\varepsilon \in (0, \varepsilon_0)$, the functional $J_\varepsilon$ satisfies the $(PS)_c$ condition restricted to $S(a)$ for $c < \Upsilon_{0,a} + \rho$.*

*Proof.* Let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\varepsilon$ restricted to $S(a)$ with $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$ and $c < \Upsilon_{0,a} + \rho$. Then, by Proposition 5.12 in [30], there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$\|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty.$$

By Lemma 3.3, if $v_n = u_n - u_\varepsilon \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, there exists $\beta > 0$ independent of $\varepsilon$ such that

$$\liminf_{n\to+\infty} |v_n|_2^2 \ge \beta.$$

Let $d_n = |v_n|_2$ satisfying that $|v_n|_2 \to d > 0$ and $|u_\varepsilon|_2 = b$, by Brézis-Lieb lemma, we obtain $a^2 = b^2 + d^2$. By Lemma 3.2, we have $b > 0$ and in its proof it was proved that $J_\varepsilon(v_n) \ge \Upsilon_{\infty,d_n} + o_n(1)$, we must have $d_n \in (0, a)$ for $n$ large enough, and so

$$c + o_n(1) = J_\varepsilon(u_n) = J_\varepsilon(v_n) + J_\varepsilon(u_\varepsilon) + o_n(1) \ge \Upsilon_{\infty,d_n} + \Upsilon_{0,b} + o_n(1).$$

By Lemma 2.6, we infer that

$$\rho + \Upsilon_{0,a} > \frac{d_n^6}{a^6}\Upsilon_{\infty,a} + \frac{b^6}{a^6}\Upsilon_{0,a}.$$

As $n \to +\infty$, using $a^2 = b^2 + d^2$, we arrive at the inequality

$$\rho > \frac{d^6}{a^6}\Upsilon_{\infty,a} + \frac{b^6 - a^6}{a^6}\Upsilon_{0,a} > \frac{d^6}{a^6}(\Upsilon_{\infty,a} - \Upsilon_{0,a}) + \frac{3a^2d^4 - 3a^4d^2}{a^6}\Upsilon_{0,a} > \frac{\beta^3}{a^6}(\Upsilon_{\infty,a} - \Upsilon_{0,a}),$$

which is contradict with (3.4). Thus, $v_n \to 0$ in $H^1(\mathbb{R}^3)$, that is, $u_n \to u_\varepsilon$ in $H^1(\mathbb{R}^3)$, which implies that $|u_\varepsilon|_2 = a$ and

$$-\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon + \phi u_\varepsilon = \lambda_\varepsilon u_\varepsilon + f(u_\varepsilon) \quad \text{in } \mathbb{R}^3,$$

where $\lambda_\varepsilon$ is the limit of some subsequence of $\{\lambda_n\}$. $\square$

## 4 Multiplicity result

Let $\delta > 0$ be fixed and $w$ be a positive solution of the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

with $J_0(w) = \Upsilon_{0,a}$. Let $\eta$ be a smooth nonincreasing cut-off function satisfying

$$\eta(s) = \begin{cases} 1, & 0 \le s \le \frac{\delta}{2}, \\ 0, & s \ge \delta. \end{cases}$$

For any $y \in M$, let us define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right), \qquad \widetilde{\Psi}_{\varepsilon,y}(x) = a \frac{\Psi_{\varepsilon,y}(x)}{|\Psi_{\varepsilon,y}|_2},$$

and denote $\Phi_\varepsilon$: $M \to S(a)$ by $\Phi_\varepsilon(y) = \widetilde{\Psi}_{\varepsilon,y}$. Clearly, $\Phi_\varepsilon(y)$ has a compact support for any $y \in M$.

**Lemma 4.1** (See [14, Chapter II, 3.2]). *Let $I$ be a $C^1$-functional defined on $C^1$-Finsler manifold $\mathcal{V}$. If $I$ is bounded from below and satisfies the $(PS)$ condition, the $I$ has at least $\operatorname{cat}_{\mathcal{V}}(\mathcal{V})$ distinct critical points.*

**Lemma 4.2** (See [8, Lemma 4.3]). *Let $\Gamma$, $\Omega^+$, $\Omega^-$ be closed sets with $\Omega^- \subset \Omega^+$. Let $\Phi : \Omega^- \to \Gamma$, $\beta : \Gamma \to \Omega^+$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\mathrm{Id} : \Omega^- \to \Omega^+$. Then $\operatorname{cat}(\Gamma) \ge \operatorname{cat}_{\Omega^+}(\Omega^-)$.*

**Lemma 4.3.** *The function $\Phi_\varepsilon$ has the following property:*

$$\lim_{\varepsilon \to 0} J_\varepsilon(\Phi_\varepsilon(y)) = \Upsilon_{0,a}, \quad \textit{uniformly in } y \in M.$$

*Proof.* To prove this lemma, we argue by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset M$, $\{y_n\}$ is a bounded sequence and $\varepsilon_n \to 0$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - \Upsilon_{0,a}| \ge \delta_0, \qquad \forall n \in \mathbb{N}.$$

Since

$$|\eta(\varepsilon_n z) w(z)|^2 \to |w(z)|^2 \quad \text{a.e. in } \mathbb{R}^3 \text{ as } n \to +\infty,$$

and

$$|\eta(\varepsilon_n z) w(z)|^2 \le |w(z)|^2,$$

by Lebesgue's dominated convergence theorem, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}^3} |\Psi_{\varepsilon_n,y_n}|^2 \mathrm{d}x = \lim_{n \to +\infty} \int_{\mathbb{R}^3} |\eta(\varepsilon_n z) w(z)|^2 \mathrm{d}z = \int_{\mathbb{R}^3} |w|^2 \mathrm{d}z = a^2.$$

Then, there exists $N > 0$ such that

$$|\Psi_{\varepsilon_n,y_n}|_2^2 \ge \frac{a^2}{2}, \qquad \forall n > N.$$

Setting $|\Psi_{\varepsilon_n,y_n}|_2^2 \geq C = \min\{\frac{a^2}{2}, |\Psi_{\varepsilon_1,y_1}|_2^2, |\Psi_{\varepsilon_2,y_2}|_2^2, \dots, |\Psi_{\varepsilon_N,y_N}|_2^2\}$.

Since

$$\lim_{n\to+\infty} F(\Phi_{\varepsilon_n}(y_n)) = \lim_{n\to+\infty} F\left(a\frac{\eta(\varepsilon_n z)w(z)}{|\eta(\varepsilon_n z)w(z)|_2}\right) = F(w) \quad \text{a.e. in } \mathbb{R}^3,$$

and by $(f_1)$ and $(f_2)$, we have that

$$|F(\Phi_{\varepsilon_n}(y_n))| = \left|F\left(a\frac{\eta(\varepsilon_n z)w(z)}{|\eta(\varepsilon_n z)w(z)|_2}\right)\right| \leq C_1|w(z)|^p + C_2|w(z)|^q,$$

thus, by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} F(\Phi_{\varepsilon_n}(y_n))\mathrm{d}x = \lim_{n\to+\infty}\int_{\mathbb{R}^3} F\left(a\frac{\eta(\varepsilon_n z)w(z)}{|\eta(\varepsilon_n z)w(z)|_2}\right)\mathrm{d}z = \int_{\mathbb{R}^3} F(w)\mathrm{d}z.$$

For almost every $z \in \mathbb{R}^3$, we deduce that

$$\begin{aligned}
&\lim_{n\to+\infty} |\nabla\Phi_{\varepsilon_n}(y_n)|^2 \\
&= \lim_{n\to+\infty} \frac{a^2}{|\Psi_{\varepsilon_n,y_n}|_2^2}|\nabla(\eta(\varepsilon_n z)w(z))|^2 \\
&= \lim_{n\to+\infty} |\nabla(\eta(\varepsilon_n z))w(z) + \eta(\varepsilon_n z)\nabla w(z)|^2 \\
&= \lim_{n\to+\infty} [\varepsilon_n^2|\nabla(\eta(\varepsilon_n z))w(z)|^2 + |\eta(\varepsilon_n z)\nabla w(z)|^2 + 2\varepsilon_n\eta(\varepsilon_n z)\nabla(\eta(\varepsilon_n z))w(z)\nabla w(z)] \\
&= \lim_{n\to+\infty} |\nabla w(z)|^2
\end{aligned}$$

and

$$\begin{aligned}
|\nabla\Phi_{\varepsilon_n}(y_n)|^2 &\leq \frac{a^2}{C}[\varepsilon_n^2|\nabla(\eta(\varepsilon_n z))w(z)|^2 + |\eta(\varepsilon_n z)\nabla w(z)|^2 + 2\varepsilon_n\eta(\varepsilon_n z)\nabla(\eta(\varepsilon_n z))w(z)\nabla w(z)] \\
&\leq \frac{a^2}{C}[C_3\varepsilon_n^2|w(z)|^2 + |\nabla w(z)|^2 + C_4\varepsilon_n^2|w(z)|^2|\nabla w(z)|^2],
\end{aligned}$$

by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} |\nabla\Phi_{\varepsilon_n}(y_n)|^2\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla w|^2\mathrm{d}z.$$

Since

$$\lim_{n\to+\infty} V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2 = \lim_{n\to+\infty} \frac{a^2V(\varepsilon_n z + y_n)}{|\Psi_{\varepsilon_n,y_n}|_2^2}|\eta(\varepsilon_n z)w(z)|^2 = 0 \quad \text{a.e. in } \mathbb{R}^3,$$

and

$$V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2 = \frac{a^2V(\varepsilon_n z + y_n)}{|\Psi_{\varepsilon_n,y_n}|_2^2}|\eta(\varepsilon_n z)w(z)|^2 \leq \frac{a^2}{C}\mu_* W(z)^2,$$

by Lebesgue's dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2\mathrm{d}x = 0.$$

Since

$$\begin{aligned}
&\lim_{n\to+\infty} \phi_{\Phi_{\varepsilon_n}(y_n)}\Phi_{\varepsilon_n}(y_n)^2 \\
&= \lim_{n\to+\infty} \frac{\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n z)w(z)\right|^2\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n r)w(r)\right|^2}{|z-r|} = \phi_w w^2 \quad \text{a.e. in } \mathbb{R}^3,
\end{aligned}$$

and by Lemma 2.2-(2), we have that

$$\phi_{\Phi_{\varepsilon_n}(y_n)}\Phi_{\varepsilon_n}(y_n)^2 = \frac{\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n z)w(z)\right|^2\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n r)w(r)\right|^2}{|z-r|}$$
$$\leq \frac{a^4}{C^4}\frac{|w(z)|^2\,|w(r)|^2}{|z-r|} \leq C_5\phi_w w^2,$$

by Lebesgue's dominated convergence theorem, there holds

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{\Phi_{\varepsilon_n}(y_n)}\Phi_{\varepsilon_n}(y_n)^2\mathrm{d}x$$
$$= \lim_{n\to+\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n z)w(z)\right|^2\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n r)w(r)\right|^2}{|z-r|}\mathrm{d}z\mathrm{d}r$$
$$= \int_{\mathbb{R}^3}\phi_w w^2\mathrm{d}z.$$

Consequently,

$$\lim_{n\to+\infty}J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = J_{0,a}(w) = \Upsilon_{0,a},$$

which is absurd. Hence, we complete the proof. □

For any $\delta > 0$, let $R = R(\delta) > 0$ be such that $M_\delta \subset B_R(0)$. Let $\chi$: $\mathbb{R}^3 \to \mathbb{R}^3$ denote by $\chi(x) = x$ for $|x| \leq R$ and $\chi(x) = \frac{Rx}{|x|}$ for $|x| \geq R$. Hereafter, we are going to consider $\beta_\varepsilon$: $S(a) \to \mathbb{R}^3$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)|u|^2\mathrm{d}x}{a^2}.$$

**Lemma 4.4.** *The function $\Phi_\varepsilon$ has the following property:*

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \textit{uniformly in } y \in M.$$

*Proof.* Suppose on the contrary that there exist $\delta_0 > 0$, $\{y_n\} \subset M$, and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0, \qquad \forall n \in \mathbb{N}. \tag{4.1}$$

By the definition of $\Phi_{\varepsilon_n}(y_n)$ and $\beta_{\varepsilon_n}$, we have that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_n z + y_n) - y_n)|\eta(\varepsilon_n z)w(z)|^2\mathrm{d}z}{|\Psi_{\varepsilon_n,y_n}(\frac{\varepsilon_n + y_n}{\varepsilon_n})|_2^2}.$$

Since $(y_n) \subset M \subset B_R(0)$,

$$\frac{(\chi(\varepsilon_n z + y_n) - y_n)|\eta(\varepsilon_n z)w(z)|^2}{|\Psi_{\varepsilon_n,y_n}(\frac{\varepsilon_n + y_n}{\varepsilon_n})|_2^2} \to 0 \quad \text{a.e. in } \mathbb{R}^3,$$

and

$$\frac{(\chi(\varepsilon_n z + y_n) - y_n)|\eta(\varepsilon_n z)w(z)|^2}{|\Psi_{\varepsilon_n,y_n}(\frac{\varepsilon_n + y_n}{\varepsilon_n})|_2^2} \leq \frac{2R}{C}|w(z)|^2,$$

by Lebesgue's dominated convergence theorem, we deduce that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \to 0, \quad \text{as } n \to +\infty,$$

which attains a contradiction with (4.1). Hence, we complete the proof. □

**Proposition 4.5.** *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset S(a)$ with $J_\varepsilon(u_n) \to \Upsilon_{0,a}$. Then, there is $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $v_n(x) = u_n(x + \widetilde{y}_n)$ has a strongly convergent subsequence in $H^1(\mathbb{R}^3)$. Moreover, up to a subsequence, $y_n = \varepsilon_n \widetilde{y}_n \to y$ in $\mathbb{R}^3$ for some $y \in M$.*

*Proof.* Firstly, we claim that there exist $R_0$, $\tau > 0$ and $\widetilde{y}_n \in \mathbb{R}^3$ such that

$$\int_{B_{R_0}(\widetilde{y}_n)} |u_n|^2 \mathrm{d}x \geq \tau \qquad \forall n \in \mathbb{N}.$$

Otherwise, owing to Lions' vanishing lemma, we have that $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for all $p \in (2, 2^*)$, which implies that $\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to 0$. Thus, $\lim_{n\to+\infty} J_{\varepsilon_n}(u_n) \geq 0$, which contradicts with $\lim_{n\to+\infty} J_{\varepsilon_n}(u_n) = \Upsilon_{0,a} < 0$.

Considering $v_n(x) = u_n(x + \widetilde{y}_n)$, up to a subsequence, we may assume that there exists $v \in H^1(\mathbb{R}^3) \setminus \{0\}$ satisfying $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Since $\{v_n\} \subset S(a)$ and $J_{\varepsilon_n}(u_n) \geq J_0(u_n) = J_0(v_n) \geq \Upsilon_{0,a}$, there holds that $J_0(v_n) \to \Upsilon_{0,a}$. By Theorem 2.8, $v_n \to v$ in $H^1(\mathbb{R}^3)$ and $v \in S(a)$.

In what follows, we are to prove that $\{y_n\}$ is bounded. Arguing by contradiction that for some subsequence $|y_n| \to +\infty$, the limit

$$\begin{aligned}
\Upsilon_{0,a} &= \lim_{n\to+\infty} J_{\varepsilon_n}(u_n) \\
&= \lim_{n\to+\infty} \left( \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(\varepsilon_n x + y_n)|v_n|^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{v_n} {v_n}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v_n)\mathrm{d}x \right) \\
&\geq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v|^2 + V_\infty |v|^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_v v^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v)\mathrm{d}x \\
&\geq \Upsilon_{\infty,a},
\end{aligned}$$

this gives a contradiction due to (3.1). Therefore, we can suppose that $y_n \to y$ in $\mathbb{R}^3$. Similarly discussed as above, we obtain

$$\Upsilon_{0,a} \geq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v|^2 + V(y)|v|^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_v v^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v)\mathrm{d}x \geq \Upsilon_{V(y),a}.$$

By Corollary 2.9, we know that $\Upsilon_{V(y),a} > \Upsilon_{0,a}$ as $V(y) > 0$. Since $V(y) \geq 0$ for all $y \in \mathbb{R}^3$, the above inequality implies that $V(y) = 0$, that is, $y \in M$. □

Let $h$: $[0, +\infty) \to [0, +\infty)$ be a function such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and set

$$\widetilde{S}(a) = \{u \in S(a) : J_\varepsilon(u) \leq \Upsilon_{0,a} + h(\varepsilon)\}. \tag{4.2}$$

In view of Lemma 4.3, the function $h(\varepsilon) = \sup_{y\in M} |J_\varepsilon(\Phi_\varepsilon(y)) - \Upsilon_{0,a}|$ satisfies that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $\Phi_\varepsilon(y) \in \widetilde{S}(a)$ for all $y \in M$.

**Lemma 4.6.** *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \leq \delta\}$. There holds*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{S}(a)} \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| = 0.$$

*Proof.* Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{S}(a)$ such that

$$\inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| = \sup_{u_n\in\widetilde{S}(a)} \inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| + o_n(1).$$

According to the above equality, it is sufficient to find a sequence $\{y_n\} \subset M_\delta$ such that

$$\lim_{n\to+\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Since $u_n \in \widetilde{S}(a)$, we obtain

$$\Upsilon_{0,a} \le J_0(u_n) \le J_{\varepsilon_n}(u_n) \le \Upsilon_{0,a} + h(\varepsilon_n) \qquad \forall n \in \mathbb{N},$$

and so,

$$u_n \in S(a) \quad \text{and} \quad J_{\varepsilon_n}(u_n) \to \Upsilon_{0,a}.$$

From Proposition 4.5, it follows that there exists $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \varepsilon_n \widetilde{y}_n \to y$ for some $y \in M$ and $v_n(x) = u_n(x + \widetilde{y}_n)$ is strongly convergent to some $v \in H^1(\mathbb{R}^3)$ with $v \neq 0$. Then, $\{y_n\} \subset M_\delta$ for $n$ large enough and

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + y_n) - y_n)|v_n|^2 \mathrm{d}z}{a^2},$$

which implies that

$$\beta_{\varepsilon_n}(u_n) - y_n = \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + y_n) - y_n)|v_n|^2 \mathrm{d}z}{a^2} \to 0 \quad \text{as } n \to +\infty.$$

The proof is completed. □

### 4.1 Proof of Theorem 1.2.

In what follows, let $\varepsilon \in (0, \varepsilon_0)$. By Lemma 4.3, for any $y \in M$, we have

$$J_\varepsilon(\Phi_\varepsilon(y)) \le \Upsilon_{0,a} + h(\varepsilon), \qquad h(\varepsilon) \to 0 \, (\varepsilon \to 0),$$

which implies that $\Phi_\varepsilon(M) \subset \widetilde{S}(a)$. By Lemma 4.6, we obtain

$$\mathrm{dist}(\beta_\varepsilon(u), M_\delta) \le \delta, \qquad \forall u \in \widetilde{S}(a),$$

which leads to $\beta_\varepsilon(\widetilde{S}(a)) \subset M_\delta$. Hence, we have that $\beta_\varepsilon \circ \Phi_\varepsilon(M) \subset M_\delta$. We define $id : M \to M_\delta$. Hereafter, let us define $W : [0,1] \times M \to M_\delta$

$$W(t, y) = t\beta_\varepsilon \circ \Phi_\varepsilon + (1-t)\,\mathrm{id}(y) \qquad t \in [0,1],$$

satisfying $W(0, y) = \mathrm{id}(y)$, $W(1, y) = \beta_\varepsilon \circ \Phi_\varepsilon$, we can conclude that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion map $\mathrm{id} : M \to M_\delta$. By Lemma 4.2, it follows that

$$\mathrm{cat}(\widetilde{S}(a)) \ge \mathrm{cat}_{M_\delta}(M).$$

Arguing as Lemma 2.4, we also have that $J_\varepsilon$ is bounded from below on $S(a)$. From Lemma 3.4, we have that the functional $J_\varepsilon$ satisfies the $(PS)_c$ condition for the $c \in (\Upsilon_{0,a}, \Upsilon_{0,a} + h(\varepsilon))$. By Lemma 4.1, there exists at least $\mathrm{cat}(S(a))$ critical points of $J_\varepsilon$ restricted to $S(a)$. Since $\widetilde{S}(a) \subset S(a)$, $\mathrm{cat}(\widetilde{S}(a)) \le \mathrm{cat}(S(a))$. Then, by the Lusternik–Schnirelmann category theory (see [19] and Theorem 5.20 of [30]), we have that $J_\varepsilon$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $S(a)$.

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