Electronic Journal of Qualitative Theory of Differential Equations
2025, No. 6, 1–36; https://doi.org/10.14232/ejqtde.2025.1.6 www.math.u-szeged.hu/ejqtde/

Critical points of locally Lipschitz functionals inside and outside the ordered interval

Xian Xu[1] and **Baoxia Qin**[✉][2]

[1]Department of Mathematics, Jiangsu Normal University, Xuzhou, Jiangsu, 221116, P. R. China
[2]School of Mathematics, Qilu Normal University, Jinan, Shandong, 250200, P. R. China

Received 25 June 2024, appeared 19 February 2025
Communicated by Roberto Livrea

Abstract. In this paper, under the condition that there exists an ordered interval composed of two internal ordered intervals which have the location similar to that of Amann's three-solution theorem, we add some simple conditions and then we obtain some results about the existence of multiple critical points inside and outside the ordered interval. The main results of this paper can be regarded as an extension of the classical Amann three-solution theorem and the mountain pass lemma on the ordered interval of Shujie Li and Zhiqiang Wang. To show our main results, we extend the method of invariant sets of descending flow that proposed by Jingxian Sun for smooth functionals to the locally Lipschitz functionals. Our main results can be applied to the study of differential inclusion problems with concave-convex nonlinearity. In this way, we partially extend some relevant results concerning the differential equation boundary value problems with a concave-convex nonlinearity that was first studied by A. Ambrosetti, H. Brezis and G. Cerami.

Keywords: locally Lipschitz functionals, critical points, differential inclusion problems.

2020 Mathematics Subject Classification: 49J52, 49J35, 58E05.

1 Introduction

In the study of nonlinear functional, ordered intervals are often used to study the existence of critical points or fixed points. It is well known that under simpler conditions, one can get the result that there is at least one critical point or fixed point in the ordered interval. If the ordered interval has a finer internal structure, one can often get multiple fixed points or critical points. A famous result of this is Amann's three-solution theorem [1]. In the case of the ordered interval composed of two pairs of upper and lower solutions, Amann used the fixed point index method to prove that there are at least three fixed points in the ordered interval. Later, Li and Wang [20] generalize this result to smooth functionals using the method of invariant set of descending flow, and establish the mountain pass theorem in the ordered interval.

✉Corresponding author. Email: qinbaoxia@126.com

In [20], as an application of their mountain pass theorem in the ordered interval, Li and Wang studied the multiple solutions of the boundary value problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1$_\lambda$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\lambda > 0$ is a parameter, $1 < q < 2$ and $g(u)$ is a superlinear term. The nonlinearity of (1.1_λ) may involve a combination of concave and convex terms. This type of nonlinearity has been studied for a long times and has been extensively studied by many authors; See [2, 10, 16, 20, 28] and the references therein. A. Ambrosetti, H. Brezis, G. Cerami in their well known paper [2] firstly considered the boundary value problem with a concave-convex terms nonlinearity

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2$_\lambda$}$$

where $0 < q < 1 < p$ and $\lambda > 0$ be a parameter. They fund a $\Lambda > 0$ such that (1.2_λ) has two positive solutions for all $\lambda \in (0, \Lambda)$, at least one positive solution for $\lambda = \Lambda$ and has no positive for all $\lambda > \Lambda$. Li and Wang [20] fund a $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$, (1.1_λ) has at least two positive solutions, at least two negative solutions and at least two sign-changing solutions. Their proof method is to combine the mountain pass lemma in the ordered interval with the invariant set method of descending flow. They first established two pairs of upper and lower solutions which have the location similar to those of H. Amann's three solution theorem. Then, by using the mountain pass theorem in the ordered interval they obtained the result of at least one positive critical point, one negative critical point and one sign-changing critical point inside the ordered interval, and by using the invariant set method of descending flow obtained the result of at least one positive critical point, one negative critical point and one sign-changing critical point outside the ordered interval.

Inspired by Li and Wang's idea of proof, in the present paper we will establish some results about multiple critical points for locally Lipschitz functionals inside and outside the ordered interval composed of two internal ordered intervals, which have the location similar to that of Amann's three-solution theorem. In the last 40 years, critical point theories and their applications for locally Lipschitz functionals have been extensively studied by many peoples; see [5–8, 12, 13, 17, 18, 23–25, 27] . Our main results in this paper can be thought as an extension of Amann's three solutions theorem and Li and Wang's mountain pass theorem in ordered intervals. By adding some simple conditions, we obtain the existence of critical points both inside and outside the ordered interval. Also, in this paper we introduce an ordering in the Banach space and give the locations of theses critical points.

To show our main results, we will employ the invariant set method of descending flow proposed by Sun [32]. For theories of invariant set of descending flow of C^1 functionals, one can also refer to [22]. According to this method, finding different critical points may be attributed to finding different invariant sets of descending flow. So far, it has been widely used to study the existence of solutions of various of elliptic equation boundary value problems. However, to our best knowledge, there are no one study the existence of critical points for locally Lipschitz functionals by using this method. There are two main difficulties to use the method of invariant sets of descending flow finding critical points of locally Lipschitz functionals. As is well known, to use the method of invariant sets of descending flow for C^1 functionals one need first construct a pseudogradient vector field over the Banach space. However, for the

Lipschitz functional case, one usually only can construct locally a pseudogradient vector field over a subset of the Banach space rather than the whole Banach space. This is the first difficulty need to overcome. In the course of the study, one need to determine whether a closed convex set is an invariant set of the descending flow generated by the pseudogradient vector field. In the cases of the functionals being of C^1, one can use the Schauder invariance condition presented in the literatures [22,32,33]. However, in the cases of the functionals being of locally Lipschitz, no one has yet used the Schauder invariance condition to establish invariant flows on convex closed sets. This is the second difficulty we face.

In this paper, we will analyze the energy of the possible critical points of the locally Lipschitz functionals in advance, construct locally a pseudogradient vector field on a neighborhood of the energy, and then extend this pseudogradient vector field to the entire Banach space. We can then use the method of invariant sets of descending flow that Sun have proposed to obtain the existence results for the critical points of the locally Lipschitz functionals. By this way we overcome the first difficulty. For the second difficulty, based on the conclusion in [25] about the relationship between the critical points on the whole space and the critical points on the closed convex sets under the Schauder invariance condition, and using the Von Neumann theorem to establish the descending flow on closed convex sets, we get the result that some closed convex sets are descending flow invariant sets. In this paper we get the theoretical results for the existence of at least one sign-changing, at least one nontrivial, at least two positive and at least two negative critical points of the locally Lipschitz functionals considered. The theoretical results can be applied to the study of the existence of sign-changing solutions, positive solutions and negative solutions for differential inclusion problems with a concave-convex nonlinearity terms. In this way, we extend the relevant results on concave-convex nonlinearity terms in literatures.

The critical points of this paper are in the sense of Chang [6]. The study of critical points of non-smooth functional has been greatly developed in the last thirty years. Some peoples have studied the critical point of the Motreanu–Panagiotopopoulos functionals [26,29,34]. This kind of functional has the form: $f := h + \psi$, with h a locally Lipschitz functional and ψ a covex,proper and l.s.c. functional. The Motreanu–Panagiotopopoulos critical point theory contain as particular cases both the the critical point theory in the sense of Chang as well as in the sense of Szulkin [34]. Obviously, how to study the critical point theory of the Motreanu–Panagiotopopoulos functionals in Banach space by using the invariant set method of descending flow is an interesting problem worthy of further study.

2 Preliminaries

In what follows we will let X and E be two real Banach spaces with the norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively. Assume that X is reflexive, E is densely and continuously embedded in X. Let X^* be the topological dual of X and $\langle\cdot,\cdot\rangle$ denote the duality pairing between X^* and X. Let P be a cone of X, that is, P is closed convex set in X, $\lambda x \in P$ for all $x \in P$ and $\lambda \geqslant 0$, and $P \cap (-P) = \{0\}$. The P is said to be generating if $X = P - P$. Let $P_1 = P \cap E$. Then P_1 is a cone in E. We assume that P_1 has a nonempty interior in the E topology, and denote its interior in the E topology by $\operatorname{int} P_1$. For each $x, y \in X$, let us define the ordering $\leqslant$ in X by

$$x \leqslant y \quad \text{if and only if} \quad y - x \in P.$$

For each $x, y \in E$, if $y - x \in \operatorname{int} P_1$, we write $x \ll y$. For $x \in X$ and $A \subset X$, let $\operatorname{dist}(x, A) = \inf_{y \in A} \|x - y\|$. For any $R > 0$, let $B(0, R) = \{x \in X : \|x\| < R\}$ and $S_R = \{x \in X : \|x\| = R\}$.

Given $B \subset A \subset E$, we write $\partial_A B$ for the boundary of B in A and $\mathrm{int}_A B$ for the interior of B in A.

Let us recall some theories concerning the sub-differential theory of locally Lipschitz functionals due to Clarke [7]. A functional $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, there exists a neighborhood U of x and a constant $k > 0$ depending on U such that $|\varphi(z) - \varphi(y)| \leqslant k\|z - y\|$ for all $z, y \in U$. For such a functional we define generalized directional derivative $\varphi^0(x; h)$ at $x \in X$ in the direction $h \in X$ by

$$\varphi^0(x;h) = \limsup_{x' \to x\ \lambda \downarrow 0^+} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \mapsto \varphi^0(x; h)$ is sublinear, continuous. So by the Hahn–Banach theorem we know that $\varphi^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leqslant \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is called the generalized or Clarke sub-differential of φ at x. A point $x \in X$ is a critical point of φ if $0 \in \partial\varphi(x)$. Let $\mathbb{K} = \{x \in X : 0 \in \partial\varphi(x)\}$.

Proposition 2.1 ([5, 12]). *1) If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz functionals, then $\partial(\varphi + \psi)(x) \subset \partial\varphi(x) + \partial\psi(x)$, while for any $\lambda \in \mathbb{R}$ we have $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$; 2) If $\varphi : X \to \mathbb{R}$ is also convex, then this sub-differential coincides with the sub-differential in the sense of convex analysis. If φ is strictly differentiable, then $\partial\varphi(x) = \{\varphi'(x)\}$; 3) If $\varphi : X \to \mathbb{R}$ is locally Lipschitz functional, $\partial\varphi(u)$ is a weakly*-compact subset of X^* which is bounded by the Lipschitz constant $k > 0$ of φ near u.*

S. T. Kyritsi and N. S. Papageorgiou [18] developed a critical point theory for non-smooth locally Lipschitz functionals defined on a closed, convex set extending this way the work of Struwe. Let $C \subset X$ be a nonempty, non-singleton, closed and convex set. For $x \in C$ we define

$$m_C(x) = \inf_{x^*} \sup_{y} \big\{ \langle x^*, x - y\rangle : y \in C, \|x - y\| < 1, x^* \in \partial\varphi(x) \big\}.$$

Evidently, $m_C(x) \geqslant 0$ for all $x \in C$. This quantity can be viewed as a measure of the generalized slope of φ at $x \in C$. If φ admits an extension $\hat{\varphi} \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$ and so we have

$$m_C(x) = \sup \big\{ \langle \varphi'(x), x - y\rangle : y \in C, \|x - y\| < 1 \big\},$$

which is the quantity used by Struwe [31, p. 147]. Also if $C = X$, then we have

$$m_C(x) = m(x) = \inf\{\|x^*\|_* : x^* \in \partial\varphi(x)\},$$

which is the quantity used by Chang [6].

Now let us introduce the outwardly directed condition and the Schauder invariance condition for set value mappings in a manner as in [25]. Let X be reflexive. As usual, we will identify X^{**} with X while $\mathsf{F} : X^* \mapsto 2^X$ will denote the duality map, given by

$$\mathsf{F}(x^*) := \{x \in X : \langle x^*, x\rangle = \|x^*\|_*^2 = \|x\|^2\}, \qquad \forall x^* \in X^*.$$

The set $\mathsf{F}(x^*)$ turns out to be nonempty, convex, and closed; see, e.g. [13, pp. 311–319]. Define

$$\nabla\varphi(x) := \mathsf{F}(\partial\varphi(x)), \qquad x \in X. \tag{2.1}$$

Clearly, $\nabla\varphi(x)$ depends on the choice of the duality pairing between X and X^* whenever it is compatible with the topology of X. If X is a Hilbert space, the duality paring becomes the scalar product and (2.1) gives the usual gradient. Write I for the identity operator on X.

Suppose C is a convex and closed set of X. Let $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of C, namely

$$\delta_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then we have

$$\partial\delta_C(x) = \big\{x^* \in X^* : \langle x^*, z - x\rangle \leqslant 0, \forall z \in C\big\}$$

The set $\partial\delta_C(x)$ is usually called normal cone to C at x.

Definition 2.2. Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous. If $(-\partial\delta_C(x)) \cap \partial\varphi(x) \subset \{0\}$ for any $x \in \partial C$ when $\operatorname{int} C \neq \emptyset$, or for any $x \in C$ when $\operatorname{int} C = \emptyset$, then we say that $\partial\varphi$ turns out to be outwardly directed on C. This clearly rewrites as

$$\forall z^* \in \partial\varphi(x)\backslash\{0\} \quad \text{there exists} \quad z \in C \quad \text{fulfilling} \quad \langle z^*, z - x\rangle < 0.$$

Definition 2.3 (Schauder invariance condition)**.** Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous. Then we say φ satisfies the Schauder invariance condition on C if $\big(I - \nabla\varphi\big)(\partial C) \subset C$ when $\operatorname{int} C \neq \emptyset$, if $\big(I - \nabla\varphi\big)(C) \subset C$ when $\operatorname{int} C = \emptyset$

Remark 2.4. Definition 2.2 and 2.3 essentially come from [25]. However, there are some subtle differences here and [25]. The well known Schauder invariance condition for a C^1-functional φ on a Hilbert space X reads as $(I - \varphi')(C) \subset C$; see [15, 22, 32, 33]. It has been extend to Banach spaces in [19]. The notation of Schauder invariance condition was firstly put forward by J. Sun in [33].

It follows from [25, Theorem 4.4 and 4.5] we have the following Lemma 2.5.

Lemma 2.5. *Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous, φ is outwardly directed on C or φ satisfies the Schauder invariance condition on C. Then $m_C(x) = 0$ if and only if $m(x) = 0$.*

Lemma 2.6 ([30], von Neumann)**.** *Let X, Y be two Hausdorff topological linear spaces, $C \subset X$, $D \subset Y$ be two convex and compact sets. Let $\psi : X \times Y \mapsto \mathbb{R}$ satisfy:*

1) $x \mapsto \psi(x, y)$ is upper semi-continuous (usc.) and concave;

2) $y \mapsto \psi(x, y)$ is lower semi-continuous (lsc.) and convex.

Then ψ has at least one saddle point $(\bar{x}, \bar{y}) \in C \times D$, that is

$$\psi(x, \bar{y}) \leqslant \psi(\bar{x}, \bar{y}) \leqslant \psi(\bar{x}, y) \quad \text{for } (x, y) \in C \times D.$$

Definition 2.7. Let D be a nonempty closed subset of X. We say that φ satisfies the non-smooth CPS-condition on D, denoted by $(CPS)_D$, if every sequence $\{x_n\} \subset D$ such that $\{\varphi(x_n)\}$ is bounded and $(1 + \|x_n\|)m_D(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence. If $D = X$, denoted simply by (CPS).

The following Lemma 2.8 comes from [4, p. 63], in which X is required to be a Hilbert space. However, we can easily see that the conclusion also holds if we assume that X is a real Banach space.

Lemma 2.8 ([4]). *Let C be a convex set of X. Then for all $x \in \operatorname{int} C$ and $y \in \bar{C}$, $\alpha x + (1-\alpha)y \subset \operatorname{int} C$ for all $\alpha \in (0,1]$.*

Proposition 2.9. *Assume that X is reflexive, C is a convex and closed set of X, $x_0 \in C$, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz. Then there exists $(x_0^*, u(x_0)) \in \partial\varphi(x_0) \times \big((x_0 - C) \cap \bar{B}(0,1)\big)$ such that $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. In particular, $m(x_0) = \langle x_0^*, u(x_0)\rangle$ for some $(x_0^*, u(x_0)) \in \partial\varphi(x_0) \times \bar{B}(x_0, 1)$ if $X = C$.*

Proof. Let X_w and X_w^* be the spaces X and X^* equipped with their weak topology respectively. Since X is reflexive, it follows from Proposition 2.1 that $\partial\varphi(x_0)$ is a compact set in X_w^*. Obviously, $\big(x_0 - C\big) \cap \bar{B}(0,1)$ is a compact set in X_w. Let $\psi : ((x_0 - C) \cap \bar{B}(0,1)) \times \partial\varphi(x_0) \mapsto \mathbb{R}$ be defined by $\psi(x, y^*) = \langle y^*, x\rangle$ for any $(x, y^*) \in ((x_0 - C) \cap \bar{B}(0,1)) \times \partial\varphi(x_0)$. It follows from Lemma 2.6 that ψ has at least one saddle point $(u(x_0), x_0^*) \in ((x_0 - C) \cap \bar{B}(0,1)) \times \partial\varphi(x_0)$, that is

$$\langle x_0^*, x\rangle \leqslant \langle x_0^*, u(x_0)\rangle \leqslant \langle x^*, u(x_0)\rangle, \qquad \forall x^* \in \partial\varphi(x_0), x \in (x_0 - C) \cap \bar{B}(0,1).$$

Hence, $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. The proof is complete. □

Remark 2.10. By using Proposition 2.9 we can find $(u(x_0), x_0^*) \in ((x_0 - C) \cap \bar{B}(0,1)) \times \partial\varphi(x_0)$ such that $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. By this way we give an unified treatment for the case $C = X$ in [6] and the case C being a proper closed and convex set in [18]. It should be pointed out that in order to prove Proposition 2.9, we only need C to be weakly sequence compact not weakly compact. A direct proof of Proposition 2.9 can be found in the Appendix.

Proposition 2.11. *Assume that X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz, $(I - \nabla\varphi)(C) \subset C$. Then*

$$m_C(x) \geqslant \min\left\{\frac{1}{2}, m(x)\right\} m(x) \quad \text{for all } x \in C. \tag{2.2}$$

Proof. It follows from of Proposition 2.9 that for each $x_0 \in C$, there exists $(u(x_0), x_0^*) \in \big(\{x_0\} - C) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$ satisfying $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. Take $y_0 \in \{x_0\} - \mathsf{F}(x_0^*)$. It follows from the condition $(I - \nabla\varphi)(C) \subset C$ that $y_0 \in C$. So, if $\|x_0 - y_0\| < 1$,

$$m_C(x_0) = \langle x_0^*, u(x_0)\rangle \geqslant \langle x_0^*, x_0 - y_0\rangle = \langle x_0^*, \mathsf{F}(x_0^*)\rangle = \|x_0^*\|_*^2 \geqslant m^2(x_0);$$

if $\|x_0 - y_0\| \geqslant 1$, let $z_0 = x_0 + \frac{y_0 - x_0}{2\|x_0 - y_0\|}$, we have $z_0 \in C$ and $\|x_0 - z_0\| = \frac{1}{2}$, and so

$$\begin{aligned} m_C(x_0) &= \langle x_0^*, u(x_0)\rangle \geqslant \langle x_0^*, x_0 - z_0\rangle \\ &= \frac{1}{2\|x_0 - y_0\|}\langle x_0^*, x_0 - y_0\rangle \\ &= \frac{1}{2\|x_0\|_*}\|x_0^*\|_*^2 = \frac{1}{2}\|x_0^*\|_* \geqslant \frac{1}{2}m(x_0) \end{aligned}$$

So, (2.2) holds. The proof is complete. □

Definition 2.12. Let D be a nonempty closed subset of X. We say that φ satisfies the nonsmooth PS-condition on D, denoted by $(PS)_D$, if every sequence $\{x_n\} \subset D$ such that $\{\varphi(x_n)\}$ is bounded and $m_D(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence. If $D = X$, denoted simply by (PS).

Remark 2.13. Assume that all conditions of Proposition 2.11 hold. Then, by (2.2), we see that φ satisfies the condition $(PS)_C$ if and only if φ satisfies the condition (PS) for each convex and closed set C. Moreover, $m_C(x_0) = 0$ if and only if $m(x_0) = 0$ for $x_0 \in C$. So, we can deduce Lemma 2.5 by (2.2) if φ satisfies the Schauder invariance condition.

Lemma 2.14 ([10]). *Assume U is bounded connected open set of $\mathbb{R}^2$ and $(0,0) \in U$, then there exists a connected component Γ' of the boundary of U, such that each one side ray l emitting from the origin satisfies $l \cap \Gamma' \neq \emptyset$.*

3 Main results

3.1 Critical points outside the ordered interval

Let $u_0, v_1 \in E$ be such that $u_0 \ll 0 \ll v_1$. Set $D_1 = [u_0, v_1]$ and $G_1 = D_1 \cap E$. Now let us introduce the following conditions:

(H_1) $\varphi^{-1}([a,b]) \cap \mathbb{K}$ is compact in E for each $a < b$;

(H_2) φ satisfies the conditions (CPS), $(CPS)_{\pm P}$ and $(CPS)_D$ for any ordered interval D in X;

(H_3) there exist subspace E_1 of E and $R_0 > 0$ such that $\dim E_1 = 2$,

$$E_1 \cap D_1 \subset B(0, R_0), \; E_1 \cap \operatorname{int} P_1 \neq \emptyset \tag{3.1}$$

and

$$\alpha_0 := \max_{u \in S_{R_0} \cap E_1} \varphi(u) < \beta_0 := \inf_{u \in D_1} \varphi(u). \tag{3.2}$$

We have the following result concerning the existence of multiple critical points outside the ordered interval D_1.

Theorem 3.1. *Suppose that (H_1)∼(H_3) hold, φ has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E G_1$. Moreover, either*

(H_4') φ is outwardly directed on $\pm P$, D_1 and $\pm P \cap D_1$, and P is generating; or

(H_5') $(I - \nabla\varphi)(\pm P) \subset \pm P$ and $(I - \nabla\varphi)(D_1) \subset D_1$.

Then φ has at least one positive critical point $\bar{u}_1$, one negative critical point $\bar{u}_2$ and one sign-changing critical point $\bar{u}_3$ outside D_1.

Remark 3.2. The condition (H_5') is stronger than (H_4'). In fact, it follows from [25, Theorem 4.5] that φ is outwardly directed on $\pm P$ if $(I - \nabla\varphi)(\pm P) \subset \pm P$.

Let $\alpha_1 = \sup_{u \in B(0,R_0) \cap E_1} \varphi(u)$, $D_0 = \varphi^{-1}\big([\alpha_0 - 1, \alpha_1 + 1]\big)$, and $K = D_0 \cap \mathbb{K}$. Assume without loss of generality that $D_0 \backslash K \neq \emptyset$. Since φ has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E G_1$, by using the conditions (H_1) we may take $\delta > 0$ small enough such that $D_0^\delta \neq \emptyset$, and

$$K_{3\delta} \cap \big(\partial_E P_1 \cup \partial_E(-P_1) \cup \partial_E G_1\big) = \emptyset \tag{3.3}$$

where $K_{3\delta} = \{x \in D_0 : \operatorname{dist}(x, K) < 3\delta\}$ and $D_0^\delta = D_0 \backslash K_\delta$.

In what follows of this Subsection 3.1, we assume that all conditions of Theorem 3.1 hold.

Lemma 3.3. *There exists a locally Lipschitz mapping $v : D_0^\delta \mapsto X$ such that $\|v(x)\| \leqslant 2(1+\|x\|)$ for any $x \in D_0^\delta$, $\langle x^*, v(x)\rangle \geqslant \frac{\gamma}{16}$ for some $\gamma > 0$ and all $x^* \in \partial\varphi(x)$. Moreover, $v : D_0^\delta \cap E \mapsto E$ is locally Lipschitz, and*

$$x - \frac{1}{1+\|x\|}v(x) \in \pm P_1 \quad \textit{for any} \;\; x \in \pm P \cap D_0^\delta, \tag{3.4}$$

$$x - \frac{1}{1+\|x\|}v(x) \in G_1 \quad \textit{for any} \;\; x \in D_1 \cap D_0^\delta, \tag{3.5}$$

$$x - \frac{1}{1+\|x\|}v(x) \in \pm P_1 \cap G_1 \quad \textit{for any} \;\; x \in \pm P \cap D_1 \cap D_0^\delta. \tag{3.6}$$

Proof. Let $S = D_0^\delta \backslash \big(P \cup (-P) \cup D_1\big)$ and $\widetilde{S} = D_0^\delta \cap (D_1 \backslash (P \cup (-P)))$. First we claim that for some $\gamma > 0$,

$$(1+\|x\|)m(x) \geqslant \gamma, \qquad \forall x \in S, \tag{3.7}$$

$$(1+\|x\|)m_P(x) \geqslant \gamma, \qquad \forall x \in (D_0^\delta \cap P)\backslash D_1, \tag{3.8}$$

$$(1+\|x\|)m_{-P}(x) \geqslant \gamma, \qquad \forall x \in (D_0^\delta \cap (-P))\backslash D_1, \tag{3.9}$$

$$(1+\|x\|)m_{D_1}(x) \geqslant \gamma, \qquad \forall x \in \widetilde{S}, \tag{3.10}$$

$$(1+\|x\|)m_{P\cap D_1}(x) \geqslant \gamma, \qquad \forall x \in D_0^\delta \cap P \cap D_1, \tag{3.11}$$

$$(1+\|x\|)m_{-P\cap D_1}(x) \geqslant \gamma, \qquad \forall x \in D_0^\delta \cap (-P) \cap D_1. \tag{3.12}$$

We only show that (3.11) holds. In a similar way we can show that (3.7)$\sim$(3.10) and (3.12) hold. Arguing by contradiction, assume that (3.11) does not hold. Then there exists an sequence $\{x_n\} \subset D_0^\delta \cap D_1 \cap P$ such that $(1+\|x_n\|)m_{P\cap D_1}(x_n) \to 0$ as $n \to \infty$. Obviously, $\{\varphi(x_n) : n = 1, 2, \ldots\} \subset [\alpha_0 - 1, \alpha_1 + 1]$. It follows from (H_2) that φ satisfies the condition $(CPS)_{D_1\cap P}$. Thus, up to a subsequence if necessary, $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in D_0^\delta \cap D_1 \cap P$. Since $m_{D_1\cap P} : D_1 \cap P \mapsto \mathbb{R}$ is lsc., we have $m_{D_1\cap P}(x_0) = 0$. It follows from (H_4') or (H_5') and Lemma 2.5 that $m(x_0) = 0$, which is a contradiction. Thus, (3.11) holds.

Pick $x_0 \in S$. It follows from Proposition 2.9 and (3.7) that there exist $(u_1(x_0), x_0^*) \in \bar{B}(x_0, 1) \times \partial\varphi(x_0)$ such that for any $y^* \in \partial\varphi(x_0)$,

$$\langle y^*, u_1(x_0)\rangle \geqslant \langle x_0^*, u_1(x_0)\rangle = m(x_0) > \frac{\gamma}{2(1+\|x_0\|)}.$$

Since the map $x \mapsto \partial\varphi(x)$ is usc. from X into X_w^*, we may take an open neighborhood $B_1(x_0, r_1(x_0))$ of x_0, such that

$$\langle y^*, u_1(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}, \qquad \forall y^* \in \partial\varphi(y),\; y \in U_1(x_0), \tag{3.13}$$

where $U_1(x_0) = B_1(x_0, r_1(x_0)) \cap S$. Since S is an open subset of D_0^δ, we may take $r_1(x_0) > 0$ small enough such that $U_1(x_0) \subset S$.

Pick $x_0 \in (D_0^\delta \cap P)\backslash D_1$. It follows from Proposition 2.9 and inequality (3.8) that there exist $(u_2(x_0), x_0^*) \in \big((x_0 - P) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$,such that for any $x^* \in \partial\varphi(x_0)$,

$$\langle x^*, u_2(x_0)\rangle \geqslant \langle x_0^*, u_2(x_0)\rangle = m_P(x_0) > \frac{\gamma}{2(1+\|x_0\|)}.$$

Again by using the fact that $x \mapsto \partial\varphi(x)$ is usc., we know that there exists an open neighborhood $B_2(x_0, r_2(x_0))$ of x_0 such that for any $y \in U_2(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_2(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.14}$$

where $U_2(x_0) = B_2(x_0, r_2(x_0)) \cap D_0^\delta$. Since $x_0 \in (D_0^\delta \cap P)\backslash D_1$ and $D_0^\delta\backslash(D_1 \cup (-P))$ is an open set of D_0^δ, we may take $r_2(x_0) > 0$ small such that $U_2(x_0) \cap \big(D_1 \cup (-P)\big) = \emptyset$.

Similarly, by (3.9) and Proposition 2.9 we can show that for each $x_0 \in (D_0^\delta \cap (-P))\backslash D_1$, there exist $U_3(x_0) := B_3(x_0, r_3(x_0)) \cap D_0^\delta$, $u_3(x_0) \in X$ with $\|u_3(x_0)\| \leqslant 1$, such that

$$x_0 - u_3(x_0) \in -P, \qquad U_3(x_0) \cap (D_1 \cup P) = \emptyset$$

and for any $y \in U_3(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_3(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.15}$$

By (3.10) we can show that for each $x_0 \in \widetilde{S}$, there exist $U_4(x_0) := B_4(x_0, r_4(x_0)) \cap D_0^\delta$, $u_4(x_0) \in X$ with $\|u_4(x_0)\| \leqslant 1$, such that

$$x_0 - u_4(x_0) \in D_1, \qquad U_4(x_0) \cap (P \cup (-P)) = \emptyset$$

and for any $y \in U_4(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_4(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.16}$$

It follows from (H_4') or (H_5') that $\partial\varphi(0) = \{0\}$. Then 0 is a critical point. Pick $x_0 \in D_0^\delta \cap D_1 \cap P$. Then $x_0 \neq 0$. By using Proposition 2.9 and (3.11) we obtain that there exist $U_5(x_0) := B_5(x_0, r_5(x_0)) \cap D_0^\delta$, $u_5(x_0) \in X$ with $\|u_5(x_0)\| \leqslant 1$, such that

$$x_0 - u_5(x_0) \in D_1 \cap P, \qquad U_5(x_0) \cap (-P) = \emptyset$$

and for any $y \in U_5(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_5(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.17}$$

Similarly, by (3.12) we can show that for each $x_0 \in D_0^\delta \cap D_1 \cap (-P)$, there exist $U_6(x_0) := B_6(x_0, r_6(x_0)) \cap D_0^\delta$, $u_6(x_0) \in X$ with $\|u_6(x_0)\| \leqslant 1$, such that

$$x_0 - u_6(x_0) \in D_1 \cap (-P), \qquad U_6(x_0) \cap P = \emptyset$$

and for any $y \in U_6(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_6(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.18}$$

By 3) in Proposition 2.1, we may assume that $\|x^*\|_* \leqslant L_{\alpha,i}$ for some $L_{\alpha,i} > 0$ and any $x \in U_i(x_\alpha)$ with $i = 1, 2, \ldots, 6$ and $x^* \in \partial\varphi(x)$. Also, we assume that for $i = 1, 2, \ldots, 6$, $B_i(x_\alpha, r_i(x_\alpha))$ has a small radius $r_i(x_\alpha) > 0$ such that $(1 + \|x\|)(1 + \|x_\alpha\|)^{-1} \leqslant 2$ for each $x \in U_i(x_\alpha)$, and

$$0 < r_i(x_\alpha) \leqslant \min\left\{\frac{1}{2}, \frac{\gamma}{64(1+\|x_\alpha\|)L_{\alpha,i}}\right\}.$$

Let

$$\begin{aligned}
\mathscr{A}_1 &= \big\{U_1(x_0) : x_0 \in S\big\},\\
\mathscr{A}_2 &= \big\{U_2(x_0) : x_0 \in (D_0^\delta \cap P)\backslash D_1\big\},\\
\mathscr{A}_3 &= \big\{U_3(x_0) : x_0 \in (D_0^\delta \cap (-P))\backslash D_1\big\},\\
\mathscr{A}_4 &= \big\{U_4(x_0) : x_0 \in \widetilde{S}\big\},\\
\mathscr{A}_5 &= \big\{U_5(x_0) : x_0 \in D_0^\delta \cap P \cap D_1\big\},\\
\mathscr{A}_6 &= \big\{U_6(x_0) : x_0 \in D_0^\delta \cap (-P) \cap D_1\big\},
\end{aligned}$$

and $\mathscr{A} = \bigcup_{i=1,2,\ldots,6} \mathscr{A}_i$. Then $\mathscr{A}$ is an open cover of D_0^δ.

By paracompactness we can find a locally finite refinement $\mathscr{B} = \{V_\alpha : \alpha \in \Lambda\}$ and a locally Lipschitz partition of unit $\{\gamma_\alpha : \alpha \in \Lambda\}$ sub-ordinate to it, with supp $\gamma_\alpha \subset V_\alpha$. For each $\alpha \in \Lambda$ we can find $x_\alpha \in D_0^\delta$ such that $V_\alpha \subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1, 2, \ldots, 6\}$, and $U_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. To this $x_\alpha \in D_0^\delta$ corresponds the element $w_\alpha^{i(\alpha)}$ such that $\|w_\alpha^{i(\alpha)}\| \leqslant 1$, and $w_\alpha^{i(\alpha)} = u_{i(\alpha)}(x_\alpha)$ if $V_\alpha \subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1, 2, \ldots, 6\}$. Since E is densely embedded in X, and so $P_1, -P_1, G_1$ are densely embedded in $P, -P, D_1$, respectively. Thus, we may take $\bar{x}_\alpha, \bar{w}_\alpha^{i(\alpha)} \in E$ with $\|\bar{w}_\alpha^{i(\alpha)}\| \leqslant 1$ such that

$$\max\left\{\|w_\alpha^{i(\alpha)} - \bar{w}_\alpha^{i(\alpha)}\|, \|x_\alpha - \bar{x}_\alpha\|\right\} < \min\left\{\frac{1}{2}, \frac{\gamma}{64(1+\|x_\alpha\|)L_{\alpha,i(\alpha)}}\right\}. \tag{3.19}$$

and

$$\bar{x}_\alpha - \bar{w}_\alpha^{i(\alpha)} \in \begin{cases} P_1 & \text{if } x_\alpha \in P \cap D_0^\delta; \\ -P_1 & \text{if } x_\alpha \in -P \cap D_0^\delta; \\ G_1 & \text{if } x_\alpha \in \widetilde{S}; \\ G_1 \cap P_1 & \text{if } x_\alpha \in P \cap D_1 \cap D_0^\delta; \\ G_1 \cap (-P_1) & \text{if } x_\alpha \in -P \cap D_1 \cap D_0^\delta. \end{cases} \tag{3.20}$$

Now, let $v : D_0^\delta \mapsto X$ be defined by

$$v(x) = (1 + \|x\|) \sum_{\alpha \in \Lambda} \gamma_\alpha(x)(\bar{w}_\alpha^{i(\alpha)} - \bar{x}_\alpha + x). \tag{3.21}$$

Then, $v : D_0^\delta \mapsto X$ is locally Lipschitz. Since $E \hookrightarrow X$, $\gamma_\alpha : D_0^\delta \cap E \mapsto \mathbb{R}$ is also locally Lipschitz. Thus, $v : D_0^\delta \cap E \mapsto E$ is locally Lipschitz.

By (3.19) and (3.21) , we have for any $x \in D_0^\delta$,

$$\|v(x)\| \leqslant (1 + \|x\|) \sum_{\alpha \in \Lambda} \gamma_\alpha(x)\big(\|\bar{w}_\alpha^{i(\alpha)}\| + \|\bar{x}_\alpha - x_\alpha\| + \|x_\alpha - x\|\big) \leqslant 2(1 + \|x\|).$$

Moreover, we have for any $x \in D_0^\delta$ and $x^* \in \partial\varphi(x)$,

$$\begin{aligned} &(1 + \|x\|)\Big|\sum_{\alpha \in \Lambda} \gamma_\alpha(x)\langle x^*, x - \bar{x}_\alpha\rangle\Big| \\ &\leqslant \sum_{\alpha \in \Lambda} \frac{1 + \|x\|}{1 + \|x_\alpha\|}(1 + \|x_\alpha\|)\gamma_\alpha(x)\|x^*\|_*\big(\|x - x_\alpha\| + \|x_\alpha - \bar{x}_\alpha\|\big) \\ &\leqslant 2\sum_{\alpha \in \Lambda} L_{\alpha,i(\alpha)}\big(\|x - x_\alpha\| + \|x_\alpha - \bar{x}_\alpha\|\big)(1 + \|x_\alpha\|) \\ &< \frac{\gamma}{16} \end{aligned}$$

and

$$\begin{aligned} \left|\sum_{\alpha \in \Lambda} \gamma_\alpha(x)(1 + \|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\rangle\right| &\leqslant \sum_{\alpha \in \Lambda} \frac{1 + \|x\|}{1 + \|x_\alpha\|}(1 + \|x_\alpha\|)\gamma_\alpha(x)\|x^*\|_*\|\bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\| \\ &\leqslant 2\sum_{\alpha \in \Lambda}(1 + \|x_\alpha\|)L_{\alpha,i(\alpha)}\|\bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\| < \frac{\gamma}{16}. \end{aligned}$$

So, by (3.13)∼(3.18) we have for any $x \in D_0^\delta$ and $x^* \in \partial\varphi(x)$,

$$\begin{aligned}\sum_{\alpha\in\Lambda} \gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)}\rangle &= \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)} + w_\alpha^{i(\alpha)}\rangle \\ &= \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(1+\|x\|)\langle x^*, w_\alpha^{i(\alpha)}\rangle \\ &\quad + \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\rangle \\ &\geqslant \frac{\gamma}{4} - \frac{\gamma}{16} \geqslant \frac{\gamma}{8}.\end{aligned}$$

Hence, we have for any $x \in D_0^\delta$ and $x^* \in \partial\varphi(x)$,

$$\begin{aligned}\langle x^*, v(x)\rangle &= \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)}\rangle \\ &\quad + \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(1+\|x\|)\langle x^*, x - \bar{x}_\alpha\rangle \geqslant \frac{\gamma}{16}.\end{aligned}$$

For a given $x \in D_0^\delta$, we assume that $x \in U_{i(\alpha)}(x_\alpha)$ for some $U_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. Recalling the construction of $\mathscr{A}$, we have

$$U_{i(\alpha)}(x_\alpha) \in \begin{cases} \mathscr{A}_2 \cup \mathscr{A}_5 & \text{if } x \in (D_0^\delta \cap P)\backslash D_1; \\ \mathscr{A}_3 \cup \mathscr{A}_6 & \text{if } x \in (D_0^\delta \cap (-P))\backslash D_1; \\ \mathscr{A}_4 \cup \mathscr{A}_5 \cup \mathscr{A}_6 & \text{if } x \in \widetilde{S}; \\ \mathscr{A}_5 & \text{if } x \in D_0^\delta \cap P \cap D_1; \\ \mathscr{A}_6 & \text{if } x \in D_0^\delta \cap (-P) \cap D_1. \end{cases} \tag{3.22}$$

It follows from (3.21) and (3.22) that (3.4)∼(3.6) hold. The proof is complete. □

Let $l_1, l_2 : X \mapsto \mathbb{R}$ be defined by

$$\begin{aligned} l_1(x) &= \frac{\text{dist}\,(x, K_\delta)}{\text{dist}\,(x, K_\delta) + \text{dist}\,(x, X\backslash K_{2\delta})}, \\ l_2(x) &= \frac{\text{dist}\,(x, \Gamma_1)}{\text{dist}\,(x, \Gamma_2) + \text{dist}\,(x, \Gamma_1)}, \end{aligned} \tag{3.23}$$

$$v_1(x) = \begin{cases} v(x), & x \in D_0^\delta; \\ 0, & \text{otherwise}, \end{cases}$$

and $V(x) = l_1(x)l_2(x)v_1(x)$ for any $x \in X$, where

$$\Gamma_1 = \varphi^{-1}\left(\left[\alpha_1 + \frac{1}{2}, +\infty\right)\right) \cup \varphi^{-1}\left(\left(-\infty, \alpha_0 - \frac{1}{2}\right]\right), \qquad \Gamma_2 = \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}\right]\right).$$

Then $l_1, l_2 : X \mapsto \mathbb{R}$ are locally Lipschitz, and so $V : X \mapsto X$ is locally Lipschitz. Since $E \hookrightarrow X$, $l_1, l_2 : E \mapsto \mathbb{R}$ are also locally Lipschitz, and so $V : E \mapsto E$ is locally Lipschitz.

Consider the following initial value problem

$$\begin{cases} \frac{du}{dt} = -V(u), \\ u(0) = v_0 \in X. \end{cases} \tag{3.24}$$

By the theories for initial value problems of ordinary equations in Banach space, we see that (3.24) has a unique solution $\sigma(t,v_0)$ in X, with its right maximal existence interval $[0,T(v_0))$, and its right maximal existence interval $[0,T_1(v_0))$ in E. Obviously, we have $T_1(v_0) \leqslant T(v_0)$. Concerning the solution $\sigma(t,v_0)$ of (3.24), we have the following Lemmas 3.4∼3.6.

Lemma 3.4. *For each $v_0 \in X$, $T(v_0) = +\infty$*

Proof. The proof is standard. For the reader's convenience we give detailed process. Arguing by contradiction, let us assume that $T(v_0) < +\infty$. By (3.24) we have

$$\|\sigma(t,v_0) - v_0\| \leqslant \int_0^t \|V(\sigma(s,v_0))\|ds \leqslant 2\int_0^t (1 + \|\sigma(s,v_0)\|)ds.$$

So, we have

$$\begin{aligned}\frac{1}{2}\|\sigma(t,v_0) - v_0\| &\leqslant \int_0^t (1 + \|\sigma(s,v_0)\|)ds \\ &\leqslant \int_0^t \|\sigma(s,v_0) - v_0\|ds + (1 + \|v_0\|)t,\end{aligned}$$

By the well known Gronwall's inequality, we have

$$\begin{aligned}\frac{1}{2}\|\sigma(t,v_0) - v_0\| &\leqslant \int_0^t (1 + \|v_0\|)e^{t-s}ds + (1 + \|v_0\|)t \\ &\leqslant (1 + \|v_0\|)(e^t - 1) + (1 + \|v_0\|)t \\ &\leqslant (1 + \|v_0\|)(t + e^t - 1) \\ &\leqslant (1 + \|v_0\|)(T(v_0) + e^{T(v_0)}).\end{aligned}$$

So, we have

$$\|\sigma(t,v_0)\| \leqslant 2(1 + \|v_0\|)(T(v_0) + e^{T(v_0)}) + 2\|v_0\| =: M_1.$$

Take $\{t_n\} \subset [0,T(v_0))$ such that $t_n \to T^-(v_0)$ and for $n = 1,2,\ldots,$

$$|t_n - t_{n-1}| < \frac{1}{2 \cdot 2^n(1 + M_1)}.$$

Then we have

$$\begin{aligned}\|\sigma(t_n,v_0) - \sigma(t_{n-1},v_0)\| &\leqslant \int_{t_{n-1}}^{t_n} \|V(\sigma(s,v_0)\|ds \\ &\leqslant 2\int_{t_{n-1}}^{t_n} (1 + \|\sigma(s,v_0)\|)ds \\ &\leqslant 2(1 + M_1)(t_n - t_{n-1}) < \frac{1}{2^n}.\end{aligned}$$

This implies that $\{\sigma(t_n,v_0)\}$ is a Cauchy sequence in X. Thus, there exists $\bar{u} \in X$ such that $\sigma(t_n,v_0) \to \bar{u}$ as $t_n \to T^-(v_0)$. Then we can show that $\sigma(t,v_0) \to \bar{u}$ as $t \to T^-(v_0)$.

Now we consider the initial value problem

$$\begin{cases}\frac{du}{dt} = -V(u), \\ u(0) = \bar{u}.\end{cases} \tag{3.25}$$

Then (3.25) has a unique solution on $[0,\bar{\delta})$ for some $\bar{\delta} > 0$, and so (3.24) has a unique solution on $[0,T(v_0) + \bar{\delta})$, which is a contradiction. Thus, we have $T(v_0) = +\infty$. The proof is complete. □

Lemma 3.5. *If* $o(v_0) := \{\sigma(t, v_0) \subset X : t \in [0, +\infty)\} \subset \Gamma_2 \backslash K_{2\delta}$, *then* $T_1(v_0) = +\infty$.

Proof. Let the operator $A : E \to X$ be defined by

$$Ax = x - \frac{1}{1 + \|x\|} V(x), \qquad \forall x \in E.$$

Then we have

$$Ax = \sum_{\alpha \in \Lambda} \gamma_\alpha(x)(\bar{x}_\alpha - \varpi_\alpha^{i(\alpha)}), \qquad \forall x \in \Gamma_2 \backslash K_{2\delta}. \tag{3.26}$$

Let

$$\mu(t) = \int_0^t (1 + \|\sigma(s, v_0)\|) ds \quad \text{for } t \in [0, +\infty).$$

Obviously, $\mu : [0, +\infty) \to [0, +\infty)$ is strictly increasing. And so, μ^{-1}, the inverse function of μ, exists. Then we have

$$\begin{cases} \frac{d}{dt}\big(e^{\mu(t)}\sigma(t, v_0)\big) = e^{\mu(t)}\big(1 + \|\sigma(t, v_0)\|\big) A\sigma(t, v_0), \\ \sigma(0, v_0) = v_0. \end{cases}$$

By direct computation, we have

$$\sigma(t, v_0) = e^{-\mu(t)} v_0 + e^{-\mu(t)} \int_0^t e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) A\sigma(s, v_0) ds. \tag{3.27}$$

where the integral is in the sense of X topology. Now we show that $T_1(v_0) = +\infty$ when $o(v_0) \subset \Gamma_2 \backslash K_{2\delta}$. Arguing by contradiction, let us assume that $T_1(v_0) < +\infty$. Take $T > T_1(v_0)$. It follows from (3.26) that $\{A(\sigma(t, v_0)) : t \in [0, T]\}$ is contained in a finite-dimensional subspace of X. Then, there exists $M_1(T) > 0$ such that for $s \in [0, T]$,

$$\|e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) A\sigma(s, v_0)\|_1 \leqslant M_1(T).$$

Let $\{t_n\} \subset [0, T_1(v_0))$ such that $t_n \to T_1^-(v_0)$ as $n \to \infty$. Note (3.27) also holds in which the integral is in the sense of E topology for any $t \in [0, T_1(v_0))$. Assume without loss of generality that $\{t_n\}$ is increasing. Then we have

$$\begin{aligned} \|\sigma(t_n, v_0) - \sigma(t_{n-1}, v_0)\|_1 \leqslant\ & |e^{-\mu(t_n)} - e^{-\mu(t_{n-1})}| \Big(\|v_0\|_1 \\ & + \int_0^{t_n} e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) \|A\sigma(s, v_0)\|_1 ds \Big) \\ & + e^{-\mu(t_{n-1})} \int_{t_{n-1}}^{t_n} e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) \|A\sigma(s, v_0)\|_1 ds \\ \leqslant\ & |e^{-\mu(t_n)} - e^{-\mu(t_{n-1})}| \big(\|v_0\|_1 + T M_1(T) \big) + M_1(T)(t_n - t_{n-1}). \end{aligned}$$

So, $\{\sigma(t_n, v_0)\}$ is a Cauchy sequence in E. Assume that that $\sigma(t_n, v_0) \to \bar{u}$ in E as $t \to T_1^-(v_0)$. Since $V : E \mapsto E$ is locally Lipschitz. Then we can easily obtain a contradiction as in the proof of $T(v_0) = +\infty$. Thus, we have $T_1(v_0) = +\infty$. The proof is complete. □

Lemma 3.6. $\varphi(\sigma(t, v_0))$ *is non-increasing in* $t \in [0, T_1(v_0))$.

Proof. Let $h(t, v_0) = \varphi(\sigma(t, v_0))$ for all $t \in [0, T_1(v_0))$. It is easy to see that $h(t, v_0)$ is locally Lipschitz in $t \in [0, T_1(v_0))$, hence differentiable almost everywhere. According to Lebourg's Mean Theorem we have

$$\begin{aligned}\frac{\partial}{\partial s}h(s,v_0) &\leqslant \max\left\{\left\langle w^*, \frac{\partial}{\partial s}\sigma(s,v_0)\right\rangle : w^* \in \partial\varphi(\sigma(s,v_0))\right\} \quad \text{a.e.}\\ &= -\min\left\{\langle w^*, V(\sigma(s,v_0))\rangle : w^* \in \partial\varphi(\sigma(s,v_0))\right\} \quad \text{a.e.}\\ &\leqslant \begin{cases} -\frac{\gamma}{16}, & \text{if } \sigma(s,v_0) \in \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}])\backslash K_{2\delta};\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{3.28}$$

Consequently, $\varphi(\sigma(t, v_0))$ is non-increasing in $t \in [0, T_1(v_0))$. The proof is complete. □

As in [22], we give the following Definition 3.7 and 3.8

Definition 3.7. A nonempty subset D of E is called an invariant set of descending flow of (3.24) if $o(v_0) \subset D$ for all $v_0 \in D$, where $o(v_0) = \{\sigma(t, v_0) : t \in [0, T_1(v_0))\}$.

Definition 3.8. Let $M \subset E$ be a connected invariant set of the descending flow of (3.24), D be an open subset of M and be an invariant set of descending flow of (3.24). Denote

$$C_M(D) = \{v_0 : v_0 \in D \text{ or } v_0 \in M\backslash D \text{ and there exists } t' \in (0, T_1(v_0)) \text{ such that } \sigma(t', v_0) \in D\}.$$

If $D = C_M(D)$, then D is called a complete invariant set of descending flow of (3.24) in M.

Lemma 3.9. *$\pm P_1$, $\operatorname{int}(\pm P_1)$, G_1 and $\operatorname{int} G_1$ are all invariant sets of descending flow of* (3.24).

Proof. 1) P_1, $-P_1$ and G_1 are all invariant sets of descending flow of (3.24).

For $u \in \partial_E G_1$, it follows from Lemma 3.3 that for $\lambda > 0$ small enough,

$$u + \lambda(-V(u)) = \lambda g(u)\left(u - \frac{v_1(u)}{1 + \|u\|}\right) + (1 - \lambda g(u))\, u \in G_1,$$

where $g(u) = l_1(u) l_2(u)\big(1 + \|u\|\big)$. It follows from the theorem due to Brezis–Martin (see [11]) that G_1 is an invariant sets of descending flow of (3.24).

In a similar way we can show that $P_1, -P_1$ are also invariant sets of descending flow of (3.24).

2) $\operatorname{int} P_1$, $\operatorname{int}(-P_1)$ and $\operatorname{int} G_1$ are all invariant sets of descending flow of (3.24)

Take $v_0 \in \operatorname{int} G_1$. Note (3.27) also holds where the integral is in the sense of E topology for $t \in [0, T_1(v_0))$. Make a variable change $\tau = e^{\mu(s)} - 1$ in (3.27). Then we have

$$s = \mu^{-1}\big(\ln(1+\tau)\big), \qquad ds = \frac{e^{-\mu(s)}}{\mu'(s)} d\tau$$

and

$$\begin{aligned} &e^{-\mu(t)} \int_0^{e^{\mu(t)}-1} A\sigma\Big(\mu^{-1}\big(\ln(1+\tau)\big), v_0\Big) d\tau \quad (\text{the integral is in the sense of } E \text{ topology})\\ &= \lim_{n\to\infty} \frac{1}{n}\big(1 - e^{-\mu(t)}\big) \sum_{k=0}^{n-1} A\sigma\left(\mu^{-1}\left(\ln(1 + \frac{k(e^{\mu(t)}-1)}{n}\right), v_0\right).\end{aligned}$$

For any $x \in G_1$, we have

$$Ax = \begin{cases} x, & \text{if } x \in \big(E \backslash \varphi^{-1}(\alpha_0 - \frac{1}{2}, \alpha_1 + \frac{1}{2})\big) \cup K_\delta; \\ x - \frac{1}{1+\|x\|} v(x), & \text{if } x \in \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}]) \backslash K_{2\delta}; \\ l(x)\Big(x - \frac{1}{1+\|x\|} v_1(x)\Big) + \big(1 - l(x)\big)x, & \text{otherwise.} \end{cases} \in G_1, \tag{3.29}$$

where $l(x) := l_1(x) l_2(x) \in (0,1)$, that is $A(G_1) \subset G_1$. Since G_1 is an invariant set of descending flow of (3.24), by (3.29) we have

$$\bar{v} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A\sigma\left(\mu^{-1}\left(\ln(1 + \frac{k(e^{\mu(t)} - 1)}{n}\right), v_0\right) \in G_1.$$

It follows from (3.27) that

$$\sigma(t, v_0) = e^{-\mu(t)} v_0 + (1 - e^{-\mu(t)})\bar{v}. \tag{3.30}$$

Since $v_0 \in \operatorname{int} G_1$ and $e^{-\mu(t)} \in (0,1)$, it follows from Lemma 2.8 and (3.30) that $\sigma(t, v_0) \in \operatorname{int} G_1$ for $t \in [0, T_1(v_0))$. Thus, $\operatorname{int} G_1$ is an invariant set of descending flow of (3.24).

Similarly, we can show that $\operatorname{int}(\pm P_1)$ are invariant sets for the descending flow of (3.24). The proof is complete. □

Lemma 3.10. *For each $v_0 \in \varphi^{-1}\big((-\infty, \alpha_1 + \frac{1}{4}]\big) \cap E$ with $\inf_{u \in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$, there exists $\tau(v_0) \geqslant 0$ such that $\sigma(\tau(v_0), v_0) \in K_{2\delta}$.*

Proof. Assume that $o(v_0) \cap K_{2\delta} = \emptyset$. It follows from (3.28) that

$$\varphi(v_0) - \varphi(\sigma(t, v_0)) = -\int_0^t \frac{\partial}{\partial s} h(s, v_0) ds \geqslant \frac{\gamma}{16} t \quad \text{for } t \in [0, T_1(v_0)).$$

It follows from Lemma 3.5 that the maximal existence interval of $\sigma(t, v_0)$ in E is $[0, +\infty)$. So, we may take $t_0 = 16\gamma^{-1}(\alpha_1 - \alpha_0 + 2)$, and have

$$\varphi(\sigma(t_0, v_0)) \leqslant \varphi(v_0) - \frac{\gamma}{16} t_0 \leqslant \alpha_1 + \frac{1}{4} - \frac{\gamma}{16} t_0 < \alpha_0 - 1,$$

which contradicts to $\inf_{u \in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$. The proof is complete. □

Similar to the proof of Lemma 3.1 in [22] we have the following Lemma 3.11.

Lemma 3.11. *Let $G \subset E$ be a connected and invariant set of* (3.24)*, and D be an open invariant subset of G. Then the following assertions hold:*

1) *$C_G(D)$ is an open subset of G;*
2) *$\partial_G C_G(D)$ is an invariant set of descending flow of* (3.24)*;*
3) *$\inf_{u \in \partial_G C_G(D)} \varphi(u) \geqslant \inf_{u \in \partial_G D} \varphi(u)$.*

Proof of Theorem 3.1. It follows from Lemma 3.9 that $\operatorname{int}(\pm P_1)$ and $\operatorname{int} G_1$ are all invariant sets of descending flow of (3.24). So, $\operatorname{int}(P_1 \cap G_1)$ and $\operatorname{int}(-P_1 \cap G_1)$ are invariant sets of descending flow of (3.24). It follows from Lemma 3.11 that $C_E(\operatorname{int} G_1)$ is an open invariant set of descending flow of (3.24). It follows from (3.2) that $C_E(\operatorname{int} G_1) \neq E$, and so $\partial_E C_E(\operatorname{int} G_1) \neq \emptyset$. By Lemma 3.11 we have

$$\inf_{u\in\partial_E C_E(\operatorname{int} G_1)} \varphi(u) \geqslant \inf_{u\in\partial_E G_1} \varphi(u) \geqslant \inf_{u\in D_1} \varphi(u) = \beta_0 > \alpha_0 - \frac{1}{4}. \tag{3.31}$$

Since $C_E(\operatorname{int} G_1)$ is open in E, $C_E(\operatorname{int} G_1) \cap E_1 \subset B(0, R_0)$ is an open and bounded subset of E_1 containing 0. It follows from Lemma 2.14 that there exists a connected component Γ' of the boundary of $C_E(\operatorname{int} G_1) \cap E_1$, such that each one side ray l emitting from the origin satisfies $l \cap \Gamma' \neq \emptyset$. Let Γ be the connected component of $\partial_E C_E(\operatorname{int} G_1)$ containing Γ'. It follows from Lemma 3.11 that Γ is an invariant set of descending flow of (3.24).

It follows from (3.1) that $S_{R_0} \cap E_1 \cap \operatorname{int} P_1 \neq \emptyset$. Thus, $\Gamma \cap \operatorname{int} P_1$ is an invariant set of descending flow of (3.24). Take $\widetilde{v}_1 \in S_{R_0} \cap E_1 \cap \operatorname{int} P_1$ and let $\widetilde{l}$ be the ray emitting from the origin and passing through $\widetilde{v}_1$. Then, we have $\widetilde{l} \cap (\Gamma \cap P_1) \neq \emptyset$, and so $\Gamma \cap P_1 \cap \varphi^{-1}\big((-\infty, \alpha_1 + \frac{1}{4}]\big) \neq \emptyset$. Take $v_0 \in \Gamma \cap P_1 \cap \varphi^{-1}\big((-\infty, \alpha_1 + \frac{1}{4}]\big)$. It follows from (3.31) that $\inf_{u\in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$, that is $o(v_0) \subset \varphi^{-1}\big([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}]\big)$.

Now we have the following two cases:

1) If $v_0 \in K_{2\delta}$, it follows from (3.3) that there must exists a u_1 with

$$\bar{u}_1 \in P_1 \cap \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1\right]\right) \cap (K\backslash\{0\})$$

and $\bar{u}_1 \in E\backslash(D_1)_\delta$.

2) If $v_0 \in \big(\Gamma \cap P_1 \cap \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}])\big)\backslash K_{2\delta}$, by Lemma 3.10 we see that $\sigma(\tau(v_0), v_0) \in K_{2\delta} \cap P_1$ for some $\tau(v_0) > 0$, also by (3.3) we see that there must exists a $\bar{u}_1$ with

$$u_1 \in P_1 \cap \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}\right]\right) \cap (K\backslash\{0\}),$$

and $\bar{u}_1 \in E\backslash(D_1)_\delta$. Hence, φ has at least one positive critical point $\bar{u}_1$ outside D_1.

Similarly, we can show that φ has at least one negative critical point $\bar{u}_2$ outside D_1.

Now we show that φ has at least one sign-changing critical point $\bar{u}_3$. Obviously, $\Gamma \cap \operatorname{int} P_1$ and $\Gamma \cap \operatorname{int}(-P_1)$ are two open invariant sets of descending flow of (3.24) in Γ. It follows from Lemma 3.11 that $C_\Gamma(\Gamma \cap \operatorname{int} P_1)$ and $C_\Gamma(\Gamma \cap \operatorname{int}(-P_1))$ are two open invariant sets of descending flow of (3.24) in Γ. By the connectedness of Γ, we see that

$$O_1 := \Gamma\backslash\big(C_\Gamma(\Gamma \cap \operatorname{int} P_1) \cup C_\Gamma(\Gamma \cap \operatorname{int}(-P_1)\big) \neq \emptyset.$$

Let

$$O_2 := \Gamma'\backslash\big(C_\Gamma(\Gamma \cap \operatorname{int} P_1) \cup C_\Gamma(\Gamma \cap \operatorname{int}(-P_1)\big).$$

Obviously, $O_2 \subset O_1$. Also by the connectedness of Γ', we have $O_2 \neq \emptyset$. Take $v_0 \in O_2$. It follows from (3.31) that $\inf_{u\in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$. Then we can show that φ has a sign-changing critical point $\bar{u}_3$. Indeed, if $v_0 \in K_{2\delta}$, by (3.31) there must exist a $\bar{u}_3 \in S \cap \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}]) \cap K$, where S is defined as in Lemma 3.3. If $v_0 \in \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}])\backslash K_{2\delta}$, it follows from Lemma 3.10 that $\sigma(\tau(v_0), v_0) \in K_{2\delta}$ for some $\tau(v_0) > 0$. By (3.3), we see that there must exists a $\bar{u}_3$ with

$$\bar{u}_3 \in (E\backslash(P \cup (-P))) \cap \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}\right]\right) \cap (K\backslash\{0\})$$

and $u_3 \in E\backslash(D_1)_\delta$. Hence, φ has at least one sign-changing critical point $\bar{u}_3$ outside D_1. The proof is complete. □

Remark 3.12. According to Remark 2.13, if (H_5') holds, then (H_2) can be substituted with the condition that φ satisfies the condition (PS). So, we have the following Corollary 3.13.

Corollary 3.13. *it Suppose that (H_1), (H_3) and (H_5') hold. Moreover, φ satisfies the condition (PS) and has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E G_1$. Then the conclusion in Theorem 3.1 holds.*

3.2 Critical points inside the ordered interval

Next in this section we will give the existence results for critical points inside the ordered interval D_1. Assume that $u_0, v_0, u_1, v_1 \in E$ such that $u_0 \ll v_0 \ll 0 \ll u_1 \ll v_1$. Set $D_2 = [u_0, v_0]$, $D_3 = [u_1, v_1]$ and $G_i = D_i \cap E$ for $i = 2, 3$.

Theorem 3.14. *Suppose that (H_1) holds, φ has no nontrivial critical point on $\partial_E G_i$ for $i = 1, 2, 3$, and φ satisfies $(CPS)_D$ for any ordered interval D in X. Moreover, either*

(H_4'') φ is outwardly directed on D_i for $i = 1, 2, 3$; or

(H_5'') $(I - \nabla\varphi)(D_i) \subset D_i$ for $i = 1, 2, 3$.

Then φ has at least one positive critical point $\bar{u}_4$, one negative critical point $\bar{u}_5$ and one critical point $\bar{u}_6$ inside D_1. Moreover, $\bar{u}_6$ is nontrivial if there exists a curve l in E_1 such that $l \cap D_2 \neq \emptyset$, $l \cap D_3 \neq \emptyset$ and $\max_{u \in l} \varphi(u) < \varphi(0)$.

Corollary 3.15. *Suppose that (H_1) holds, φ has no nontrivial critical point on $\partial_E G_i$ for $i = 1, 2, 3$, φ satisfies (PS) and $(I - \nabla\varphi)(D_i) \subset D_i$ for $i = 1, 2, 3$. Then the conclusion in Theorem 3.14 holds.*

Set

$$\begin{aligned}
\widetilde{K}_r &= \{x \in D_1 : \varphi(x) = r, m(x) = 0\} \text{ for any } r \in \mathbb{R}, \\
(\widetilde{K}_r)_\delta &= \{x \in D_1 : d(x, \widetilde{K}_r) < \delta\}, (\widetilde{K}_r)^c_\delta = D_1 \backslash (\widetilde{K}_r)_\delta \text{ for any } r \in \mathbb{R}, \delta > 0, \\
D_1(r, \varepsilon, \delta) &= \{x \in D_1 : r - \varepsilon \leqslant \varphi(x) \leqslant r + \varepsilon, x \in (\widetilde{K}_r)^c_\delta\} \text{ for any } r \in \mathbb{R}, \delta > 0, \varepsilon > 0.
\end{aligned}$$

Lemma 3.16. *Assume that all conditions in Theorem 3.14 hold. Let $r \in \mathbb{R}, \delta > 0$ be given. Then there exist $\bar{\varepsilon} > 0$, $\gamma > 0$ and a locally Lipschitz mapping $\widetilde{v} : D_1(r, \bar{\varepsilon}, \delta) \mapsto D_1$ such that $\|\widetilde{v}(x)\| \leqslant 2(1 + \|x\|)$ for any $x \in D_1(r, \bar{\varepsilon}, \delta)$, $\langle x^*, \widetilde{v}(x)\rangle \geqslant \frac{\gamma}{16}$ for any $x \in D_1(r, \bar{\varepsilon}, \delta)$ and $x^* \in \partial\varphi(x)$. Moreover, $\widetilde{v} : D_1(r, \bar{\varepsilon}, \delta) \cap E \mapsto E$ is locally Lipschitz, and for $i = 1, 2, 3$,*

$$x - \frac{1}{1 + \|x\|}\widetilde{v}(x) \in G_i \quad \textit{for any } x \in D_i \cap D_1(r, \bar{\varepsilon}, \delta). \tag{3.32}$$

Proof. The proof is similar to Lemma 3.3. Now we only sketch it. First we can show that for $i = 1, 2, 3$ and some $\bar{\varepsilon}, \gamma > 0$,

$$(1 + \|x\|) m_{D_i}(x) \geqslant \gamma, \qquad \forall x \in D_1(r, \bar{\varepsilon}, \delta) \cap D_i. \tag{3.33}$$

Then, by using (3.33), as the proof in Lemma 3.3, we can find a $\widetilde{u}_i(x) \in X$ for $i = 1, 2, 3$, an open neighborhood $\widetilde{U}_1(x)$ for $x \in D_1(r, \bar{\varepsilon}, \delta) \backslash (D_2 \cup D_3)$, an open neighborhood $\widetilde{U}_2(x)$ for $x \in D_2 \cap D_1(r, \bar{\varepsilon}, \delta)$, and an open neighborhood $\widetilde{U}_3(x)$ for $x \in D_3 \cap D_1(r, \bar{\varepsilon}, \delta)$ such that, $\|\widetilde{u}_i(x)\| \leqslant 1, x - \widetilde{u}_i(x) \in D_i$,

$$\widetilde{U}_1(x) \cap (D_2 \cup D_3) = \emptyset, \qquad \widetilde{U}_2(x) \cap D_3 = \emptyset, \qquad \widetilde{U}_3(x) \cap D_2 = \emptyset,$$

and

$$\langle y^*, \widetilde{u}_i(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}, \qquad \forall y^* \in \partial\varphi(y),\ y \in \widetilde{U}_i(x), \tag{3.34}$$

where $\widetilde{U}_i(x) = B(x, r_i(x)) \cap D_1(r, \bar{\varepsilon}, \delta)$ for $i = 1, 2, 3$.

Let $\mathscr{A}_1 = \{\widetilde{U}_1(x) : x \in D_1(r, \bar{\varepsilon}, \delta) \backslash (D_2 \cup D_3)\}$, $\mathscr{A}_i = \{\widetilde{U}_i(x) : x \in D_i \cap D_1(r, \bar{\varepsilon}, \delta)\}$ for $i = 2, 3$ and $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$. The collection $\mathscr{A}$ is an open cover of the set $D_1(r, \bar{\varepsilon}, \delta)$. So, by the paracompactness we can find a locally finite refinement $\{V_\alpha : \alpha \in \Lambda\}$ and a locally Lipschitz partition of unit $\{\gamma_\alpha : \alpha \in \Lambda\}$ sub-ordinate to it with $\operatorname{supp} \gamma_\alpha \subset V_\alpha$ for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, we can find $x_\alpha \in D_1(r, \bar{\varepsilon}, \delta)$ such that $V_\alpha \subset \widetilde{U}_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1, 2, 3\}$, and $\widetilde{U}_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. To this $x_\alpha \in D_1(r, \bar{\varepsilon}, \delta)$ corresponds the element $w_\alpha^{i(\alpha)}$ such that $\|w_\alpha^{i(\alpha)}\| \leqslant 1$, and $w_\alpha^{i(\alpha)} = \bar{u}_{i(\alpha)}(x_\alpha)$ if $V_\alpha \subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1, 2, 3\}$. Since E is densely embedded in X, and so G_1, G_2, G_3 are densely embedded in D_1, D_2, D_3, respectively. Thus, we may take $\bar{x}_\alpha, \bar{w}_\alpha^{i(\alpha)} \in E$ with $\|\bar{w}_\alpha^{i(\alpha)}\| \leqslant 1$ such that

$$\max\left\{\|w_\alpha^{i(\alpha)} - \bar{w}_\alpha^{i(\alpha)}\|, \|x_\alpha - \bar{x}_\alpha\|\right\} < \min\left\{\frac{1}{2}, \frac{\gamma}{64(1+\|x_\alpha\|)L_{\alpha,i(\alpha)}}\right\}. \tag{3.35}$$

and

$$\bar{x}_\alpha - \bar{w}_\alpha^{i(\alpha)} \in G_i \quad \text{if } x_\alpha \in D_i \cap D_1(r, \bar{\varepsilon}, \delta) \text{ for } i = 1, 2, 3. \tag{3.36}$$

Now, let $\widetilde{v} : D_1(r, \bar{\varepsilon}, \delta) \mapsto X$ be defined by

$$\widetilde{v}(x) = (1+\|x\|) \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(\bar{w}_\alpha^{i(\alpha)} - \bar{x}_\alpha + x). \tag{3.37}$$

Then, by (3.34), (3.35) and (3.37) we have $\|\widetilde{v}(x)\| \leqslant 2(1+\|x\|)$, and $\langle x^*, \widetilde{v}(x)\rangle \geqslant \frac{\gamma}{16}$ for any $x \in D_1(r, \bar{\varepsilon}, \delta)$ and $x^* \in \partial\varphi(x)$.

For a given $x \in D_1(r, \bar{\varepsilon}, \delta)$, we assume that $x \in U_{i(\alpha)}(x_\alpha)$ for some $U_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. Recalling the construction of $\mathscr{A}$, we have

$$U_{i(\alpha)}(x_\alpha) \in \begin{cases} \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 & \text{if } x \in D_1(r, \bar{\varepsilon}, \delta) \backslash (D_2 \cup D_3); \\ \mathscr{A}_2 & \text{if } x \in D_2 \cap D_1(r, \bar{\varepsilon}, \delta); \\ \mathscr{A}_3 & \text{if } x \in D_3 \cap D_1(r, \bar{\varepsilon}, \delta). \end{cases} \tag{3.38}$$

It follows from (3.36), (3.37), (3.38) that (3.32) holds. The proof of is complete. □

Since φ has no nontrivial critical point on $\partial_E G_i$ for $i = 1, 2, 3$, by using the conditions (H_1) we may take $\delta > 0$ small enough such that

$$(\widetilde{K}_r)_{3\delta} \cap (\partial_{G_1} G_2 \cup \partial_{G_1} G_3) = \emptyset. \tag{3.39}$$

Let $O = G_1 \backslash (G_2 \cup G_3)$ and $\widetilde{K}_r^0 = \widetilde{K}_r \cap O$.

Lemma 3.17. *Assume that all the conditions in Theorem 3.14 hold. Let $r \in \mathbb{R}, \delta > 0$ be such that (3.39) holds. Then there exists $\bar{\varepsilon}_0 > 0$ such that for any $0 < \varepsilon < \frac{1}{3}\bar{\varepsilon}_0$ and any compact set $B \subset \varphi^{r+\varepsilon} \cap G_1$, there exists $\eta \in C([0,1] \times G_1, G_1)$ such that*

1) *$\eta(t, x) = x$ for $t = 0$; or $x \notin G_1 \cap D_1(r, \bar{\varepsilon}_0, \delta)$;*

2) *$\eta(1, B \backslash (\widetilde{K}_r^0)_{3\delta}) \subset (O \cap \varphi^{r-\varepsilon}) \cup \operatorname{int}_{G_1} G_2 \cup \operatorname{int}_{G_1} G_3$;*

3) $\eta(t,\cdot)$ *is a homeomorphism of* G_1 *for* $t \in [0,1]$*;*

4) $\varphi(\eta(\cdot,x))$ *is non-increasing for any* $x \in G_1$*;*

5) $\eta(1,G_2) \subset G_2$, $\eta(1,G_3) \subset G_3$*;*

6) $\eta(1,\mathrm{int}_{G_1} G_2) \subset \mathrm{int}_{G_1} G_2$, $\eta(1,\mathrm{int}_{G_1} G_3) \subset \mathrm{int}_{G_1} G_3$.

Proof. It follows from Lemma 3.16 that there exists $\widetilde{v} : D_1(r,\bar{\varepsilon},\delta) \to X$ such that such that $\|\widetilde{v}(x)\| \leqslant 2(1+\|x\|)$ for any $x \in D_1(r,\bar{\varepsilon},\delta)$, $\langle x^*,\widetilde{v}(x)\rangle \geqslant \frac{\gamma}{16}$ for some $\gamma > 0$ and all $x^* \in \partial\varphi(x)$, and (3.32) holds. Since $\widetilde{K}_r$ is compact in X, we may choose $R_0 > 0$ such that $\|x\| \leqslant R_0$ for all $x \in (\widetilde{K}_r)_{3\delta}$. Take

$$0 < \bar{\varepsilon}_0 < \min\left\{\bar{\varepsilon}, \frac{\gamma\delta}{32(1+R_0)}\right\}. \tag{3.40}$$

Take $0 < \varepsilon < \frac{1}{3}\bar{\varepsilon}_0$. Let $l_i : D_1 \to \mathbb{R}$ for $i = 1,2$ be defined by

$$l_1(x) = \frac{d\big(x, D_1\backslash\varphi_{r-\bar{\varepsilon}_0}^{r+\bar{\varepsilon}_0}\big)}{d\big(x, D_1\backslash\varphi_{r-\bar{\varepsilon}_0}^{r+\bar{\varepsilon}_0}\big) + d\big(x, D_1 \cap \varphi_{r-2\varepsilon}^{r+2\varepsilon}\big)},$$

and

$$l_2(x) = \frac{d\big(x, D_1 \cap (\widetilde{K}_r)_\delta\big)}{d\big(x, D_1 \cap (\widetilde{K}_r)_\delta\big) + d\big(x, D_1\backslash(\widetilde{K}_r)_{2\delta}\big)}.$$

Then $l_i : D_1 \to \mathbb{R}$ for $i = 1,2$ is locally Lipschitz continuous in X. It is easy to see that $l_i : G_1 \to \mathbb{R}$ for $i = 1,2$ is locally Lipschitz continuous in E. Let $V(x) = l_1(x)l_2(x)\widetilde{v}_1(x)$ for all $x \in D_1$, where

$$\widetilde{v}_1(x) = \begin{cases} \widetilde{v}(x), & x \in D_1(r,\bar{\varepsilon},\delta); \\ 0, & \text{otherwise}. \end{cases}$$

Then $V : D_1 \to X$ is locally Lipschitz, $V : G_1 \to E$ is also locally Lipschitz.

Consider the initial value problem in X

$$\begin{cases} \frac{d\sigma}{dt} = -V(\sigma(t,u)), \\ \sigma(0) = u \in G_1. \end{cases} \tag{3.41}$$

Obviously, (3.41) has a unique solution $\sigma(t,u)$ in X, with its right maximal existence interval $[0,T(u))$, where $0 < T(u) \leqslant +\infty$. Since $V : G_1 \to E$ is also locally Lipschitz, (3.41) has a unique solution $\sigma(t,u)$ in E, with its right maximal existence interval $[0,T_1(u))$, where $0 < T_1(u) \leqslant T(u) = +\infty$.

In a similar way as in the proof of Lemmas 3.4, 3.6, 3.9,we can show the following conclusions hold: 1) $T(u) = +\infty$; 2). G_1, G_2, G_3, $\mathrm{int}_{G_1} G_2$ and $\mathrm{int}_{G_1} G_3$ are invariant sets for the flow of (3.41). Moreover, the following inequality hold: for any $[t_1,t_2] \subset [0,T_1(u))$,

$$\begin{aligned} \varphi\big(\sigma(t_1,u)\big) - \varphi\big(\sigma(t_2,u)\big) &= -\int_{t_1}^{t_2} \frac{d\varphi(\sigma(s,u))}{ds} ds \\ &\geqslant \frac{\gamma}{16}(t_2 - t_1) \ \text{ if } \ \sigma(s,u) \in G_1 \cap D_1(r,2\varepsilon,2\delta) \text{ for } s \in [t_1,t_2]. \end{aligned} \tag{3.42}$$

Let $B \subset \varphi^{r+\varepsilon} \cap G_1$ be a compact set, $u_0 \in B\backslash(K_r^0)_{3\delta}$ and $o(u_0) = \{\sigma(t,u_0) : t \in [0,T_1(u_0))\}$. Then we have the following cases:

1) $o(u_0) \cap \operatorname{int}_{G_1} G_2 \neq \emptyset$. Assume that $u_1 := \sigma(t(u_0), u_0) \in \operatorname{int}_{G_1} G_2$ for some $t(u_0) \in [0, T_1(u_0))$. Take a neighborhood $U(u_1)$ of u_1 in G_1 such that $U(u_1) \subset \operatorname{int}_{G_1} G_2$. By the continuous dependence of ordinary differential equations on initial data, there exists an open neighborhood $U(u_0)$ of u_0, such that $\sigma(t(u_0), u) \in U(u_1) \subset \operatorname{int}_{G_1} G_2$ for any $u \in U(u_0)$. Since $\operatorname{int}_{G_1} G_2$ is an invariant set, $\sigma(t, u) \in \operatorname{int}_{G_1} G_2$ for all $u \in U(u_0)$ and $t \geqslant t(u_0)$.

2) $o(u_0) \cap \operatorname{int}_{G_1} G_3 \neq \emptyset$. A similar argument as above yields that, there is a neighborhood $U(u_0)$ of u_0 and $t(u_0) > 0$, such that $\sigma(t, u) \in \operatorname{int}_{G_1} G_3$ for any $u \in U(u_0)$ and $t \geqslant t(u_0)$.

3) $o(u_0) \cap \left(\operatorname{int}_{G_1} G_2 \cup \operatorname{int}_{G_1} G_3\right) = \emptyset$. In this case we have the following two subcases:

3a) $o(u_0) \cap (\widetilde{K}_r^0)_{2\delta} = \emptyset$. First we show that $o(u_0) \cap \{x \in X : \varphi(x) < r - \varepsilon\} \neq \emptyset$. Assume indirectly that this is not the case, then $o(u_0) \cap \{x \in X : \varphi(x) < r - \varepsilon\} = \emptyset$. Assume without loss of generality that $o(u_0) \subset \varphi^{-1}([r-\varepsilon, r+\varepsilon])$. Let $Ax = x - \frac{1}{1+\|x\|} V(x)$ for all $x \in D_1$. Note that

$$Ax = \sum_{\alpha \in \Lambda} \gamma_\alpha(x)(\bar{x}_\alpha - \bar{w}_\alpha), \qquad \forall x \in D_1 \cap \varphi^{-1}([r-\varepsilon, r+\varepsilon]) \backslash (\widetilde{K}_r)_{2\delta}.$$

So, in a similar way as in the proof of Lemma 3.5 we can easily show that $T_1(u_0) = +\infty$. Then, we can take $t_1 = \frac{32\varepsilon}{\gamma}$. By (3.42) we have

$$vr(\sigma(t_1, u_0)) \leqslant \varphi(u_0) - \frac{\gamma}{16} t_1 \leqslant \varphi(u_0) - \frac{\gamma}{16} t_1 < r - \varepsilon. \tag{3.43}$$

which is a contradiction. Thus, $o(u_0) \cap \{x \in X : \varphi(x) < r - \varepsilon\} \neq \emptyset$. Assume that $u_1 := \sigma(t(u_0), u_0) \in \{x \in X : \varphi(x) < r - \varepsilon\}$ for some $t(u_0) \geqslant 0$. Take a neighborhood $U(u_1)$ of u_1 such that $U(u_1) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$. Then, there exists an open neighborhood $U(u_0)$ of u_0, such that $\sigma(t(u_0), u) \in U(u_1) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$ for any $u \in U(u_0)$. Since $\sigma(t, u)$ is non-increasing in $t \in [0, +\infty)$, we have $\sigma(t, u) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$ for any $u \in U(u_0)$ and $t \geqslant t(u_0)$.

3b) $o(u_0) \cap (\widetilde{K}_r^0)_{2\delta} \neq \emptyset$. In this case, we may take $[t_1, t_2] \subset [0, +\infty)$ such that $\sigma(t_1, u_0) \in \partial(\widetilde{K}_r)_{3\delta}$, $\sigma(t_2, u_0) \in \partial(\widetilde{K}_r)_{2\delta}$, and $\sigma(t, u_0) \in (\widetilde{K}_r^0)_{3\delta} \backslash (\widetilde{K}_r^0)_{2\delta}$ for $t \in [t_1, t_2]$. By (3.41) we have

$$\begin{aligned}
\|\sigma(t_2, u_0) - \sigma(t_1, u_0)\| &\leqslant \int_{t_1}^{t_2} \|V(\sigma(s, u_0)\| ds \\
&\leqslant 2 \int_{t_1}^{t_2} \big(1 + \|\sigma(s, u_0)\|\big) ds \\
&\leqslant 2\big(1 + R_0\big)(t_2 - t_1).
\end{aligned}$$

So, we have

$$t_2 - t_1 \geqslant \frac{\|\sigma(t_2, u_0) - \sigma(t_1, u_0)\|}{2\big(1 + R_0\big)} \geqslant \frac{\delta}{2\big(1 + R_0\big)}. \tag{3.44}$$

Now we show that there must exist $t(u_0) \in [t_1, t_2]$ such that $\sigma(t(u_0), u_0) \in \{x \in X : \varphi(x) < r - \varepsilon\}$. Assume indirectly that this is not the case, then

$$\{\sigma(t, u_0) : t \in [t_1, t_2]\} \cap \{x \in X : \varphi(x) < r - \varepsilon\} = \emptyset.$$

Then, by (3.42) and (3.44) we have

$$\begin{aligned}
\varphi(\sigma(t_2, u_0)) &\leqslant \varphi(\sigma(t_1, u_0)) - \frac{\gamma}{16}(t_2 - t_1) \\
&\leqslant \varphi(u_0) - \frac{\gamma}{16}(t_2 - t_1) \\
&\leqslant r + \varepsilon - \frac{\gamma\delta}{32\big(1 + R_0\big)} \\
&< r + \varepsilon - \bar{\varepsilon}_0 < r - \varepsilon,
\end{aligned}$$

which is a contradiction. Thus, there must exist $t(u_0) \in [t_1, t_2]$ such that $\sigma(t(u_0), u_0) \in \{x \in X : \varphi(x) < r - \varepsilon\}$. Then, as the arguments in the case 3a), we can show that there exists a open neighborhood of $U(u_0)$ of u_0 such that $\sigma(t, u) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$ for any $u \in U(u_0)$ and $t \geqslant t(u_0)$.

Let $\mathscr{B} = \{U(u_0) : u_0 \in B \backslash (\widetilde{K}_r^0)_{3\delta}\}$. Then $\mathscr{B}$ is an open cover of $B \backslash (\widetilde{K}_r^0)_{3\delta}$. Since $B \backslash (\widetilde{K}_r^0)_{3\delta}$ is a compact set, then there exist finite sets of $\mathscr{B}$, say $U(u_1), U(u_2), \dots, U(u_m)$, such that $B \backslash (\widetilde{K}_r^0)_{3\delta} \subset \bigcup_{i=1}^m U(u_i)$. Let $T = \max\{t(u_1), t(u_2), \dots, t(u_m)\}$ and $\eta(Tt, u)$ for all $t \in [0,1]$ and $u \in B \backslash (\widetilde{K}_r^0)_{3\delta}$. Then we can easily check that all conclusions hold. The proof is complete. $\square$

Proof of Theorem 3.14. Let

$$r = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1]) \cap O} \varphi(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], G_1) : \gamma(0) \in \operatorname{int}_{G_1} G_2, \gamma(1) \in \operatorname{int}_{G_1} G_3\}.$$

Now we show that $\widetilde{K}_r \cap O \neq \emptyset$. Arguing by contradiction that $O \cap \widetilde{K}_r = \emptyset$. Take $\delta > 0$ small enough such that (3.39) holds. By using Lemma 3.17, there exists $\bar{\varepsilon}_0 > 0$ such that for any $0 < \varepsilon < \frac{1}{3}\bar{\varepsilon}_0$ and any compact set $B \subset \varphi^{r+\varepsilon} \cap G_1$, there exists $\eta \in C([0,1] \times G_1, G_1)$ such that the conclusions 1)∼6) in Lemma 3.17 hold.

Take $\gamma \in \Gamma$ such that $\sup_{u \in \gamma([0,1]) \cap O} \varphi(u) < r + \varepsilon$. Let $\gamma_1 = \eta(1, \gamma(\cdot))$. Since

$$\eta(1, \operatorname{int}_{G_1} G_2) \subset \operatorname{int}_{G_1} G_2, \qquad \eta(1, \operatorname{int}_{G_1} G_3) \subset \operatorname{int}_{G_1} G_3,$$

$$\gamma(0) \in \operatorname{int}_{G_1} G_2, \qquad \gamma(1) \in \operatorname{int}_{G_1} G_3,$$

then we have $\gamma_1(0) \in \operatorname{int}_{G_1} G_2$ and $\gamma_1(1) \in \operatorname{int}_{G_1} G_3$. So, $\gamma_1 \in \Gamma$.

On the other hand, we have by 2) in Lemma 3.17,

$$r \leqslant \sup_{u \in \gamma_1([0,1]) \cap O} \varphi(u) \leqslant \sup_{u \in (\varphi^{r-\varepsilon} \cup \operatorname{int}_{G_1} G_2 \cup \operatorname{int}_{G_1} G_3) \cap O} \varphi(u) \leqslant \sup_{u \in \varphi^{r-\varepsilon}} \varphi(u) \leqslant r - \varepsilon,$$

which is a contradiction. So, $\widetilde{K}_r \cap O \neq \emptyset$.

If there exists a curve l in E_1 such that $l \cap D_2 \neq \emptyset$, $l \cap D_3 \neq \emptyset$ and $\max_{u \in l} \varphi(u) < \varphi(0)$, then we have $r < \varphi(0)$. So, $\widetilde{K}_r \cap (O \backslash \{0\}) \neq \emptyset$. The proof is complete. $\square$

Remark 3.18. Here in Theorem 3.14 we have proved a three critical points theorem in ordered interval, which can be thought as a mountain pass theorem in ordered interval. It should be pointed out that Theorem 3.14 can also be obtained by the method showing Theorem 3.1.

3.3 Critical points inside and outside the ordered interval

By Theorem 3.1 and 3.14 we can get the following Theorem 3.19.

Theorem 3.19. *Suppose that (H_1)∼(H_3) hold, φ has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E(D_i \cap E)$ for $i = 1, 2, 3$. Moreover, either*

(H_4) φ is outwardly directed on $\pm P$, D_i for $i = 1, 2, 3$ and $\pm P \cap D_1$; or

(H_5) $(I - \nabla\varphi)(\pm P) \subset \pm P$ and $(I - \nabla\varphi)(D_i) \subset D_i$ for $i = 1, 2, 3$.

Then φ has at least one positive critical point $\bar{u}_1$, one negative critical point $\bar{u}_2$ and one sign-changing critical point $\bar{u}_3$ outside D_1; 2) φ has at least one positive critical point $\bar{u}_4$, one negative critical point $\bar{u}_5$ and one critical point $\bar{u}_6$ inside D_1; moreover, $\bar{u}_6$ is nontrivial if there exists a curve l in E_1 such that $l \cap D_2 \neq \emptyset$, $l \cap D_3 \neq \emptyset$ and $\max_{u\in l} \varphi(u) < \varphi(0)$.

Remark 3.20. Obviously, by the method above we can study the existence of multiple critical points inside and outside the ordered interval which not contain the origin. Moreover, we can give results for at least one or two critical points exist outside the ordered interval, and we can also use the concept of parallel pair of upper and lower solutions to establish the results for multiple critical points. For the sake of brevity, we will only list two of such kind results and not give their proofs.

Theorem 3.21. *Let $u_0 \ll v_1$, $D_1 = [u_0, v_1]$ and $G_1 = D_1 \cap E$. Suppose that (H_1) and (H_3) hold, φ satisfies the condition (PS) and has no critical points on $\partial_E G_1$, $(I - \nabla\varphi)(D_1) \subset D_1$. Then φ has at least one critical point outside D_1, and at least one critical point inside D_1.*

Theorem 3.22. *Let $u_0 \ll u_1 \ll v_1$, $D_1 = [u_0, v_1]$, $D_2 = [u_1, v_1]$, $G_1 = D_1 \cap E$, $G_2 = D_2 \cap E$ and $O_1 = \{x \in X : x \geqslant u_1\}$. Suppose that (H_1) and (H_3) hold, φ satisfies the condition (PS) and has no critical points on $\partial_E G_1 \cup \partial_E(O_1 \cap E) \cup \partial_E G_2$, $(I - \nabla\varphi)(D_1) \subset D_1$, $(I - \nabla\varphi)(D_2) \subset D_2$, and $(I - \nabla\varphi)(O_1) \subset O_1$. Then φ has at least two critical points $\bar{u}_1, \bar{u}_2$ outside D_1 and at least one critical point $\bar{u}_3$ inside D_1 such that $\bar{u}_1 \gg u_1$, $\bar{u}_2 \not\gg u_1$, and $u_1 \ll \bar{u}_3 \ll v_1$.*

Remark 3.23. Obviously, we can also study the existence of fixed points for set value operators using the methods described above. Moreover, our main results also be applicable for φ being of C^1 class, and some of our main results are new even for the case of φ being of C^1 class.

4 Application to the differential inclusion problem with a convex-concave nonlinearity

As the application of Theorem 3.19, in this section we will show multiple solutions of a differential inclusion problem with a convex-concave nonlinearity. The main result of this section extend some relevant results concerning the differential equation boundary value problems with a concave-convex nonlinearity that was first studied by A. Ambrosetti, H. Brezis and G. Cerami [2].

Let

$$\|u\| = \left(\int_\Omega \|Du\|^p dx\right)^{\frac{1}{p}}, \qquad |u|_k = \left(\int_\Omega |u|^k dx\right)^{\frac{1}{k}}$$

be the standard norms of $W_0^{1,p}(\Omega)$, respectively $L^k(\Omega)$ for $1 < k < p^*$. Let $X = W_0^{1,p}(\Omega)$ and $E = C_0^1(\Omega)$.

Consider the following Dirichlet problem for differential inclusion problem

$$\begin{cases} -\mathrm{div}\big(\|Du(x)\|^{p-2}Du(x)\big) - \lambda|u(x)|^{q-2}u(x) \in \partial j(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with a smooth boundary, $1 < q < p < +\infty$, the reaction term $\partial j(x,s)$ is the generalized gradient of a non-smooth potential $s \mapsto j(x,s)$, which is subject to the following conditions.

(H_j) $j : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function and there exist constants $a_1 > 0$, $q < p < r < p^*$ such that

(i) $j(x, \cdot)$ is locally Lipschitz for almost every $x \in \Omega$, $j(x,z) \geqslant 0$ for all $z \in \mathbb{R}$ and $\partial j(x,0) = \{0\}$ a.e. on Ω;

(ii) $|\xi| \leqslant a_1(1+|s|^{r-1})$ a.e. in Ω and for all $s \in \mathbb{R}$, $\xi \in \partial j(x,s)$;

(iii) there exist constants $\mu > p$ and $M > 0$ such that

$$\inf_{x\in\Omega} j(x,M) > 0 \quad \text{and} \quad \mu j(x,z) \leqslant -j^o(x,z;-z) \quad \text{a.e. on } \Omega \text{ all } z \geqslant M;$$

(iv) there exist $M_+ > 0$, $M_- < 0$ such that

$$\max \partial j(x, M_+^{\frac{1}{p-1}} e) < \frac{M_+}{2} \quad \text{and} \quad \min \partial j(x, M_-^{\frac{1}{p-1}} e) > \frac{M_-}{2} \quad \text{a.e. in } \Omega,$$

where $e \in W_0^{1,p}(\Omega)$ such that $-\Delta_p e = 1$.

(v) both $\min \partial j(x,z) + m|z|^{p-2}z$ and $\max \partial j(x,z) + m|z|^{p-2}z$ are nondecreasing in $z \in \mathbb{R}$ for a.e. $x \in \Omega$ and some $m \geqslant 0$.

Remark 4.1. (H_j) (iii) was firstly put forward by [27]. It is a super-linear condition; see (4.14) below. So, (4.1) has a concave-convex nonlinearity. The condition (H_j) (iv) assures the existence of a pair of strict upper and lower solution of (4.1).

For $\lambda > 0$, we introduce the energy functional $\varphi_\lambda : X \mapsto \mathbb{R}$ by

$$\varphi_\lambda(u) = \frac{1}{p}\|u\|_p^p - \frac{\lambda}{q}|u|_q^q - \int_\Omega j(x,u(x))dx,$$

Let $P = \{u \in X : u(x) \geqslant 0 \text{ a.e. } x \in \Omega\}$ and $P_1 = P \cap E$. Given $\lambda > 0$, we say that $u \in X$ is a (weak) solution of (4.1$_\lambda$) if $\Delta_p u \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, and

$$-\Delta_p u(x) \in \lambda |u(x)|^{q-2}u(x) + \partial j(x,u(x)) \quad \text{for almost every } x \in \Omega.$$

Let $K_\lambda = \{x \in X : 0 \in \partial\varphi_\lambda(x)\}$ for $\lambda > 0$.

Theorem 4.2. *Assume (H_j) holds. Then there exists $\lambda^* > 0$, such that for $\lambda \in (0,\lambda^*)$, (4.1) has at least two positive solutions $\bar{u}_1, \bar{u}_2 \in \operatorname{int} P_1$, two negative solutions $\bar{u}_3, \bar{u}_4 \in \operatorname{int}(-P_1)$, one sign-changing solution $\bar{u}_5 \in C_0^1(\bar{\Omega})$ and one nontrivial solution $\bar{u}_6 \in C_0^1(\bar{\Omega})$.*

Example 4.3. Assume $\partial j(x,u) := g(u)$ satisfies:

(a) $g : \mathbb{R} \to \mathbb{R}$ is continuous;

(b) $|g(u)| \leqslant a_1(1+|u|^{r-1})$ for some $a_1 > 0$ and $q < p < r < p^*$;

(c) there exists $\mu > p$ and $M \geqslant 0$ such that $0 < \mu G(u) \leqslant u g(u)$ for all $u \geqslant M$, where $G(u) = \int_0^u g(s)ds$;

(d) $\lim_{u\to 0} \frac{g(u)}{|u|^{p-2}u} = 0$;

(e) there exists $m_0 \geqslant 0$ such that $g(u) + m_0|u|^{p-2}u$ is nondecreasing in u.

Note $e \in L^\infty(\Omega)$ if $e \in W_0^{1,p}(\Omega)$ and $-\Delta_p e = 1$. By the condition (d) we can see the condition (H_j) (iv) holds. Thus, all of the conditions in (H_j) hold if all conditions (a)∼(e) hold. According to Theorem 4.2, we see that, for small enough $\lambda > 0$, the following boundary value problem

$$\begin{cases} -\mathrm{div}\big(\|Du(x)\|^{p-2}Du(x)\big) - \lambda|u(x)|^{q-2}u(x) = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two positive solutions $\bar{u}_1, \bar{u}_2 \in \operatorname{int} P_1$, two negative solutions $\bar{u}_3, \bar{u}_4 \in \operatorname{int}(-P_1)$, one sign-changing solution $\bar{u}_5 \in C_0^1(\bar{\Omega})$ and one nontrivial solution $\bar{u}_6 \in C_0^1(\bar{\Omega})$.

We should point out that in this case $\bar{u}_6$ may be taken as a sign-changing solution. The result of this kind has been obtained in [20]. Hence, Theorem 4.2 can be thought as an extension of some main results in [20].

Example 4.4. Let

$$j(s) = \begin{cases} \frac{a}{r_1}|s|^{r_1}, & \text{if } s < 1, \\ \frac{b}{r_2}|s|^{r_2} + \big(\frac{a}{r_1} - \frac{b}{r_2}\big), & \text{if } s \geqslant 1, \end{cases}$$

where $p < r_1 < r_2 < p^*$, $0 < a < b$. Let $Q = [a, b]$, then we have

$$\partial j(s) = \begin{cases} a|s|^{r_1-2}s, & \text{if } s < 1, \\ Q, & \text{if } s = 1, \\ b|s|^{r_2-2}s, & \text{if } s > 1. \end{cases}$$

It is easy to check that (H_j) holds for the above j.

To show Theorem 4.2 we will apply Theorem 3.19. To this end, we need to check that all conditions in Theorem 3.19 hold. By now, many of these proofs are standard and well known. Nevertheless, for the convenience of the reader, we will give detailed proofs of some lemmas. Some of our proofs refer to [17, 27]. In what follows we will assume that $m = 0$. It is not difficult to show the result holds in the case of $m > 0$.

Let $A : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ be defined by

$$\langle A(u), v\rangle = \int_\Omega \|Du(z)\|^{p-2}(Du(z), Dv(z))_{\mathbb{R}^N} dz \quad \text{for } u, v \in W_0^{1,p}(\Omega).$$

The following Lemma 4.5 can be found in [17, p. 435].

Lemma 4.5. *The mapping $A : W^{1,p} \mapsto W^{-1,p'}(\Omega)$ is continuous and has the $(S)_+$ property, i.e., if $\{u_n\}$ is a sequence in $W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

Recall some facts about the spectrum of the p-Laplacian with Dirichlet boundary condition. Consider the nonlinear eigenvalue problem

$$\begin{cases} -\mathrm{div}\big(\|Du(x)\|^{p-2}Du(x)\big) = \lambda|u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Let λ_1 be the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Then λ_1 is positive, isolated and simple. There is the following variational characterization of λ_1 using Rayleigh quotient:

$$\lambda_1 = \inf\left\{\frac{\|Du\|_p^p}{|u|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right\}.$$

This minimum is actually realized at normalized eigenfunction $\widetilde{u}_1$. The Ljusternik–Schnirelmann theory gives, in addition to λ_1, a whole strictly increasing sequence of positive numbers $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ for which there exist nontrivial solutions for problem (4.2). In what follows we let $\widetilde{u}_2 \in W_0^{1,p}(\Omega)$ be a nontrivial solutions for problem (4.2) corresponding to λ_2, and $E_1 = \operatorname{span}\{\widetilde{u}_1, \widetilde{u}_2\}$.

Lemma 4.6. *If $u \in K_\lambda$, then $u \in C_0^1(\bar{\Omega})$ and u solves (4.1). Moreover, if $u \in \pm P \cap K_\lambda$ and $u \neq 0$, then $u \in \operatorname{int}(\pm P_1) \cap K_\lambda$.*

Proof. The proof is similar to Proposition 3.1 and 3.2 in [17]. Obviously, $u \mapsto \frac{1}{p}\|u\|^p$ is a C^1-functional whose derivative is the operator A. Aubin–Clarke's Theorem ensures that the functional

$$u \mapsto \int_\Omega j(x,u)dx$$

is Lipschitz continuous on any bounded subset of $L^r(\Omega)$ and its gradient is included in the set

$$N(u) = \{w \in L^{r'}(\Omega) : w(x) \in \partial j(x, u(x)) \text{ for almost every } x \in \Omega\}.$$

Since X continuously embedded in $L^r(\Omega)$, the function φ_λ turns out to be locally Lipschitz on X. So, we have

$$\partial\varphi_\lambda(u) \subset A(u) - \lambda|u|^{q-2}u - N(u). \tag{4.3}$$

Now, if $u \in X$ complies with $0 \in \partial\varphi_\lambda(u)$ then

$$A(u) = \lambda|u|^{q-2}u + w \quad \text{in } X^*$$

for some $w \in N(u)$. Hence, $\Delta_p u \in L^{r'}(\Omega)$ and u solves (4.1). By the condition (H_j) (ii) and (4.3) we get the estimate

$$-u\Delta_p u \leqslant a_1(|u| + |u|^r) \quad \text{a.e. in } \Omega.$$

Hence, by [12, Theorem 1.5.5], we have $u \in L^\infty(\Omega)$. From (H_j) (ii) it follows $\Delta_p u \in L^\infty(\Omega)$. So, by [12, Theorem 1.5.6], we have $u \in C_0^1(\bar{\Omega})$.

Let $u \in P \cap K_\lambda$ and $u \neq 0$. By (H_j) (v), we can find a constant $c_0 > 0$

$$\Delta_p u = -\lambda u^{q-1} - w \leqslant c_0 u^{p-1}$$

for some $w \in \partial j(x,u)$. The Vázquez maximum principle yields $u \in \operatorname{int} P_1$.

Similarly, if $u \in -P \cap K_\lambda$ and $u \neq 0$, then $u \in \operatorname{int}(-P_1) \cap K_\lambda$. The proof is complete. □

Lemma 4.7. *If $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded, and either $(1 + \|x_n\|)m(x_n) \to 0$, or $(1 + \|x_n\|)m_{\pm P}(x_n) \to 0$, or $(1 + \|x_n\|)m_D(x_n) \to 0$ as $n \to +\infty$, where D is an ordered interval in X, then $\{x_n\}$ has a convergent subsequence.*

Proof. We only consider the case of $(1+\|x_n\|)m(x_n) \to 0$ as $n \to +\infty$. In a similar way we can prove other cases.

Since $\{x_n\}$ is bounded, by passing to a subsequence if necessary, we may assume

$$x_n \rightharpoonup x \quad \text{in } W_0^{1,p}(\Omega), \quad x_n \to x \quad \text{in } L^k(\Omega) \quad \text{for } 1 < k < p^*, \quad x_n(z) \to x(z) \quad \text{a.e. on } \Omega$$

and $|x_n(z)| \leqslant g(z)$ a.e. on Ω, for all $n \geqslant 1$, with $g \in L^r(\Omega)$. Take $x_n^* \in \partial\varphi_\lambda(x_n)$ such that $m(x_n) = \|x_n^*\|_*$ for $n \geqslant 1$. Then we have

$$x_n^* = A(x_n) - \lambda|x_n|^{q-2}x_n - u_n \tag{4.4}$$

with $u_n \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, satisfying $u_n(x) \in \partial\varphi(x, x_n(x))$ a.e. on Ω.

Now, we can deduce from $(1+\|x_n\|)m(x_n) \to 0$ that $|\langle x_n^*, x_n - x\rangle| \leqslant \frac{1}{n}\|x_n - x\|$. This reads

$$\left|\langle A(x_n), x_n - x\rangle - \lambda \int_\Omega |x_n|^{q-2}x_n(x_n - x)dz - \int_\Omega u_n(x_n - x)dz\right| \leqslant \frac{1}{n}\|x_n - x\|.$$

Then, we have

$$\lambda \int_\Omega |x_n|^{q-2}x_n(x_n - x)dz \to 0 \quad \text{and} \quad \int_\Omega u_n(x_n - x)dz \to 0 \quad \text{as } n \to \infty,$$

and so,

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0.$$

It follows from Lemma 4.5 that $x_n \to x$ in $W_0^{1,p}(\Omega)$. The proof is complete. □

Lemma 4.8. *For $\lambda > 0$ small enough, the functional $\varphi_\lambda : X \mapsto \mathbb{R}$ satisfies the conditions (CPS), $(CPS)_{\pm P}$ and $(CPS)_D$ for any ordered interval $D \subset X$.*

Proof. The proof is similar to claim 1 of Theorem 3.1 in [27]. We only prove that $\varphi_\lambda : X \mapsto \mathbb{R}$ satisfies the conditions $(CPS)_P$ and $(CPS)_D$ for any interval $D \subset X$.

Let $c_1 > 0$ is the best embedding constant of $L^p(\Omega) \hookrightarrow L^q(\Omega)$. Assume that $\widetilde{\lambda} > 0$ be such that

$$\left(\frac{\mu}{p} - 1\right) - \frac{\widetilde{\lambda} c_1^p}{\lambda_1^{\frac{q}{p}}}\left(\frac{\mu}{q} - 1\right) > 0.$$

Let $\lambda \in (0, \widetilde{\lambda})$. In what follows $c_2, \ldots, c_8$ denote some positive constants. Let $\{x_n\} \subset P$ be such that $|\varphi_\lambda(x_n)| \leqslant M_1$ for some $M_1 > 0$, and $(1+\|x_n\|)m_P(x_n) \to 0$ as $n \to \infty$. By Proposition 2.9 that $m_P(x_n) = \langle x_n^*, u(x_n)\rangle$ for some $x_n^* \in \partial\varphi_\lambda(x_n)$ and $u(x_n) \in (x_n - P) \cap \bar{B}(0,1)$. So, we have

$$\langle x_n^*, x_n - y\rangle \leqslant \langle x_n^*, u(x_n)\rangle \quad \text{for all } y \in P, \|x_n - y\| < 1. \tag{4.5}$$

Let $y_n = x_n + \frac{1}{2\|x_n\|}x_n$ for $n \geqslant 1$. Then, $y_n \in P$, $\|x_n - y_n\| < 1$. And so, by (4.5) we have

$$\begin{aligned}
(1+\|x_n\|)\langle x_n^*, x_n - y_n\rangle &= -\frac{(1+\|x_n\|)}{2\|x_n\|}\langle x_n^*, x_n\rangle \\
&= -\frac{(1+\|x_n\|)}{2\|x_n\|}\left(\langle A(x_n), x_n\rangle - \lambda|x_n|_q^q + \int_\Omega j^o(z, x_n(z); -x_n(z))dz\right) \\
&\leqslant (1+\|x_n\|)\langle x_n^*, u(x_n)\rangle =: \varepsilon_n \quad \text{with } \varepsilon_n \downarrow 0.
\end{aligned}$$

So we obtain

$$-\left(\langle A(x_n), x_n\rangle - \lambda|x_n|_q^q + \int_\Omega j^o(z, x_n(z); -x_n(z))dz\right) \leqslant \frac{2\|x_n\|}{(1+\|x_n\|)}\varepsilon_n.$$

which leads

$$-\|Dx_n\|_p^p + \lambda|x_n|_q^q - \int_\Omega j^o(z, x_n(z); -x_n(z))dz \leqslant \frac{2\|x_n\|}{(1+\|x_n\|)}\varepsilon_n = \varepsilon_n', \varepsilon_n' \downarrow 0. \tag{4.6}$$

Assume without loss of generality that $\varepsilon_n' \leqslant 1$ for all n. Since $|\varphi_\lambda(x_n)| \leqslant M_1$ for all $n \geqslant 1$, we have

$$\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda}{q}|x_n|_q^q - \int_\Omega j(z, x_n(z))dz \leqslant M_1. \tag{4.7}$$

By (4.6) and (4.7) we obtain that if $\|x_n\| \geqslant 1$,

$$\begin{aligned}&\left(\frac{\mu}{p}-1\right)\|Dx_n\|_p^p - \lambda\left(\frac{\mu}{q}-1\right)|x_n|_q^q - \int_\Omega \left(\mu j(z, x_n(z)) + j^o(z, x_n(z); -x_n(z))\right)dz \\ &\leqslant \mu M_1 + \varepsilon_n'.\end{aligned} \tag{4.8}$$

By (H_j) (ii)(iii) we have for some $\beta_1 > 0$ (see [27, 2531]),

$$\int_\Omega \left(\mu j(z, x_n(z)) + j^o(z, x_n(z); -x_n(z))\right)dz \geqslant -\beta_1. \tag{4.9}$$

It follows from (4.8) and (4.9) that for $n \geqslant 1$,

$$\left[\left(\frac{\mu}{p}-1\right) - \frac{\lambda c_1^q}{\lambda_1^{\frac{q}{p}}}\left(\frac{\mu}{q}-1\right)\right]\|Dx_n\|_p^p \leqslant \mu M_1 + 1 + \beta_1.$$

Then, we infer that $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded. So, by Lemma 4.7 we see that $\{x_n\}$ has a convergent subsequence. Thus, φ_λ satisfies the condition $(CPS)_P$.

Now we show that φ_λ satisfies the condition $(CPS)_D$ for any ordered interval $D \subset X$. Let $\{x_n\} \subset D$ be such that $|\varphi_\lambda(x_n)| \leqslant M_2$ for some $M_2 > 0$ and $(1+\|x_n\|)m_D(x_n) \to 0$ as $n \to \infty$. Obviously, we see that $\{|x_n|_\mu\}$ is bounded. Take $s > 1$ such that $r < s < \min\left\{p^*, \frac{p\max\{N,p\}+\mu)}{\min\{N,p\}}\right\}$. By (H_j) (ii) we have for a.e. $x \in \Omega$, and $z \in \mathbb{R}$,

$$j(x, z) \leqslant c_2 + c_3|z|^s, c_2, c_3 > 0.$$

Take

$$\theta = \begin{cases}\frac{p^*(s-\mu)}{s(p^*-\mu)}, & N > p,\\ 1-\frac{\mu}{s}, & N \leqslant p.\end{cases}$$

Then we have $0 < \theta < 1$, and

$$\frac{1}{s} = \frac{1-\theta}{\mu} + \frac{\theta}{p^*}.$$

Thus, by using the interpolation inequality and the Sobolev embedding theorem, we obtain

$$|x_n|_s \leqslant |x_n|_\mu^{1-\theta}|x_n|_{p^*}^\theta \leqslant c_4\|x_n\|^\theta \quad \text{for some } c_4 > 0.$$

Since $|\varphi_\lambda(x_n)| \leqslant M_1$, $L^s(\Omega) \hookrightarrow L^p(\Omega)$ and $L^s(\Omega) \hookrightarrow L^q(\Omega)$, by using the well known Young's inequality we have

$$\begin{aligned}\frac{1}{p}\|Dx_n\|_p^p &\leqslant \frac{1}{q}|x_n|_q^q + c_2|\Omega| + c_3|x_n|_s^s + M_1\\ &\leqslant c_5 + c_6|x_n|_s^s \leqslant c_7 + c_8\|Dx_n\|_p^{\theta s}.\end{aligned}$$

If $N > p$, then $Ns < p(N+\mu)$, and so we have

$$\theta s = \frac{p^*(s-\mu)}{p^*-\mu} = \frac{pN}{N-p}\cdot\frac{(s-\mu)(N-p)}{pN-N\mu+p\mu} < \frac{pN}{N-p}\cdot\frac{(s-\mu)(N-p)}{Ns-N\mu} = p.$$

If $N \leqslant p$, then $s < p+\mu$. So,

$$\theta s = \Big(1-\frac{\mu}{s}\Big)s = s-\mu < p.$$

Thus, in both cases we have $\theta s < p$. Consequently, $\{\|x_n\|\}$ is bounded. So, by Lemma 4.6 we see that $\{x_n\}$ has a convergent subsequence. Thus, φ_λ satisfies the condition $(CPS)_D$. The proof is complete. □

By the condition (H$_j$)(iv), we may take $\rho_+ > 1$ and $\lambda_+ > 0$ such that

$$\max \partial j(x, M_+^{\frac{1}{p-1}}e) < \frac{M_+}{2\rho_+} \text{ and } \lambda_+(M_+)^{\frac{q-p}{p-1}}\|e\|_\infty^{q-1} < \frac{1}{2\rho_+}.$$

Then we have for $\lambda \in (0,\lambda_+)$,

$$\rho_+^{-1}\big(-\Delta_p(M_+^{\frac{1}{p-1}}e)\big) > \lambda|M_+^{\frac{1}{p-1}}e|^{q-2}(M_+^{\frac{1}{p-1}}e) + \max\partial j(x, M_+^{\frac{1}{p-1}}e) \quad \text{a.e. in } \Omega. \tag{4.10}$$

Similarly, there exist $\rho_- > 1, \lambda_- > 0$ such that for $\lambda \in (0,\lambda_-)$,

$$\rho_-\big(-\Delta_p(M_-^{\frac{1}{p-1}}e)\big) < \lambda|M_-^{\frac{1}{p-1}}e|^{q-2}(M_-^{\frac{1}{p-1}}e) + \min\partial j(x, M_-^{\frac{1}{p-1}}e) \quad \text{a.e. in } \Omega.$$

Let $\rho = \min\{\rho_+,\rho_-\}$ and $\lambda^* = \min\{\lambda_-,\lambda_+,\widetilde{\lambda}\}$. For each $\lambda \in (0,\lambda^*)$, let

$$\varepsilon_0(\lambda) = \left(\frac{\lambda^{\frac{1}{p-q}}}{(\rho\lambda_1)^{\frac{1}{p-q}}\|\widetilde{u}_1\|_\infty}\right)^{p-1}.$$

Then for each $\varepsilon \in (0,\varepsilon_0(\lambda))$, we have

$$\begin{aligned}\rho\big(-\Delta_p(\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)\big) &= \rho\varepsilon\lambda_1\widetilde{u}_1^{p-1} < \lambda(\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)^{q-1}\\ &\leqslant \lambda(\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)^{q-1} + \min\partial j(x,\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1) \quad \text{a.e. in } \Omega,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\rho^{-1}\big(-\Delta_p(-\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)\big) &= -\varepsilon\lambda_1|\widetilde{u}_1|^{p-2}\widetilde{u}_1\\ &> -\lambda|\varepsilon^{\frac{1}{p-1}}\bar{u}_1|^{q-2}\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1 + \max\partial j(x,-\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1) \quad \text{a.e. in } \Omega.\end{aligned}$$

Since $\widetilde{u}_1, e \in P_1$, there exists $\varepsilon_1(\lambda) \in (0,\varepsilon_0(\lambda))$ such that for each $\lambda \in (0,\lambda^*)$ and $\varepsilon \in (0,\varepsilon_1(\lambda))$,

$$M_-^{\frac{1}{p-1}}e \ll -\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1 \ll 0 \ll \varepsilon^{\frac{1}{p-1}}\widetilde{u}_1 \ll M_+^{\frac{1}{p-1}}e.$$

Let

$$u_0 = M_-^{\frac{1}{p-1}}e, v_0 = -\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1, u_1 = \varepsilon^{\frac{1}{p-1}}\widetilde{u}_1, v_1 = M_+^{\frac{1}{p-1}}e \tag{4.12}$$

and $D_1 = [u_0,v_1], D_2 = [u_0,v_0], D_3 = [u_1,v_1]$.

Lemma 4.9.

1) *There exists $R_0 > 0$ such that for $\lambda \in (0, \lambda^*)$,*

$$\sup_{u \in S_{R_0} \cap E_1} \varphi_\lambda(u) < \varphi(0). \tag{4.13}$$

2) *Let $l_\varepsilon = S_\varepsilon \cap E_1$ such that $l_\varepsilon \cap G_2 \neq \emptyset$ and $l_\varepsilon \cap G_3 \neq \emptyset$. Then there exits $\bar{\varepsilon}_0 > 0$ such that $\alpha(\varepsilon) := \max_{u \in l_\varepsilon} \varphi_\lambda(u) < 0$ for $\varepsilon \in (0, \bar{\varepsilon}_0)$.*

Proof. 1) The proof is similar to claim 3 of Theorem 3.1 in [27]. For almost all $x \in \Omega$ and all $z \in \mathbb{R}$, the function $s \mapsto \frac{1}{s^\mu} j(x, sz)$ is locally Lipschitz on $(0, +\infty)$. Using the mean value theorem for locally Lipschitz functions, for $s > 1$ we can find $\theta \in (1, s)$ such that

$$\begin{aligned} \frac{1}{s^\mu} j(x, sz) - j(x, z) &\in \Big(-\frac{\mu}{\theta^{\mu+1}} j(x, \theta z) + \frac{1}{\theta^\mu} \partial_z j(x, \theta z) z \Big)(s-1) \\ &= \frac{s-1}{\theta^{\mu+1}} \big(-\mu j(x, \theta z) + \partial_z j(x, \theta z) \theta z \big). \end{aligned}$$

By (H_j) (iii), for almost all $x \in \Omega$ and all $z \geqslant M$, we have

$$\frac{1}{s^\mu} j(x, sz) - j(x, z) \geqslant \frac{s-1}{\theta^{\mu+1}} \big(-\mu j(x, \theta z) - j^o(x, \theta z; -\theta z) \big) \geqslant 0.$$

Then for almost all $x \in \Omega$ and all $z \geqslant M$, we have

$$j(x, z) = j\left(x, \frac{z}{M} M\right) \geqslant \left(\frac{z}{M}\right)^\mu j(x, M) \geqslant \left(\frac{z}{M}\right)^\mu \inf_{x \in \Omega} j(x, M).$$

Take $p_1 \in (p, \mu)$. So, it is seen that for a given $\eta > 0$ we can find a constant $c_\eta > 0$ such that

$$j(x, z) \geqslant \frac{\eta}{p} z^{p_1} - c_\eta \quad \text{for a.e. } x \in \Omega. \tag{4.14}$$

Let $\bar{u} \in S_1 := \{u \in W_0^{1,p}(\Omega) : \|Du\|_p = 1\}$. It follows from (4.14) that for $t \in [0, +\infty)$,

$$\begin{aligned} \varphi_\lambda(t\bar{u}) &= \frac{1}{p} \|D(t\bar{u})\|_p^p - \frac{\lambda t^q}{q} |\bar{u}|_q^q - \int_\Omega j(x, t\bar{u}(x)) dx \\ &\leqslant \frac{t^p}{p} - \frac{\lambda t^q}{q} |\bar{u}|_q^q - \frac{\eta t^{p_1}}{p} |\bar{u}|_{p_1}^{p_1} + c_\eta |\Omega|. \end{aligned} \tag{4.15}$$

This implies that

$$\lim_{t \to +\infty} \varphi_\lambda(t\bar{u}) = -\infty.$$

Since $E_1 \cap S_1$ is compact, there exists $R_0 > 0$ such that (4.13) holds.

2) For each $u \in l_\varepsilon$, let $\bar{u} = \frac{u}{\|u\|} \in S_1$. Since $j(x, z) \geqslant 0$ for a.e. $x \in \Omega$ and $z \in \mathbb{R}$, as in the proof of (4.15) we have

$$\begin{aligned} \varphi_\lambda(\varepsilon\bar{u}) &= \frac{1}{p} \|D(\varepsilon\bar{u})\|_p^p - \frac{\lambda \varepsilon^q}{q} |\bar{u}|_q^q - \int_\Omega j(x, \bar{u}(x)) dx \\ &\leqslant \frac{\varepsilon^p}{p} - \frac{\lambda \varepsilon^q}{q} |\bar{u}|_q^q \end{aligned}$$

Thus, there exists $\bar{\varepsilon}_0 > 0$ such that $\alpha(\varepsilon) < 0$ for $\varepsilon \in (0, \bar{\varepsilon}_0)$. The proof is complete. □

Lemma 4.10. *φ_λ is outwardly directed on $\pm P$, D_i and $\pm P \cap D_i$ for $i = 1,2,3$ and $\lambda \in (0,\lambda^*)$. Moreover, φ_λ has no critical point at $\partial_E(\pm P_1)$, and $\partial_E(D_i \cap E)$ for $i = 1,2,3$ and $\lambda \in (0,\lambda^*)$, where D_i be defined by (4.12).*

Proof. We only prove that φ_λ is outwardly directed on $[u_1, v_1]$ and has no critical point on $\partial_E([u_1,v_1] \cap E)$ for $\lambda \in (0,\lambda^*)$. In the same way we can show other cases.

Assume to the contrary that $\partial_E([u_1,v_1] \cap E) \cap K_\lambda \neq \emptyset$. Take $u \in \partial_E([u_1,v_1] \cap E) \cap K_\lambda$. Then we have $u = (-\Delta_p)^{-1}(\lambda|u|^{q-2}u + w)$ for some $w \in N(u)$. It follows from $u_1, v_1 \in E$ and $u_1 \leqslant u \leqslant v_1$ that $u \in L^\infty(\Omega)$. Then we have by (4.10) and (4.11),

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{q-2}u + w \geqslant \lambda|u|^{q-2}u + \min \partial j(x,u) \\ &\geqslant \lambda|u_1|^{q-2}u_1 + \min \partial j(x,u_1) \geqslant \rho\big(-\Delta_p u_1\big), \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{q-2}u + w \leqslant \lambda|u|^{q-2}u + \max \partial j(x,u) \\ &\leqslant \lambda|v_1|^{q-2}v_1 + \max \partial j(x,v_1) \leqslant \rho^{-1}\big(-\Delta_p v_1\big). \end{aligned}$$

As a consequence of the weak comparison theorem [35] we obtain $u \geqslant \rho^{\frac{1}{p-1}} u_1 \gg u_1$ and $v_1 \geqslant \rho^{\frac{1}{p-1}} u \gg u$. This implies that $u \in \operatorname{int}([u_1,v_1] \cap E)$, which is a contradiction. Hence, φ_λ has no critical point on $\partial_E([u_1,v_1] \cap E)$.

The following elementary inequality is well known:

$$\big(|y|^{p-2}y - |h|^{p-2}h, y-h\big)_{\mathbb{R}^N} \geqslant \begin{cases} c_1(p)(|y|+|h|)^{p-2}|y-h|^2 & \text{if } 1<p<2, \\ c_2(p)|y-h|^p & \text{if } p \geqslant 2 \end{cases}$$

for all $y, h \in \mathbb{R}^N$, where $c_1(p), c_2(p) > 0$ are constants.

For $u^* = Au - \lambda|u|^{q-2}u - w \in \partial\varphi_\lambda(u)$, we let $v = (-\Delta_p)^{-1}\big(\lambda|u|^{q-2}u + w\big)$, where $w \in L^{r'}(\Omega)$ and $w \in \partial j(x,u)$. If $u^* \neq 0$, then $u \neq v$, and so

$$\begin{aligned} \langle u^*, u-v\rangle &= \langle Au - \lambda|u|^{q-2}u - w, u-v\rangle \\ &= \langle Au + \Delta_p v, u-v\rangle = \langle Au, u-v\rangle + \langle \Delta_p v, u-v\rangle \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u-v) + \int_\Omega (u-v)\Delta_p v \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u-v) - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u-v) \\ &\geqslant \begin{cases} c_1(p)\int_\Omega(|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2 & \text{if } 1<p<2, \\ c_2(p)\int_\Omega |\nabla(u-v)|^p & \text{if } p \geqslant 2 \end{cases} \\ &> 0. \end{aligned}$$

This implies that φ_λ is outwardly directed on $[u_1, v_1]$. The proof is complete. □

Lemma 4.11. *For any $a, b \in \mathbb{R}$ with $a < b$, $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is compact in E.*

Proof. We follow some ideas in Lemma 3.2 and 3.3 in [3]. For each $u \in K_\lambda \cap \varphi_\lambda^{-1}([a,b])$, it follows from the proof of Lemma 4.6 that $u = (-\Delta_p)^{-1}(\lambda|u|^{q-2}u + w)$ for some $w \in N(u)$. By a similar way as the proof of Lemma 4.8 we can prove that $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in X. Let

$$B(\lambda) := \big\{\lambda|u|^{q-2}u + w : u \in K_\lambda \cap \varphi_\lambda^{-1}([a,b]), w \in N(u)\big\}.$$

If $p > N$ then $X \hookrightarrow L^\infty(\Omega)$. So, $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in $L^\infty(\Omega)$ if $p > N$. It follows the condition (H$_j$) (ii) that the set $B(\lambda)$ is bounded in $L^\infty(\Omega)$. According to [21], there exists $0 < \alpha < 1$ and $c_1 > 0$ such that

$$\|(-\Delta_p)^{-1}u\|_{C^{1,\alpha}} \leqslant c_1\|u\|_\infty^{\frac{1}{p-1}} \quad \text{for all } u \in L^\infty(\Omega). \tag{4.16}$$

Hence, $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in $C^{1,\alpha}(\bar{\Omega})$, and is compact in E.

If $1 < p \leqslant N$, take $p^* > \tilde{r} > \frac{(r-1)N}{p}$. According to [14] we have

$$\|(-\Delta_p)^{-1}u\|_\infty \leqslant c_2\|u\|_{\tilde{r}}^{\frac{1}{p-1}} \quad \text{for all } u \in L^{\tilde{r}}(\Omega).$$

So, for each $u \in K_\lambda \cap \varphi_\lambda^{-1}([a,b])$, $u = (-\Delta_p)^{-1}\big(\lambda|u|^{q-2}u + w\big)$ for some $w \in N(u)$, we have

$$\|u\|_\infty = \|(-\Delta_p)^{-1}(\lambda|u|^{q-2}u + w)\|_\infty \leqslant c_2\|(\lambda|u|^{q-2}u + w)\|_{\tilde{r}}^{\frac{1}{p-1}}. \tag{4.17}$$

It follows from the Sobolev embedding theorem and the condition (H$_j$) (ii) that the set $B(\lambda)$ is bounded in $L^{\tilde{r}}(\Omega)$. Thus, it follows from (4.17) that $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in $L^\infty(\Omega)$. Then, by (4.16) , we see that $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is compact in E. The proof is complete. □

Proof of Theorem 4.2. It follows from Lemmas 4.5∼4.11 that all conditions in Theorem 3.19 hold. According to Theorem 3.19, (4.1) has at least two positive solutions $\bar{u}_1, \bar{u}_2$, two negative solutions $\bar{u}_3, \bar{u}_4$, one sign-changing solution $\bar{u}_5$ and one nontrivial solutions $\bar{u}_6$. The proof is complete. □

Remark 4.12. There have been some papers studied the existence for sign-changing solutions of differential inclusion problems; see [3, 8, 17, 36] and the references therein. For example, by combing variational methods with truncation techniques the paper [17] obtained the existence of positive, negative and nodal solutions to differential inclusion problems with a parameter. Here, our method is different to that in [8, 17].

Remark 4.13. Here, we cannot be sure that the nontrivial solution $\bar{u}_6$ is a sign-changing solution. How to get the nontrivial solution to be a sign-changing solution under the condition of $q < p$ is a problem that needs to be further discussed.

5 Appendix: Another proof of Proposition 2.9

Proof. We follow some ideas to show von Neumann–Sion Saddle-point Theorem. Set $D = \partial\varphi(x_0)$. Let X_w^* and X_w denote the spaces X^* and X furnished their weak topology, respectively. Then $\big((x_0 - C) \cap \bar{B}(0,1)\big)$ is a sequentially compact set in the space X_w. Define $h : D \to \mathbb{R}$ by $h(y^*) = \sup_{x \in \left((x_0 - C)\cap \bar{B}(0,1)\right)} \langle y^*, x\rangle$ for $y^* \in D$. Then $h : D \to \mathbb{R}$ is lower weakly semi-continuous. Note that D is a compact set of X_w^*. Hence, $\alpha = \min_{y^* \in D} \max_{x \in ((x_0 - C)\cap \bar{B}(0,1))} \langle y^*, x\rangle$ exists.

For each $\delta > 0$ and $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$, let $g_x(y^*) = \langle y^*, x\rangle - \alpha + \delta$ for all $y^* \in D$. For each $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$ and $\varepsilon > 0$, let

$$G_{x,\varepsilon} = \{y^* \in D : g_x(y^*) > \varepsilon\}.$$

Since g_x is continuous in X_w^*, $G_{x,\varepsilon}$ is an open subset in X_w^* for each $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$ and $\varepsilon > 0$. Note that the inequalities

$$g_x(y^*) \leqslant 0, \qquad \forall x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$$

has no solution in D. Then, for each $y^* \in D$, there at least exist $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$ such that $g_x(y^*) > 0$. Hence, $\mathscr{A} = \{G_{x,\varepsilon} : x \in \big((x_0 - C) \cap \bar{B}(0,1)\big), \varepsilon > 0\}$ is an open cover of D in X_w^*. Since D is compact in X_w^*, then there exist finite sets in $\mathscr{A}$, say $G_{x_1,\varepsilon_1}, G_{x_2,\varepsilon_2}, \ldots, G_{x_m,\varepsilon_m}$, such that $D \subset \bigcup_{i=1}^m G_{x_i,\varepsilon_i}$. Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$. Then we have $D \subset \bigcup_{i=1}^m G_{x_i,\varepsilon_0}$.

Let $T : D \to \mathbb{R}^m$ be defined by

$$T(y^*) = \big(g_{x_1}(y^*), g_{x_2}(y^*), \ldots, g_{x_m}(y^*)\big).$$

Set $A = \{(y_1, y_2, \ldots, y_m) : y_i < \varepsilon_0, i = 1, 2, \ldots, m\}$. Then, we have $\mathrm{co}\,\big(T(D)\big) \cap A = \emptyset$. In fact, for any $y = (y_1, y_2, \ldots, y_m) \in \mathrm{co}\,\big(T(D)\big)$, there exist $z_1, z_2, \ldots, z_\gamma \in T(D)$ and $t_i \geqslant 0$ for $1 \leqslant i \leqslant \gamma$, $\sum_{i=1}^m t_i = 1$ such that $y = \sum_{i=1}^\gamma t_i z_i$. Moreover, for each $z_i \in T(D)$, $1 \leqslant i \leqslant \gamma$, there exist $y_i^* (1 \leqslant i \leqslant \gamma)$ such that

$$z_i = \big(g_{x_1}(y_i^*), g_{x_2}(y_i^*), \ldots, g_{x_m}(y_i^*)\big), \qquad i = 1, 2, \ldots, \gamma.$$

Thus, we have

$$y_j = \sum_{i=1}^\gamma t_i g_{x_j}(y_i^*) = g_{x_j}\left(\sum_{i=1}^\gamma t_i y_i^*\right), \qquad j = 1, 2, \ldots, m.$$

Since $\sum_{i=1}^\gamma t_i y_i^* \in D$, then there exists $j_0 \in \{1, 2, \ldots, m\}$ such that $y_{j_0} = g_{x_{j_0}}\big(\sum_{i=1}^\gamma t_i y_i^*\big) > \varepsilon_0$. This implies that $y \notin A$, and so $\mathrm{co}\,(T(D)) \cap A = \emptyset$.

By using the Eidelheit convex separation theorem, there exists $\lambda_1', \lambda_2', \ldots, \lambda_n'$, such that $\sum_{i+1}^m \lambda_i' y_i \leqslant 1$ for all $(y_1, y_2, \ldots, y_m) \in A$, and $\sum_{i=1}^m \lambda_i' g_{x_i}(y^*) \geqslant 1$ for each $y^* \in D$ and $T(y^*) = \big(g_{x_1}(y^*), g_{x_2}(y^*), \ldots, g_{x_m}(y^*)\big) \in T(D)$. It is easy to see that $\lambda_i' \geqslant 0$ and $(\lambda_1', \lambda_2', \ldots, \lambda_m') \neq (0, 0, \ldots, 0)$. Let $\lambda = \sum_{i=1}^m \lambda_i'$ and $\lambda_i = \frac{\lambda_i'}{\lambda}$ for $i = 1, 2, \ldots, m$. Then we have for all $y^* \in D$,

$$g(y^*) := \sum_{i=1}^m \lambda_i g_{x_i}(y^*) \geqslant \frac{1}{\lambda} > 0.$$

Note that

$$g(y^*) = \sum_{i=1}^m \lambda_i \big(\langle y^*, x_i \rangle - \alpha + \delta\big) = \left\langle y^*, \sum_{i=1}^m \lambda_i x_i \right\rangle - \alpha + \delta.$$

Thus, we have for all $y^* \in D$,

$$\left\langle y^*, \sum_{i=1}^m \lambda_i x_i \right\rangle > \alpha - \delta.$$

So, we have

$$\min_{y^* \in D} \left\langle y^*, \sum_{i=1}^m \lambda_i x_i \right\rangle \geqslant \alpha - \delta.$$

Since $\delta > 0$ is arbitrarily given, we have

$$\max_{x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)} \min_{y^* \in D} \langle y^*, x \rangle \geqslant \alpha.$$

On the other hand, we have

$$\min_{z^*\in D}\langle z^*,x\rangle \leqslant \langle y^*,x\rangle, \qquad \forall y^*\in D, x\in \big((x_0-C)\cap \bar{B}(0,1)\big),$$

and so

$$\max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\ \min_{z^*\in D}\langle z^*,x\rangle \leqslant \max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\langle y^*,x\rangle.$$

Hence,

$$\max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\ \min_{z^*\in D}\langle z^*,x\rangle \leqslant \min_{y^*\in D}\ \max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\langle y^*,x\rangle = \alpha.$$

Then we have

$$\max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\ \min_{y^*\in D}\langle y^*,x\rangle = \min_{y^*\in D}\ \max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\langle y^*,x\rangle = \alpha.$$

Moreover, there exists $(x_0^*,x_0)\in D\times\big((x_0-C)\cap\bar{B}(0,1)\big)$ such that

$$\max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\ \min_{y^*\in D}\langle y^*,x\rangle = \langle x_0^*,x_0\rangle = \min_{y^*\in D}\ \max_{x\in\big((x_0-C)\cap\bar{B}(0,1)\big)}\langle y^*,x\rangle.$$

This implies that for all $x\in\big((x_0-C)\cap\bar{B}(0,1)\big)$ and $y^*\in D$,

$$\langle x_0^*,x\rangle \leqslant \langle x_0^*,x_0\rangle \leqslant \langle y^*,x_0\rangle.$$

The proof is complete. □

Declarations

- **Data availability** Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
- **Conflict of interests** On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgements

This paper is supported by NSFC 11871250 and Natural Science Foundation of Shandong Province ZR2021MA087, ZR2023MA039.

References

[1] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, *J. Functional Analysis* **11**(1972), 346–384. https://doi.org/10.1016/0022-1236(72)90074-2; MR0358470; Zbl 0244.47046

[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **122**(1994), No. 2, 519–543. https://doi.org/10.1006/jfan.1994.1078; MR1276168; Zbl 0805.35028

[3] T. Bartsch, Z. Liu, On a superlinear elliptic p-Laplacian equation, *J. Differential Equations* **198**(2004), No. 1, 149–175. https://doi.org/10.1016/j.jde.2003.08.001; MR2037753; Zbl 1087.35034

[4] H. H. Bauschke, P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, Second edition, Springer, New York, 2017. https://doi.org/10.1007/978-3-319-48311-5; MR3616647; Zbl 1359.26003

[5] S. Carl, V. K. Le, D. Motreanu, *Nonsmooth variational problems and their inequalities*, Springer Monogr. Math., Springer, New York, 2007. https://doi.org/10.1007/978-0-387-46252-3; MR2267795; Zbl 1109.35004

[6] K.-C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80**(1981) 102–129. https://doi.org/10.1016/0022-247X(81)90095-0; MR0614246; Zbl 0487.49027

[7] F. H. Clarke, *Optimization and nonsmooth analysis*, Classics in Applied Mathematics, Vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990. https://doi.org/10.1137/1.9781611971309; MR1058436; Zbl 0696.49002

[8] F. Colasuonno, A. Iannizzotto, D. Mugnai, Three solutions for a Neumann partial differential inclusion via nonsmooth Morse theory, *Set-Valued Var. Anal.* **25**(2017), No. 2, 405–425. https://doi.org/10.1007/s11228-016-0387-2; MR3649577; Zbl 1371.49012

[9] N. Costea, A. Kristály, C. Varga, *Variational and monotonicity methods in nonsmooth analysis* Front. Math., Birkhäuser/Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-81671-1; MR4368827; Zbl 1490.49001

[10] E. N. Dancer, Z. Zhang, Fucik spectrum, sign-changing, and multiple solutions for semilinear elliptic boundary value problems with resonance at infinity. *J. Math. Anal. Appl.* **250**(2000), No. 2, 449–464. https://doi.org/10.1006/jmaa.2000.6969; MR1786075; Zbl 0974.35028

[11] K. Deimling, *Ordinary differential equations in Banach spaces*, Lecture Notes in Mathematics, Vol. 596, Springer-Verlag, Berlin–Heidelberg–New York, 1977. https://doi.org/10.1007/BFb0091636; MR0463601; Zbl 0361.34050

[12] L. Gasiński, N. S. Papageorgiou, *Nonsmooth critical point theory and nonlinear boundary value problems*, Ser. Math. Anal. Appl., Vol. 8, Chapman and Hall/CRC Press, Boca Raton, 2005. MR2092433; Zbl 1058.58005

[13] L. Gasiński, N. S. Papageorgiou, *Nonlinear analysis*, Ser. Math. Anal. Appl., Vol. 9, Chapman and Hall/CRC Press, Boca Raton, 2005. https://doi.org/10.1201/9781420035049; Zbl 1086.47001

[14] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, 2nd Edition, Springer, Berlin, 2001. https://doi.org/10.1007/978-3-642-61798-0; MR1814364; Zbl 1042.35002;

[15] H. Hofer, Variational and topological methods in partially ordered Hilbert space, *Math. Ann.* **261**(1982), 493–514. https://doi.org/10.1007/BF01457453; MR0682663; Zbl 0488.47034

[16] S. Hu, N. S. Papageorgiou, Semilinear Robin problems with indefinite potential and competition phenomena, *Acta Appl. Math.* **168**(2020), 187–216. https://doi.org/10.1007/s10440-019-00284-y; MR4128682; Zbl 1448.35208

[17] A. Iannizzotto, S. A. Marano, D. Motreanu, Positive, negative, and nodal solutions to elliptic differential inclusions depending on a parameter, *Adv. Nonlinear Stud.* **13**(2013), No. 2, 431–445. https://doi.org/10.1515/ans-2013-0210; MR3086880; Zbl 1271.49010

[18] S. Th. Kyritsi, N. S. Papageorgiou, Nonsmooth critical point theory on closed convex sets and nonlinear hemivariational inequalities, *Nonlinear Anal.* **61**(2005), 373–403. https://doi.org/10.1016/j.na.2004.12.001; MR2123083; Zbl 1067.49005

[19] Y. Li, Almost critical points for functionals defined on convex sets, *An. Univ. București Mat.* **47**(1998), 121–126. MR1794897; Zbl 0976.49011

[20] S. Li, Z.-Q. Wang, Mountain pass theorem in order intervals and multiple solutions for semilinear elliptic Dirichlet problems. *J. Anal. Math.* **81**(2000), 373–396. https://doi.org/10.1007/BF02788997; MR1785289; Zbl 0962.35065

[21] G. M. Liebermann, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12**(1988), 1203–1219. https://doi.org/10.1016/0362-546X(88)90053-3; MR0969499

[22] Z. Liu, J. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, *J. Differential Equations* **172**(2001), No. 2, 257–299. https://doi.org/10.1006/jdeq.2000.3867; MR1829631; Zbl 0995.58006

[23] R. Livrea, S. A. Marano, Non-smooth critical point theory, in: D. Y. Gao, D. Motreanu (Eds.), *Handbook of nonconvex analysis and applications*, International Press, Somerville, 2010, pp. 353–407. MR2768814; Zbl 1216.58004

[24] S. A. Marano, S. J. N. Mosconi, Multiple solutions to elliptic inclusions via critical point theory on closed convex sets, *Discrete Contin. Dyn. Syst.* **35**(2015), 3087–3102. https://doi.org/10.3934/dcds.2015.35.3087; MR3343555; Zbl 1344.49024

[25] S. A. Marano, S. J. N. Mosconi, Critical points on closed convex sets vs. critical points and applications, *J. Convex Anal.* **22**(2015), No. 4, 1107–1124. MR3436703; Zbl 1334.49052

[26] A. Moameni, Critical point theory on convex subsets with applications in differential equations and analysis. *J. Math. Pures Appl. (9)* **141**(2020), 266–315. https://doi.org/10.1016/j.matpur.2020.05.005; MR4134457; Zbl 1446.58008

[27] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou, Positive solutions and multiple solutions at non-resonance, resonance and near resonance for hemivariational inequalities with p-Laplacian, *Trans. Amer. Math. Soc.* **360**(2008), No. 5, 2527–2545. https://doi.org/10.1090/S0002-9947-07-04449-2; MR2373324; Zbl 1143.35076

[28] D. Motreanu, D. O'Regan, N. S. Papageorgiou, A unified treatment using critical point methods of the existence of multiple solutions for superlinear and sublinear Neumann problems, *Commun. Pure Appl. Anal.* **10**(2011), No. 6, 1791–1816. https://doi.org/10.3934/cpaa.2011.10.1791; MR2805340; Zbl 1234.35080

[29] D. Motreanu, P. D. Panagiotopoulos, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities*, Nonconvex Optimization and its Applications, Vol. 29, Kluwer Academic Publishers, Dordrecht, 1999. https://doi.org/10.1007/978-1-4615-4064-9; Zbl 1060.49500

[30] J. von Neumann, Über ein okonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, *Erg. Math. Kolloqu.* **8**(1937), 73–83. Zbl 0017.03901

[31] M. Struwe, *Variational methods*, Springer, fourth edition, Berlin, 2008. https://doi.org/10.1007/978-3-540-74013-1

[32] J. Sun, *On some problems about nonlinear operators*, Ph.D. thesis, Shandong University, Jinan, 1984.

[33] J. Sun, The Schauder condition in the critical point theory, *Chinese Sci. Bull.* **31**(1986), 1157–1162. MR0866081; Zbl 0603.47045

[34] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3**(1986), No. 2, 77–109. MR0837231; Zbl 0612.58011

[35] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(1984) 191–202. https://doi.org/10.1007/BF01449041; MR0768629; Zbl 0561.35003

[36] Z. Zhang, J. Chen, S. Li, Construction of pseudo-gradient vector field and sign-changing multiple solutions involving p-Laplacian, *J. Differential Equations* **201**(2004), No. 2, 287–303. https://doi.org/10.1016/j.jde.2004.03.019; MR2059609; Zbl 1079.35035