Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **69**, 1–27; https://doi.org/10.14232/ejqtde.2024.1.69 www.math.u-szeged.hu/ejqtde/

Weak solutions for a class of quasilinear elliptic equations containing the $p(\cdot)$-Laplacian and the mean curvature operator in a variable exponent Sobolev space

Junichi Aramaki ✉ [1]

[1] Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama, 350-0394, Japan

Received 11 August 2024, appeared 6 December 2024
Communicated by Patrizia Pucci

Abstract. In this paper, we consider the equation for a class of nonlinear operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions in a bounded domain Ω of $\mathbb{R}^N$, under the hypothesis $p(x) > 1$ in $\overline{\Omega}$. More precisely, we are concerned with the problem under the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of one, two and infinitely many nontrivial weak solutions of the equation according to the conditions on given functions.

Keywords: $p(\cdot)$-Laplacian type operator, mean curvature operator, mixed boundary value problem, variable exponent Sobolev space.

2020 Mathematics Subject Classification: 35D30, 35A01, 35J62, 35J57.

1 Introduction

In this paper, we consider the following equation

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \nabla u(x))] = f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \nabla u(x)) = g(x, u(x)) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary $\partial\Omega = \Gamma$ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

✉ Corresponding author. Email: aramaki@hctv.ne.jp

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi}) = \nabla_{\xi} A(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p \in C(\overline{\Omega})$ with $1 < p(x)$ for $x \in \overline{\Omega}$. Then the operator div $[\boldsymbol{a}(x, \nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian

$$\Delta_{p(x)} u(x) = \text{div}\,[|\nabla u(x)|^{p(x)-2} \nabla u(x)]$$

and the mean curvature operator

$$\text{div}\,[(1 + |\nabla u(x)|^2)^{(p(x)-2)/2} \nabla u(x)].$$

These generalities bring about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying some conditions.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [31]), in electrorheological fluids (Diening [10], Halsey [19], Mihăilescu and Rădulescu [22], Růžička [24]).

Since we can only find a few of papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). See Aramaki [2, 5]. We are convinced of the reason for existence of this paper.

Fan [13] considered the problem (1.1) when $A(x,\boldsymbol{\xi}) = \frac{1}{p(x)} |\boldsymbol{\xi}|^{p(x)}$ and $\Gamma_2 = \emptyset$, and derived the existence of a nontrivial weak solution to (1.1). Yücedağ [29] and Mashiyev et al. [21] and many authors extended the result to the case where $A(x,\boldsymbol{\xi})$ satisfies the $p(\cdot)$-uniform convexity. In Aramaki [3] and Dai and Hao [8], the authors treated the Kirchhoff-type operator in the case where $A(x,\boldsymbol{\xi})$ satisfies the $p(\cdot)$-uniform convexity. Here the $p(\cdot)$-uniform convexity of $A(x,\boldsymbol{\xi})$ means that

$$A\left(x, \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) + c|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} \leq \frac{1}{2} A(x,\boldsymbol{\xi}) + \frac{1}{2} A(x,\boldsymbol{\eta}) \tag{1.3}$$

for a.e. $x \in \Omega$ and all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$ with some constant $c > 0$. However, even in the case where $A(x,\boldsymbol{\xi}) = \frac{1}{p(x)} |\boldsymbol{\xi}|^{p(x)}$, in general, if $1 < p(x) < 2$ in a non-empty subset of Ω, then this $p(\cdot)$-uniform convexity does not hold. Of course, if $p(x) \geq 2$ in Ω, then (1.3) holds.

In this paper, we give up this condition, but we assume that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is uniformly monotone (see (A.2) below in Section 3), because we think that this hypothesis is more natural for the $p(\cdot)$-Laplacian and the mean curvature operator, and allow not only the case $2 \leq p(x)$ in $\overline{\Omega}$, but also the case $1 < p(x)$ in $\overline{\Omega}$. To overcome this, if we apply a version of the idea of Glowinski and A. Marroco [18] who treated the case $p(x) = p =$ const., then we get Proposition 3.7 below. So our results are new, because the results contain the case $1 < p(x)$ in $\overline{\Omega}$.

We derive that there exist one, two and infinitely many nontrivial weak solutions. We use the standard Mountain-Pass Theorem, Ekeland variational principle and the Symmetric Mountain-Pass Theorem, respectively (cf. Aramaki [4, 6], [21]).

This paper is also an extension of the articles [13] to the case of mixed boundary value problem and of a class of operators containing the $p(\cdot)$-Laplacian and the mean curvature operator with the case where $p(x) > 1$ in $\overline{\Omega}$.

The paper is organized as follows. In Section 2, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the assumptions to the main theorems. In Section 4, we state the main theorems (Theorem 4.3, 4.5 and 4.6) on the existence of at least one, two and infinitely many nontrivial weak solutions according to the hypotheses on given functions f and g. The proofs of these main theorems are given in Section 5.

2 Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [15], Kováčik and Rácosník [20] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define $C(\overline{\Omega}) = \{p; p \text{ is a continuous function on } \overline{\Omega}\}$, and for any $p \in C(\overline{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x \in \Omega} p(x) \text{ and } p^- = p^-(\Omega) = \inf_{x \in \Omega} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0; \rho_{p(\cdot)} \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

where ∇u is the gradient of u, that is, $\nabla u = (\partial_1 u, \dots, \partial_N u)$, $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}.$$

The following three propositions are well known (see Fan et al. [16], Fan and Zhao [17], Zhao et al. [30]).

Proposition 2.1. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$, and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \dots)$. Then we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1(=1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1(=1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let* $p \in C_+(\overline{\Omega})$, *where*

$$C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); p^- > 1\}.$$

For any $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\int_\Omega |u(x)v(x)|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\overline{\Omega})$, $p'(\cdot)$ *denote the conjugate exponent of* $p(\cdot)$, *that is,* $p'(x) = p(x)/(p(x) - 1)$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with* $C^{0,1}$*-boundary and let* $p \in C_+(\overline{\Omega})$. *Then we have the following properties.*

(i) *The spaces* $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If* $q(x) \in C(\overline{\Omega})$ *with* $q^- \ge 1$ *satisfies that* $q(x) \le p^*(x)$ *for all* $x \in \Omega$, *then the embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, *where* $\hookrightarrow$ *means that the embedding is continuous.*

(iii) *If* $q(x) \in C(\overline{\Omega})$ *with* $q^- \ge 1$ *satisfies that* $q(x) < p^*(x)$ *for all* $x \in \Omega$, *then the embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

Next we consider the trace (cf. Fan [14]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p \in C(\overline{\Omega})$ with $p^- \ge 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $\gamma(u) = u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma) = \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\}$$

for $f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then we can see that $(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later we also write $F\big|_\Gamma = g$ by $F = g$ on Γ. Moreover, for $i = 1, 2$, we denote

$$(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\|g\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}; f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in C_+(\Gamma) := \{q \in C(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma_x$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma_x \\ \text{satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma_x < \infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma_x \le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma_x.$$

Similarly as Proposition 2.1, we have the following proposition.

Proposition 2.4. *Let $q \in C(\Gamma)$ with $q^- \ge 1$, and let $u, u_n \in L^{q(\cdot)}(\Gamma)$. Then we have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{q(\cdot),\Gamma}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} > 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(iii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

(iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to 0 \Longleftrightarrow \rho_{q(\cdot),\Gamma}(u_n) \to 0$.

(v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to \infty \Longleftrightarrow \rho_{q(\cdot),\Gamma}(u_n) \to \infty$.

The Hölder inequality also holds for functions on Γ.

Proposition 2.5. *Let $q \in C(\Gamma)$ with $q^- > 1$. Then the following inequality holds.*

$$\int_\Gamma |f(x)g(x)| d\sigma_x \le 2\|f\|_{L^{q(\cdot)}(\Gamma)}\|g\|_{L^{q'(\cdot)}(\Gamma)} \quad \text{for all } f \in L^{q(\cdot)}(\Gamma), g \in L^{q'(\cdot)}(\Gamma).$$

Proposition 2.6. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}.$$

In particular, If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_i)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}$ for $i = 1, 2$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

The following proposition follows from Yao [28, Proposition 2.6].

Proposition 2.7. *Let $p \in C_+(\overline{\Omega})$. Then if $q \in C_+(\Gamma)$ satisfies $q(x) \leq p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well-defined, continuous and*

$$\|u\|_{L^{q(\cdot)}(\Gamma)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)} \text{ for } u \in W^{1,p(\cdot)}(\Omega)$$

for some constant $C > 0$.

In particular, if $q(x) < p^\partial(x)$ for all $x \in \Gamma_2$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is compact.

Now we consider the weighted variable exponent Lebesgue space. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x) > 0$ a.e. $x \in \Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u) = \int_\Omega a(x)|u(x)|^{p(x)}dx \text{ for any measurable function } u \text{ in } \Omega.$$

Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega) = \left\{u; u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u) < \infty\right\}$$

equipped with the norm

$$\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. [13, Proposition 2.5]).

Proposition 2.8. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$. For $u, u_n \in L^{p(\cdot)}_{a(\cdot)}(\Omega)$, we have the following.*

(i) *For $u \neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \lambda \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}\left(\frac{u}{\lambda}\right) = 1$.*

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1\ (= 1, > 1) \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}(u) < 1\ (= 1, > 1)$.

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(v) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{(p(\cdot),a(\cdot))}(u_n - u) = 0$.

(vi) $\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}(u_n) \to \infty$ *as* $n \to \infty$.

The author of [13] also derived the following proposition (cf. [13, Theorem 2.1]).

Proposition 2.9. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and $p \in C_+(\overline{\Omega})$. Moreover, let $a \in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x) > 0$ a.e. $x \in \Omega$ and $\alpha \in C_+(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ satisfies*

$$1 \leq q(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \quad \text{for all } x \in \overline{\Omega},$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact.

Similarly, let $q \in C(\Gamma)$ with $q^- \geq 1$ and let $b(x)$ be a measurable function with respect to σ on Γ with $b(x) > 0$ σ-a.e. $x \in \Gamma$. We define a modular

$$\rho_{(q(\cdot),b(\cdot)),\Gamma}(u) = \int_\Gamma b(x)|u(x)|^{q(x)} d\sigma_x.$$

Then the weighted Lebesgue space on Γ is defined by

$$L^{q(\cdot)}_{b(\cdot)}(\Gamma) = \{u; u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma b(x) \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma_x \leq 1\right\}.$$

Then $L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ is a Banach space.

Then we have the following proposition.

Proposition 2.10. *Let $q \in C(\Gamma)$ with $q^- \geq 1$. For $u, u_n \in L^{q(\cdot)}_{b(\cdot)}(\Gamma)$, we have the following.*

(i) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1\ (= 1, > 1) \iff \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < 1\ (= 1, > 1)$.

(ii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} > 1 \implies \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \leq \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \leq \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 \implies \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \leq \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \leq \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = 0 \iff \lim_{n\to\infty} \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \to \infty$ *as* $n \to \infty \iff \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition plays an important role in the present paper.

Proposition 2.11. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. Assume that $0 < b \in L^{\beta(\cdot)}(\Gamma)$, $\beta \in C_+(\Gamma)$. If $r \in C(\Gamma)$ satisfies*

$$1 \leq r(x) < \frac{\beta(x) - 1}{\beta(x)} p^\partial(x) \quad \text{for all } x \in \Gamma,$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact.

The following proposition is due to Edmunds and Rákosník [11, Lemma 2.1].

Proposition 2.12. *Let $q \in L^\infty(\Omega)$ and p be a measurable function on Ω such that $1 \leq p(x) \leq \infty$ and $1 \leq q(x)p(x) \leq \infty$. Assume that $f \in L^{p(\cdot)}(\Omega)$ with $f \neq 0$. Then we have the following.*

$$\|f\|_{L^{q(\cdot)p(\cdot)}(\Omega)} \leq 1 \Longrightarrow \|f\|^{q^+}_{L^{q(\cdot)p(\cdot)}(\Omega)} \leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|^{q^-}_{L^{q(\cdot)p(\cdot)}(\Omega)}.$$
$$\|f\|_{L^{q(\cdot)p(\cdot)}(\Omega)} \geq 1 \Longrightarrow \|f\|^{q^-}_{L^{q(\cdot)p(\cdot)}(\Omega)} \leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|^{q^+}_{L^{q(\cdot)p(\cdot)}(\Omega)}.$$

In particular, if $q(x) = q =$ const., then $\||f|^q\|_{L^{p(\cdot)}(\Omega)} = \|f\|^q_{L^{qp(\cdot)}(\Omega)}$.

Define a space by
$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$
Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We get the following Poincaré-type inequality (cf. Ciarlet and Dinca [7]).

Proposition 2.13. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and let $p \in C_+(\overline{\Omega})$. Then there exists a constant $C = C(\Omega, N, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \textit{for all } u \in X.$$

In particular, $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see Aramaki [1, Lemma 2.5].
Thus we can define the norm on X so that

$$\|v\|_X = \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \quad \text{for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.13.

3 Assumptions to the main theorems

In this section, we state the assumptions to the main theorems. Let $p \in C_+(\overline{\Omega})$ be fixed.
Throughout this paper, we assume the following.

(A.0) Let $A : \Omega \times \mathbb{R}^N \to [0, \infty)$ be a function satisfying that for a.e. $x \in \Omega$ the function $A(x, \cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x, \boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$ the function $A(\cdot, \boldsymbol{\xi}) : \Omega \ni x \mapsto A(x, \boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x, \mathbf{0}) = 0$ and put $\boldsymbol{a}(x, \boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})$. Then $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$.

Moreover, we assume the following structure conditions. There exist constants $C_0, k_0 > 0$, nonnegative functions $h_0 \in L^{p'(\cdot)}(\Omega)$ and $h_1 \in L^1(\Omega)$ with $h_1(x) \geq 1$ for a.e. $x \in \Omega$ such that the following conditions hold.

(A.1) $|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq C_0(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.2) $\boldsymbol{a}(x, \mathbf{0}) = \mathbf{0}$ for a.e. $x \in \Omega$ and

$$(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \geq \begin{cases} k_0 h_1(x)|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)} & \text{if } p(x) \geq 2, \\ k_0 h_1(x)(1 + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi} - \boldsymbol{\eta}|^2 & \text{if } p(x) < 2 \end{cases}$$

for a.e. $x \in \Omega$ and all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$.

(A.3) A is $p(\cdot)$-subhomogeneous in the sense of

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi}\le p(x)A(x,\boldsymbol{\xi})+h_1(x) \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^N \text{ and a.e. } x\in\Omega.$$

Lemma 3.1. *Under (A.0) and (A.2), there exists a constant $c>0$ such that*

$$\frac{1}{2}A(x,\boldsymbol{\xi})+\frac{1}{2}A(x,\boldsymbol{\eta})-A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)\ge\begin{cases}c\,h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} & \text{if } p(x)\ge 2,\\ c\,h_1(x)(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 & \text{if } p(x)<2\end{cases}$$

for a.e. $x\in\Omega$ and all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N$.

In particular, $A(x,\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$.

Proof. Since

$$A(x,\boldsymbol{\eta})-A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)=\int_0^1\boldsymbol{a}\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}+s\left(\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)\right)\cdot\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}ds,$$

and

$$A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)-A(x,\boldsymbol{\xi})=\int_0^1\boldsymbol{a}\left(x,\boldsymbol{\xi}+s\left(\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)\right)\cdot\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}ds,$$

it follows from (A.0) and (A.2) that

$$\begin{aligned}&\frac{1}{2}A(x,\boldsymbol{\xi})+\frac{1}{2}A(x,\boldsymbol{\eta})-A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)\\&=\frac{1}{2}\int_0^1\left(\boldsymbol{a}\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)-\boldsymbol{a}\left(x,\boldsymbol{\xi}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)\right)\cdot\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}ds\\&\ge\begin{cases}\frac{1}{2}k_0h_1(x)\left|\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|^{p(x)} & \text{if } p(x)\ge 2,\\ \frac{1}{2}k_0h_1(x)\int_0^1\left(1+\left|\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|+\left|\boldsymbol{\xi}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|\right)^{p(x)-2}\left|\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|^2ds & \text{if } p(x)<2\end{cases}\\&\ge\begin{cases}\left(\frac{1}{2}\right)^{p^++1}k_0h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} & \text{if } p(x)\ge 2,\\ \frac{1}{4}k_0h_1(x)(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 & \text{if } p(x)<2.\end{cases}\end{aligned}$$

In particular, since $A(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2})\le\frac{1}{2}A(x,\boldsymbol{\xi})+\frac{1}{2}A(x,\boldsymbol{\eta})$ and $A(x,\boldsymbol{\xi})$ is continuous with respect to $\boldsymbol{\xi}$, it is well known that $A(x,\boldsymbol{\xi})$ is convex. □

Example 3.2.

(i) $A(x,\boldsymbol{\xi})=\frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $h\in L^1(\Omega)$ satisfying $h(x)\ge 1$ for a.e. $x\in\Omega$.

(ii) $A(x,\boldsymbol{\xi})=\frac{h(x)}{p(x)}((1+|\boldsymbol{\xi}|^2)^{p(x)/2}-1)$ with $h\in L^{p'(\cdot)}(\Omega)$ satisfying $h(x)\ge 1$ for a.e. $x\in\Omega$.

Then $A(x,\boldsymbol{\xi})$ and $\boldsymbol{a}(x,\boldsymbol{\xi})=\boldsymbol{\nabla}_{\boldsymbol{\xi}}A(x,\boldsymbol{\xi})$ satisfy the above assumptions (A.0)–(A.3).

Proof. In the case (i), $A(x,\boldsymbol{\xi})$ is clearly differentiable with respect to $\boldsymbol{\xi}$ for $\boldsymbol{\xi}\ne\boldsymbol{0}$ and $\boldsymbol{a}(x,\boldsymbol{\xi})=h(x)|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi}$ for $\boldsymbol{\xi}\ne\boldsymbol{0}$. Since $p(x)>1$, if we define $\boldsymbol{a}(x,\boldsymbol{0})=\boldsymbol{0}$, then we see that $A(x,\boldsymbol{\xi})$ is of C^1-class with respect to $\boldsymbol{\xi}$, so (A.0) holds. (A.1) easily holds. If we use the well-known inequality (cf. Thelin [25]): there exists a constant $k_0>0$ such that

$$(|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi}-|\boldsymbol{\eta}|^{p(x)-2}\boldsymbol{\eta})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\ge\begin{cases}k_0|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} & \text{if } p(x)\ge 2,\\ k_0(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 & \text{if } p(x)<2,\end{cases}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$, then we see that (A.2) holds. We can easily see that (A.3) holds.

In the case (ii), clearly $A(x, \boldsymbol{\xi})$ is of C^1-class with respect to $\boldsymbol{\xi}$ and $\boldsymbol{a}(x, \boldsymbol{\xi}) = h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}\boldsymbol{\xi}$.

If $p(x) \geq 2$, since $|\boldsymbol{\xi}| \leq 1 + |\boldsymbol{\xi}|^{p(x)-1}$, we have

$$|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq h(x)2^{(p^+-2)/2}(1 + |\boldsymbol{\xi}|^{p(x)-2})|\boldsymbol{\xi}| \leq 2^{p^+/2}(h(x) + h(x)|\boldsymbol{\xi}|^{p(x)-1}).$$

If $p(x) < 2$,

$$|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq h(x)|\boldsymbol{\xi}|^{p(x)-2}|\boldsymbol{\xi}| = h(x)|\boldsymbol{\xi}|^{p(x)-1}.$$

Thus (A.1) with $h_0 = h_1 = h$ holds. We show that (A.2) holds. We have

$$\begin{aligned}
&(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\
&\quad = h(x) \int_0^1 \frac{d}{ds} \Big[(1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2}(s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta})\Big]\, ds \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\
&\quad = h(x) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\
&\qquad + h(x)(p(x) - 2) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-4)/2} |(s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta})|^2 ds.
\end{aligned}$$

If $p(x) \geq 2$, it follows from DiBenedetto [9, p. 14] that

$$(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \geq h(x) \int_0^1 |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^{p(x)-2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \geq k_0 h(x)|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)}.$$

If $p(x) < 2$, we have

$$\begin{aligned}
&(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\
&\quad \geq h(x) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\
&\qquad + h(x)(p(x) - 2) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-4)/2} |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2 |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 ds \\
&\quad \geq h(x)(p(x) - 1) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\
&\quad \geq (p^- - 1)h(x)(1 + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p(x)-2} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2.
\end{aligned}$$

Thus (A.2) holds. We show that (A.3) holds.

$$\begin{aligned}
\boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} &= h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2} |\boldsymbol{\xi}|^2 \\
&= h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}(1 + |\boldsymbol{\xi}|^2 - 1) \\
&= h(x)(1 + |\boldsymbol{\xi}|^2)^{p(x)/2} - h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2} \\
&= p(x)A(x, \boldsymbol{\xi}) + h(x)(1 - (1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}) \\
&\leq p(x)A(x, \boldsymbol{\xi}) + h(x).
\end{aligned}$$

If $p(x) \geq 2$, then we can delete the last term $h(x)$, however if $p(x) < 2$, then we can not delete the last term $h(x)$ since $\{(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}; \boldsymbol{\xi} \in \mathbb{R}^N\} = [0, 1]$. □

Remark 3.3.

(i) When $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

(ii) In many papers (for example, [29], [21], [6], [4]), the authors assume that $\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \le p(x)A(x,\boldsymbol{\xi})$ instead of (A.3). However, in the above Example 3.2 we saw that if the example (ii) satisfies $1 < p(x) < 2$ in a subset of Ω with positive measure, then we have to assume (A.3).

Lemma 3.4. *Under (A.0)–(A.2), we have the following.*

(i) $|A(x,\boldsymbol{\xi})| \le C_0(h_0(x)|\boldsymbol{\xi}| + h_1(x)|\boldsymbol{\xi}|^{p(x)})$ *for a.e.* $x\in\Omega$ *and all* $\boldsymbol{\xi}\in\mathbb{R}^N$.

(ii) *There exist constants* $c>0$ *and* $C\ge 0$ *such that*

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \ge ch_1(x)|\boldsymbol{\xi}|^{p(x)} - Ch_1(x) \text{ for a.e. } x\in\Omega \text{ and all } \boldsymbol{\xi}\in\mathbb{R}^N.$$

In particular, if $p^- \ge 2$, *then we can take* $C=0$.

Proof. (i) From (A.0) and (A.1), we have

$$|A(x,\boldsymbol{\xi})| = |A(x,\boldsymbol{\xi}) - A(x,\mathbf{0})| = \left|\int_0^1 \frac{d}{dt}A(x,t\boldsymbol{\xi})dt\right| = \left|\int_0^1 \boldsymbol{a}(x,t\boldsymbol{\xi})\cdot\boldsymbol{\xi}dt\right| \le C_0(h_0(x)|\boldsymbol{\xi}| + h_1(x)|\boldsymbol{\xi}|^{p(x)}).$$

(ii) Since it follows from (A.2) with $\boldsymbol{\eta}=\mathbf{0}$ that

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \ge \begin{cases} k_0h_1(x)|\boldsymbol{\xi}|^{p(x)} & \text{if } p(x)\ge 2,\\ k_0h_1(x)(1+|\boldsymbol{\xi}|)^{p(x)-2})|\boldsymbol{\xi}|^2 & \text{if } p(x)<2,\end{cases}$$

it suffices to show that when $p(x)<2$, we have $(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge c'|\boldsymbol{\xi}|^{p(x)} - C'$ for some constant $c',C'>0$. Using an elementary inequality $(a+b)^q \le 2^q(a^q+b^q)$ for real numbers $a,b\ge 0$ and $q>0$, we have

$$(1+|\boldsymbol{\xi}|)^{2-p(x)} \le 2^{2-p(x)}(|\boldsymbol{\xi}|^{2-p(x)}+1) \le 2^{2-p^-}|\boldsymbol{\xi}|^{2-p(x)} + 2^{2-p^-}.$$

Thereby, $|\boldsymbol{\xi}|^{2-p(x)} \ge 2^{p^--2}(1+|\boldsymbol{\xi}|)^{2-p(x)} - 1$. When $|\boldsymbol{\xi}|\le 1$, since $p(x)-1>0$, we have

$$\begin{aligned}(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 &= (1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^{2-p(x)}|\boldsymbol{\xi}|^{p(x)}\\ &\ge (1+|\boldsymbol{\xi}|)^{p(x)-2}(2^{p^--2}(1+|\boldsymbol{\xi}|)^{2-p(x)}-1)|\boldsymbol{\xi}|^{p(x)}\\ &= 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - (1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^{p(x)}\\ &\ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - (2|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^{p(x)}\\ &\ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - 2^{p^+-2}|\boldsymbol{\xi}|^{2(p(x)-1)}\\ &\ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - 2^{p^+-2}.\end{aligned}$$

When $|\boldsymbol{\xi}|\ge 1$, we have $(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge (2|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)}$. Therefore, we have $(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - 2^{p^+-2}$ for all $\boldsymbol{\xi}\in\mathbb{R}^N$. □

For the function $h_1\in L^1(\Omega)$ with $h_1(x)\ge 1$ for a.e. $x\in\Omega$, we define a modular

$$\rho_{p(\cdot),h_1(\cdot)}(v) = \rho_{p(\cdot),h_1(\cdot),\Omega}(v) = \int_\Omega h_1(x)|\nabla v(x)|^{p(x)}dx \quad \text{for } v\in Y,$$

where Y is our basic space defined by

$$Y = Y(\Omega) = \{v \in X; \rho_{p(\cdot),h_1(\cdot)}(v) < \infty\}, \tag{3.1}$$

the space X is defined by (2.1), equipped with the norm

$$\|v\|_Y = \inf\left\{\lambda > 0; \rho_{p(\cdot),h_1(\cdot)}\left(\frac{v}{\lambda}\right) \le 1\right\}.$$

Then Y is a Banach space (see Proposition 3.5 below). We note that $C_0^\infty(\Omega) \subset Y$. Since

$$\rho_{p(\cdot),h_1(\cdot)}(v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)}\nabla v),$$

we have

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}. \tag{3.2}$$

We have the following propositions.

Proposition 3.5. *The space $(Y, \|\cdot\|_Y)$ is a separable and reflexive Banach space.*

For the proof, see [4, Lemma 2.12].

Proposition 3.6. *Let Y be the above Banach space defined by (3.1) and X be the space defined by (2.1). Then we have the following properties.*

(i) *$Y \hookrightarrow X$ and $\|v\|_X \le \|v\|_Y$ for all $v \in Y$.*

(ii) *Let $v \in Y$. Then $\|v\|_Y > 1 (= 1, < 1) \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(v) > 1 (= 1, < 1)$.*

(iii) *Let $v \in Y$. Then $\|v\|_Y > 1 \Longrightarrow \|v\|_Y^{p^-} \le \rho_{p(\cdot),h_1(\cdot)}(v) \le \|v\|_Y^{p^+}$.*

(iv) *Let $v \in Y$. Then $\|v\|_Y < 1 \Longrightarrow \|v\|_Y^{p^+} \le \rho_{p(\cdot),h_1(\cdot)}(v) \le \|v\|_Y^{p^-}$.*

(v) *Let $u_n, u \in Y$. Then $\lim_{n\to\infty} \|u_n - u\|_Y = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot),h_1(\cdot)}(u_n - u) = 0$.*

(vi) *Let $u_n \in Y$. Then $\|u_n\|_Y \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(u_n) \to \infty$ as $n \to \infty$.*

The following proposition fulfills an important role in this paper. In the following, we denote positive constants by c, c', C, C' which may vary from line to line, and put $\Omega_1 = \{x \in \Omega; p(x) \ge 2\}$, $\Omega_2 = \{x \in \Omega; p(x) < 2\}$.

Proposition 3.7. *Under (A.0)–(A.2), there exist positive constants c and C such that*

$$\begin{aligned}\int_\Omega (\boldsymbol{a}(x, \nabla u(x)) - \boldsymbol{a}(x, \nabla v(x))) \cdot (\nabla u(x) - \nabla v(x))dx &\ge c\rho_{h_1(\cdot),p(\cdot),\Omega_1}(u - v) \\ &+ \left\{c(C + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{h_1(\cdot),p(\cdot),\Omega_2}(u - v)\right\}^{2/p^+(\Omega_2)} \\ &\wedge \left\{c(C + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{h_1(\cdot),p(\cdot),\Omega_2}(u - v)\right\}^{2/p^-(\Omega_2)}\end{aligned}$$

for $u, v \in Y$. Here and from now on, we denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for real numbers a and b.

In particular, if $v = 0$ and $\|u\|_Y < 1$, then we have

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx \ge c_1(\rho_{h_1(\cdot),p(\cdot),\Omega_1}(u) + \rho_{h_1(\cdot),p(\cdot),\Omega_2}(u)^{2/p^-})$$

for some constant $c_1 > 0$. We also get the following estimate.

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx \ge c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C\|h_1\|_{L^1(\Omega)} \quad \text{for all } u \in Y. \tag{3.3}$$

Proof. For brevity of notation, for $u, v \in Y$, we put

$$J(u(x); v(x)) = (\boldsymbol{a}(x, \nabla u(x)) - \boldsymbol{a}(x, \nabla v(x))) \cdot (\nabla u(x) - \nabla v(x)).$$

We decompose the integral of $J(u(x); v(x))$ over Ω as follows.

$$\int_\Omega J(u(x); v(x))dx = \int_{\Omega_1} J(u(x); v(x))dx + \int_{\Omega_2} J(u(x); v(x))dx.$$

We can easily see that when $|\Omega_1| > 0$, it follows from (A.2) that

$$\int_{\Omega_1} J(u(x); v(x))dx \geq k_0 \int_{\Omega_1} h_1(x)|\nabla u(x) - \nabla v(x)|^{p(x)}dx.$$

When $|\Omega_2| > 0$, it follows from (A.2) that

$$\begin{aligned}(h_1(x)^{1/p(x)} + h_1(x)^{1/p(x)}|\nabla u(x)| + h_1(x)^{1/p(x)}|\nabla v(x)|)^{2-p(x)} J(u(x); v(x))\\ \geq k_0 |h_1(x)^{1/p(x)} \nabla u(x) - h_1(x)^{1/p(x)} \nabla v(x)|^2.\end{aligned}$$

By integrating $p(x)/2$-powers of the above inequality over Ω_2, we have

$$\begin{aligned}&\int_{\Omega_2} k_0^{p(x)/2} |h_1(x)^{1/p(x)} \nabla u(x) - h_1(x)^{1/p(x)} \nabla v(x)|^{p(x)}dx\\ &\quad\leq \int_{\Omega_2} (h_1(x)^{1/p(x)} + h_1(x)^{1/p(x)}|\nabla u(x)| + h_1(x)^{1/p(x)}|\nabla v(x)|)^{(2-p(x))p(x)/2}\\ &\qquad\qquad \times J(u(x); v(x))^{p(x)/2}dx.\end{aligned}$$

We note that

$$(h_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2} \in L^{2/(2-p(\cdot))}(\Omega_2),$$

and $(J(u(\cdot); v(\cdot))^{p(\cdot)/2} \in L^{2/p(\cdot)}(\Omega_2)$, and $(2 - p(x))/2 + p(x)/2 = 1$. By the Hölder inequality (Proposition 2.2), we have

$$\begin{aligned}&k_1 \int_{\Omega_2} h_1(x)^{1/p(x)} |\nabla u(x) - h_1(x)^{1/p(x)} \nabla v(x)|^{p(x)}dx\\ &\quad\leq 2\|(h_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2}\|_{L^{2/(2-p(\cdot))}(\Omega_2)}\\ &\qquad\qquad \times \|J(u(\cdot); v(\cdot))^{p(\cdot)/2}\|_{L^{2/p(\cdot)}(\Omega_2)},\end{aligned}$$

where $k_1 = k_0^{p^+(\Omega_2)/2} \wedge k_0^{p^-(\Omega_2)/2}$. We choose $C > 1$ so that $C\|h_1(\cdot)^{1/p(\cdot)}\|_{L^{p(\cdot)}(\Omega_2)} \geq 1$. Then $\|Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)} \geq 1$ by the definition of $L^{p(\cdot)}$-norm. By Proposition 2.12,

$$\begin{aligned}&\|(Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2}\|_{L^{2/(2-p(\cdot))}(\Omega_2)}\\ &\qquad \leq \|Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)}^{((2-p(\cdot))p(\cdot)/2)^+(\Omega_2)}.\end{aligned}$$

Here since $(2 - p(x))p(x)/2 = -\frac{1}{2}(p(x) - 1)^2 + \frac{1}{2}$, we see that $(2 - p(x))p(x)/2)^+(\Omega_2) = (2 - p^-(\Omega_2))p^-(\Omega_2)/2$. Since it follows from Proposition 2.12 that

$$\|Ch_1(\cdot)^{1/p(\cdot)}\|_{L^{p(\cdot)}(\Omega_2)} \leq C\|h_1\|_{L^1(\Omega)}^{1/p^+(\Omega_2)} \vee \|h_1\|_{L^1(\Omega)}^{1/p^-(\Omega_2)} =: C_1,$$

and $\|h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)} \le \|u\|_Y$ and $\|h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)} \le \|v\|_Y$, we have

$$\|(Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2}\|_{L^{2/(2-p(\cdot))}(\Omega_2)} \le (C_1 + \|u\|_Y + \|v\|_Y)^{(2-p^-(\Omega_2))p^-(\Omega_2)/2}.$$

Using Proposition 2.12,

$$\|J(u(\cdot);v(\cdot))^{p(\cdot)/2}\|_{L^{2/p(\cdot)}(\Omega_2)} \le \|J(u(\cdot);v(\cdot))\|_{L^1(\Omega_2)}^{p^+(\Omega_2)/2} \vee \|J(u(\cdot);v(\cdot))\|_{L^1(\Omega_2)}^{p^-(\Omega_2)/2}.$$

Hence we have

$$\begin{aligned}\int_{\Omega_2} J(u(x);v(x))dx &= \|J(u(\cdot);v(\cdot))\|_{L^1(\Omega_2)} \\ &\ge \left\{(C_1 + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} k_1 \int_{\Omega_2} h_1(x)|\nabla u(x) - \nabla v(x)|^{p(x)}dx\right\}^{2/p^+(\Omega_2)} \\ &\quad \wedge \left\{(C_1 + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} k_1 \int_{\Omega_2} h_1(x)|\nabla u(x) - \nabla v(x)|^{p(x)}dx\right\}^{2/p^-(\Omega_2)}.\end{aligned}$$

In particular case where $v = 0$ and $\|u\|_Y < 1$,

$$\begin{aligned}\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx &\ge k_0 \int_{\Omega_1} h_1(x)|\nabla u(x)|^{p(x)}dx \\ &\quad + (C_1+1)^{p^-(\Omega_2)-2} k_1^{2/p^+(\Omega_2)} \wedge k_1^{2/p^-(\Omega_2)} \left\{\int_{\Omega_2} h_1(x)|\nabla u(x)|^{p(x)}dx\right\}^{2/p^-(\Omega_2)}.\end{aligned}$$

For the estimate (3.3), it suffices to use Lemma 3.4 (ii). This completes the proof. □

Here we state the assumptions on functions f and g in (1.1).

(f.1) $f = f(x,t)$ is a real Carathéodory function on $\Omega \times \mathbb{R}$ and there exist $1 \le a \in L^{\alpha(\cdot)}(\Omega)$ with $\alpha \in C_+(\overline{\Omega})$, and $q \in C_+(\overline{\Omega})$ with

$$q(x) < \frac{\alpha(x)-1}{\alpha(x)} p^*(x) \quad \text{for all } x \in \overline{\Omega}$$

such that

$$|f(x,t)| \le C_1(1 + a(x)|t|^{q(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where C_1 is a positive constant and $p^+ < q^-$.

(f.2) There exist $\theta > p^+$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) \le f(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Omega,$$

where

$$F(x,t) = \int_0^t f(x,s)ds. \tag{3.4}$$

(f.3) $f(x,t) = o(|t|^{p^+-1})$ uniformly as $t \to 0$.

(g.1) $g = g(x,t)$ is a real Carathéodory function on $\Gamma_2 \times \mathbb{R}$ and there exist $1 \leq b \in L^{\beta\cdot)}(\Gamma_2)$ with $\beta \in C_+(\overline{\Gamma_2})$, and $r \in C_+(\overline{\Gamma_2})$ with

$$r(x) < \frac{\beta(x)-1}{\beta(x)} p^{\partial}(x) \quad \text{for all } x \in \overline{\Gamma_2}$$

such that

$$|g(x,t)| \leq C_2(1 + b(x)|t|^{r(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and } \sigma\text{-a.e. } x \in \Gamma_2,$$

where C_2 is a positive constant and $p^+ < r^-$.

(g.2) Let θ and t_0 be as in (f.2). That is, there exist $\theta > p^+(\Omega_1) \vee 2p^+(\Omega_2)/p^-(\Omega_2)$ and $t_0 > 0$ such that

$$0 < \theta G(x,t) \leq g(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Gamma_2,$$

where

$$G(x,t) = \int_0^t g(x,s)ds. \tag{3.5}$$

(g.3) $g(x,t) = o(|t|^{p^+-1})$ uniformly as $t \to 0$.

Lemma 3.8. *Under (f.1)–(f.3) and (g.1)–(g.3), we have the following.*

(i) *For any $\lambda > 0$, there exists a constant $C_1' > 0$ such that*

$$|F(x,t)| \leq \frac{\lambda}{p^+}|t|^{p^+} + C_1' a(x)|t|^{q(x)} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)} \quad \textit{for a.e. } x \in \Omega, t \in \mathbb{R}.$$

(ii) *For any $\lambda > 0$, there exists a constant $C_2' > 0$ such that*

$$|G(x,t)| \leq \frac{\lambda}{p^+}|t|^{p^+} + C_2' b(x)|t|^{r(x)} \quad \textit{for } \sigma\textit{-a.e. } x \in \Gamma_2, t \in \mathbb{R}.$$

Proof. From (f.3), for any $\lambda > 0$, there exists $\delta \in (0,1)$ such that

$$|f(x,t)| \leq \lambda |t|^{p^+-1} \quad \text{for a.e. } x \in \Omega, t \in (-\delta, \delta).$$

Hence we have

$$|F(x,t)| \leq \frac{\lambda}{p^+}|t|^{p^+} \quad \text{for a.e. } x \in \Omega, t \in (-\delta, \delta).$$

On the other hand, from (f.1), we have

$$|F(x,t)| \leq C_1(|t| + \frac{a(x)}{q(x)}|t|^{q(x)}) \leq C_2' a(x)|t|^{q(x)} \text{ for a.e. } x \in \Omega, |t| \geq \delta.$$

If we choose $C_2'' > 0$ so that $C_2''\delta^{q^+} \geq \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)}$, then

$$C_2'' a(x)|t|^{q(x)} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)} \geq C_2''\delta^{q^+} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)} \geq 0$$

for a.e. $x \in \Omega$ and $|t| \geq \delta$. Hence $|F(x,t)| \leq (C_2' + C_2'')a(x)|t|^{q(x)} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)}$ for a.e. $x \in \Omega$ and $|t| \geq \delta$. It suffices to put $C_1' = C_2' + C_2''$.

Similarly (ii) holds. □

Define a functional on Y by

$$I(u) = \Phi(u) - J(u) - K(u) \quad \text{for } u \in Y, \tag{3.6}$$

where

$$\Phi(u) = \int_\Omega A(x, \nabla u(x))dx, \tag{3.7}$$

$$J(u) = \int_\Omega F(x, u(x))dx,\ F(x,t) \quad \text{is defined by (3.4)}, \tag{3.8}$$

$$K(u) = \int_{\Gamma_2} G(x, u(x))d\sigma_x,\ G(x,t) \quad \text{is defined by (3.5)}. \tag{3.9}$$

Proposition 3.9. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then the functionals* $\Phi, J, K \in C^1(Y, \mathbb{R})$ *and the Fréchet derivatives* Φ', J' *and* K' *satisfy the following equalities.*

$$\langle \Phi'(u), v\rangle = \int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x)dx, \tag{3.10}$$

$$\langle J'(u), v\rangle = \int_\Omega f(x, u(x))v(x)dx, \tag{3.11}$$

$$\langle K'(u), v\rangle = \int_{\Gamma_2} g(x, u(x))v(x)d\sigma_x \tag{3.12}$$

for all $u, v \in Y$*. Here and hereafter, we write the duality* $\langle \cdot, \cdot\rangle_{Y^*,Y}$ *by simply* $\langle \cdot, \cdot\rangle$*.*

For the proof, see [4, Proposition 4.2].

Proposition 3.10. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then we have the following.*

(i) *The functionals* J *and* K *are weakly continuous in* Y*, that is, if* $u_n \to u$ *weakly in* Y *as* $n \to \infty$*, then* $J(u_n) \to J(u)$ *and* $K(u_n) \to K(u)$ *as* $n \to \infty$*.*

(ii) *The functional* Φ *is sequentially weakly lower semi-continuous in* Y*, that is, if* $u_n \to u$ *weakly in* Y *as* $n \to \infty$*, then* $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$*.*

(iii) $\Phi(u) - \Phi(v) \ge \langle \Phi'(v), u - v\rangle$ *for all* $u, v \in Y$*.*

For the proof, see [4, Proposition 4.4].

Lemma 3.11. *Under (f.1)–(f.3) and (g.1)–(g.3), there exist constants* c_1*,* c_2*,* C_3 *and* C_4 *such that for* $u \in Y$ *with* $\|u\|_Y < 1$*, the following inequalities hold.*

(i) *We have*

$$J(u) \le \frac{\lambda}{p^+} c_1 \|u\|_Y^{p^+} + C_3 \|u\|_Y^{q^-} - \|h_1/p\|_{L^1(\Omega)}.$$

(ii) *We have*

$$K(u) \le \frac{\lambda}{p^+} c_2 \|u\|_Y^{p^+} + C_4 \|u\|_Y^{r^-}.$$

Proof. From Lemma 3.8,

$$J(u) \le \frac{\lambda}{p^+}\int_\Omega |u(x)|^{p^+}dx + C_3\int_\Omega a(x)|u(x)|^{q(x)}dx - \|h_1/p\|_{L^1(\Omega)}.$$

Here it suffices to note that

$$\int_\Omega |u(x)|^{p^+}dx \le C\|u\|_Y^{p^+}$$

with some constant $C > 0$, and

$$\int_\Omega a(x)|u(x)|^{q(x)}dx \le C'\|u\|_Y^{q^-}.$$

(ii) follows from the similar arguments as (i). □

Proposition 3.12. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then there exist constants $c, c_1, c_2 > 0$ and $C_1', C_2' > 0$ such that for $u \in Y$ with $\|u\|_Y < 1$,*

$$I(u) \ge (c - \lambda c_1 - \lambda c_2)\,\|u\|_Y^{p^+} - C_1'\|u\|_Y^{q^-} - C_2'\|u\|_Y^{r^-}.$$

In particular, there exists $\rho \in (0,1)$ such that

$$\inf_{\|u\|_Y=\rho} I(u) > 0. \tag{3.13}$$

Proof. Let $\|u\|_Y < 1$. It follows from (A.3) and Proposition 3.7 that

$$\begin{aligned}\Phi(u) = \int_\Omega A(x,\nabla u(x))dx &\ge \int_\Omega \frac{1}{p(x)}\boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)dx - \|h_1/p\|_{L^1(\Omega)}\\ &\ge c\|u\|_Y^{p^+} - \|h_1/p\|_{L^1(\Omega)}.\end{aligned}$$

From Lemma 3.11,

$$I(u) = \Phi(u) - J(u) - K(u) \ge (c - \lambda c_1 - \lambda c_2)\|u\|_Y^{p^+} - C_1'\|u\|_Y^{q^-} - C_2'\|u\|_Y^{r^-}.$$

If we choose $\lambda > 0$ small enough so that $c'' := c - \lambda c_1 - \lambda c_2 > 0$, then we have

$$I(u) \ge \|u\|_Y^{p^+}(c'' - C_1'\|u\|_Y^{q^- - p^+} - C_2'\|u\|_Y^{r^- - p^+}).$$

Since $q^- > p^+$ and $r^- > p^+$, if $\|u\|_Y = \rho > 0$ is small, then we have $\inf_{\|u\|_Y=\rho} I(u) > 0$. □

Proposition 3.13. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then there exists a constant $C_4 > 0$ such that*

$$I(u) \ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} + \frac{1}{\theta}\langle I'(u), u\rangle - C_4 \quad \text{for all } u \in Y.$$

Proof. From (A.3) and Lemma 3.4 (ii), for $u \in Y$, we have

$$\begin{aligned}\Phi(u) - \frac{1}{\theta}\langle\Phi'(u),u\rangle &= \int_\Omega A(x,\nabla u(x))dx - \frac{1}{\theta}\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)dx\\ &\ge \int_\Omega\left(\frac{1}{p(x)} - \frac{1}{\theta}\right)\boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)dx - \|h_1/p\|_{L^1(\Omega)}\\ &\ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\left(c\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx - C\int_\Omega h_1(x)dx\right) - \|h_1/p\|_{L^1(\Omega)}\\ &\ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right)c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C_1\|h_1\|_{L^1(\Omega)}\end{aligned}$$

for some constant $C_1 > 0$.

On the other hand, it follows from (f.2) that

$$0 < \theta F(x,t) \le f(x,t)t \text{ for a.e. } x \in \Omega, t \in \mathbb{R} \setminus (-t_0, t_0).$$

Put $\Omega_u = \{x \in \Omega; |u(x)| > t_0\}$. Then $\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \ge 0$ for a.e. $x \in \Omega_u$. For $x \in \Omega \setminus \Omega_u$, we have

$$\left| \frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \right| \le C_2 (t_0 + a(x) t_0^{q^+} \vee t_0^{q^-}).$$

Hence we have

$$\begin{aligned}
\frac{1}{\theta}\langle J'(u), u\rangle - J(u) &= \int_{\Omega_u} \left(\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \right) dx \\
&\quad + \int_{\Omega \setminus \Omega_u} \left(\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \right) dx \\
&\ge -C_2 \int_{\Omega \setminus \Omega_u} (t_0 + a(x) t_0^{q^+} \vee t_0^{q^-}) dx \\
&\ge -C_2 t_0 |\Omega| - C_2 t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)}.
\end{aligned}$$

Similarly we have

$$\frac{1}{\theta}\langle K'(u), u\rangle - K(u) \ge -C_3 t_0 |\Gamma_2| - C_3 t_0^{r^+} \vee t_0^{r^-} \|b\|_{L^1(\Gamma_2)}.$$

Thus we have

$$\begin{aligned}
I(u) - \frac{1}{\theta}\langle I'(u), u\rangle &= \Phi(u) - \frac{1}{\theta}\langle \Phi'(u), u\rangle - \left(J(u) - \frac{1}{\theta}\langle J'(u), u\rangle \right) \\
&\quad - \left(K(u) - \frac{1}{\theta}\langle K'(u), u\rangle \right) \ge \left(\frac{1}{p^+} - \frac{1}{\theta} \right) c \|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C_4
\end{aligned}$$

for some constant C_4. □

Proposition 3.14. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then the functional I satisfies the Palais–Smale condition, that is, if a sequence $\{u_n\} \subset Y$ satisfies that $\lim_{n\to\infty} I(u_n) = \gamma \in \mathbb{R}$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$, then $\{u_n\}$ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset Y$ satisfy that $\lim_{n\to\infty} I(u_n) = \gamma \in \mathbb{R}$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$.

Step 1. $\{u_n\}$ is bounded in Y. Indeed, if it is false, then passing to a subsequence, we can assume that $\lim_{n\to\infty} \|u_n\|_Y = \infty$. By proposition 3.13, we have

$$I(u_n) \ge \left(\frac{1}{p^+} - \frac{1}{\theta} \right) k_0 \|u_n\|_Y^{p^-} - \frac{1}{\theta} \|I'(u_n)\|_{Y^*} \|u_n\|_Y - C_4$$

for large n. Since $\frac{1}{p^+} - \frac{1}{\theta} > 0$ and $p^- > 1$ and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$, we have $I(u_n) \to \infty$ as $n \to \infty$. This is a contradiction.

Step 2. Since Y is a reflexive Banach space from Proposition 3.5, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u \in Y$ such that $u_{n'} \to u$ weakly in Y as $n' \to \infty$. Since $\{u_{n'} - u\}$ is bounded in Y and $\lim_{n'\to\infty} \|I'(u_{n'})\|_{Y^*} = 0$, we see that

$$\langle I'(u_{n'}), u_{n'} - u\rangle \to 0 \text{ as } n' \to \infty.$$

By Proposition 2.9, $u_{n'} \to u$ strongly in $L^{q(\cdot)}_{a(\cdot)}(\Omega)$ and $L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ as $n' \to \infty$. From (f.1), using the Hölder inequality,

$$\begin{aligned}
&\left| \int_\Omega f(x, u_{n'}(x))(u_{n'}(x) - u(x))dx \right| \\
&\quad \le \int_\Omega C_1(1 + a(x)|u_{n'}(x)|^{q(x)-1})|u_{n'}(x) - u(x)|dx \\
&\quad \le C_1 \int_\Omega (a(x)^{1/q(x)}|u_{n'}(x) - u(x)| + a(x)^{1/q'(x)}|u_{n'}(x)|^{q(x)-1}a(x)^{1/q(x)}|u_{n'}(x) - u(x)|)dx \\
&\quad \le 2C_1\|1\|_{L^{q'(\cdot)}(\Omega)}\|a^{1/q(\cdot)}|u_{n'} - u|\|_{L^{q(\cdot)}(\Omega)} \\
&\qquad + 2C_1\|a^{1/q'(\cdot)}|u_{n'}(\cdot)|^{q(\cdot)-1}\|_{L^{q'(\cdot)}(\Omega)}\|a^{1/q(\cdot)}|u_{n'} - u|\|_{L^{q(\cdot)}(\Omega)}.
\end{aligned}$$

Since

$$\rho_{q'(\cdot)}(a^{1/q'(\cdot)}|u_{n'}|^{q(\cdot)-1}) = \int_\Omega a(x)|u_{n'}(x)|^{q(x)}dx$$

is bounded, we see that $\|a^{1/q'(\cdot)}|u_{n'}|^{q(\cdot)-1}\|_{L^{q'(\cdot)}(\Omega)}$ is bounded. Since $\|u_{n'} - u\|_{L^{q(\cdot)}_{a(\cdot)}(\Omega)} \to 0$ as $n' \to \infty$, we see that

$$\lim_{n'\to\infty} \langle J'(u_{n'}), u_{n'} - u\rangle = \lim_{n'\to\infty} \int_\Omega f(x, u_{n'}(x))(u_{n'}(x) - u(x))dx = 0.$$

Similarly, we have

$$\lim_{n'\to\infty} \langle K'(u_{n'}), u_{n'} - u\rangle = \lim_{n'\to\infty} \int_{\Gamma_2} g(x, u_{n'}(x))(u_{n'}(x) - u(x))d\sigma_x = 0.$$

Thus we have

$$\lim_{n'\to\infty} \langle \Phi'(u_{n'}), u_{n'} - u\rangle = \lim_{n'\to\infty} (\langle J'(u_{n'}), u_{n'} - u\rangle + \langle K'(u_{n'}), u_{n'} - u\rangle + \langle I'(u_{n'}), u_{n'} - u\rangle) = 0.$$

Since $u_{n'} \to u$ weakly in Y, we have $\lim_{n'\to\infty}\langle \Phi'(u), u_{n'} - u\rangle = 0$, so

$$\lim_{n'\to\infty} \langle \Phi'(u_{n'}) - \Phi'(u), u_{n'} - u\rangle = 0.$$

Since $\{u_{n'}\}$ is bounded in Y, it follows from Proposition 3.7 that

$$\int_\Omega h_1(x)|\nabla u_{n'}(x) - \nabla u(x)|^{p(x)}dx \to 0 \quad \text{as } n' \to \infty,$$

so $u_{n'} \to u$ strongly in Y. □

4 Main theorems

In this section, we state the main theorems (Theorem 4.3, 4.5 and 4.6).

Definition 4.1. We say $u \in Y$ is a weak solution of (1.1) if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x)dx = \int_\Omega f(x, u(x))v(x)dx + \int_{\Gamma_2} g(x, u(x))v(x)d\sigma_x \tag{4.1}$$

for all $v \in Y$.

Remark 4.2. Since $C_0^\infty(\Omega) \subset Y$, if $u \in Y$ satisfies (4.1), then the equation (1.1) holds in the distribution sense.

Now we obtain the following three theorems.

Theorem 4.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying* (1.2). *Under the hypotheses (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3), the problem* (1.1) *has a nontrivial weak solution.*

Remark 4.4. This theorem extends the result of [8] in which the authors considered the case where $A(x,\xi) = \frac{1}{p(x)}|\xi|^{p(x)}$, $\Gamma_2 = \emptyset$ and $p^- \geq 2$. This theorem is new and also an extension to the case $p^- > 1$.

We impose one more assumption.

(f.4) For any $\delta' \in (0,1)$, the function $f(x,t)$ satisfies the following inequality.

$$f(x,t) \geq \begin{cases} ct^{m-1} & \text{for } t \in [\delta', 1], \\ 0 & \text{for } t \in [0,\infty) \setminus [\delta', 1], \end{cases}$$

where $c > 0$ and $0 < m < 1$.

For example, A function $f(x,t) = \chi_{\delta'}(t)|t|^{m-2}t + a(x)|t|^{q(x)-2}t$, where $\chi_{\delta'} \in C_0(\mathbb{R})$ satisfying that $0 \leq \chi_{\delta'} \leq 1$,

$$\chi_{\delta'}(t) = \begin{cases} 0 & \text{for } |t| \leq \delta'/2 \\ 1 & \text{for } \delta' \leq |t| \leq 1 \end{cases}$$

and that a function a is as in (f.1) verifies (f.1)–(f.4).

Theorem 4.5. *In addition to the hypotheses of Theorem 4.3, assume that (f.4) also holds. Then the problem* (1.1) *has at least two nontrivial weak solutions.*

Finally, in addition to the hypotheses of Theorem 4.3, we assume the following hypotheses.

(A.4) $A(x,\xi)$ is even with respect to ξ, that is, $A(x,-\xi) = A(x,\xi)$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

(f.5) $f(x,t)$ is odd with respect to t, that is, $f(x,-t) = -f(x,t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

(g.4) $g(x,t)$ is odd with respect to t, that is, $g(x,-t) = -g(x,t)$ for σ-a.e. $x \in \Gamma_2$ and all $t \in \mathbb{R}$.

Then we can derive that there exist infinitely many weak solutions of (1.1).

Theorem 4.6. *In addition to the hypotheses of Theorem 4.3, assume that (A.4), (f.5) and (g.4) also hold. Then the problem* (1.1) *has infinitely many nontrivial weak solutions.*

5 Proofs of Theorem 4.3, 4.5 and 4.6

In this section, we give proofs of Theorem 4.3, 4.5 and 4.6. Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold.

The proofs of Theorem 4.3, 4.5 and 4.6 consist of some lemma and propositions.

Lemma 5.1. *Under the hypotheses of Theorem 4.3, we have the following.*

(i) $|F(x,t)| \le C_1'(1 + a(x)|t|^{q(x)})$ *for a.e.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ *with some constant* $C_1' > 0$.

(ii) *There exists* $\gamma \in L^{\alpha(\cdot)}(\Omega)$ *such that* $\gamma(x) > 0$ *a.e.* $x \in \Omega$ *and* $F(x,t) \ge \gamma(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and a.e.* $x \in \Omega$, *where* $\alpha(\cdot)$ *and* t_0 *are as in (f.1) and (f.2), respectively.*

(iii) $|G(x,t)| \le C_2'(1 + b(x)|t|^{r(x)})$ *for* σ*-a.e.* $x \in \Gamma_2$ *and all* $t \in \mathbb{R}$ *with some constant* $C_2' > 0$.

(iv) *There exists* $\delta \in L^{\beta(\cdot)}(\Gamma_2)$ *such that* $\delta(x) > 0$ σ*-a.e.* $x \in \Gamma_2$ *and* $G(x,t) \ge \delta(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and* σ*-a.e.* $x \in \Gamma_2$, *where* $\beta(\cdot)$ *and* t_0 *are as in (g.1) and (g.2), respectively.*

Proof. (i) easily follows from (f.1).

(ii) From (f.2), for $t \ge t_0$,

$$0 < \theta F(x,t) \le f(x,t)t. \tag{5.1}$$

Put $\gamma(x) = F(x,t_0)t_0^{-\theta}$. Then $\gamma(x) > 0$ for a.e. $x \in \Omega$ and it follows from (ii) that

$$\gamma(x) \le C_1'(1 + a(x)t_0^{q(x)})t_0^{-\theta} \le C_1'(1 + a(x)t_0^{q^+} \vee t_0^{q^-})t_0^{-\theta}.$$

So $\gamma \in L^{\alpha(\cdot)}(\Omega)$. From (5.1),

$$\frac{\theta}{\tau} \le \frac{f(x,\tau)}{F(x,\tau)} = \frac{\frac{\partial F}{\partial \tau}(x,\tau)}{F(x,\tau)}.$$

Integrating this inequality over (t_0, t), we have

$$\theta \log \frac{t}{t_0} \le \log \frac{F(x,t)}{F(x,t_0)} \quad \text{for all } t \ge t_0.$$

This implies that $F(x,t) \ge \gamma(x)t^\theta$ for all $t \ge t_0$.

(iii) and (iv) follow from the similar argument as (i) and (ii) using (g.1) and (g.2), respectively. □

5.1 Proof of Theorem 4.3

For a proof of Theorem 4.3, we apply the following standard Mountain-Pass Theorem (cf. Willem [26]).

Proposition 5.2. *Let* $(V, \|\cdot\|_V)$ *be a Banach space and* $I \in C^1(V, \mathbb{R})$ *be a functional satisfying the Palais–Smale condition. Assume that* $I(0) = 0$, *and there exist* $\rho > 0$ *and* $z_0 \in V$ *such that* $\|z_0\|_V > \rho$, $I(z_0) \le I(0) = 0$ *and*

$$\alpha := \inf\{I(u); u \in V \text{ with } \|u\|_V = \rho\} > 0.$$

Let $G := \{\varphi \in C([0,1], V); \varphi(0) = 0, \varphi(1) = z_0\} \ne \emptyset$ *and* $\beta = \inf\{\max I(\varphi([0,1])); \varphi \in G\}$. *Then* $\beta \ge \alpha$ *and* β *is a critical value of* I.

We apply Proposition 5.2 with $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$. By Proposition 3.9 and Proposition 3.14, the functional I satisfies that $I \in C^1(Y, \mathbb{R})$ and the Palais–Smale condition holds. Since $\Phi(0) = J(0) = K(0) = 0$, we have $I(0) = 0$. According to (3.13),

$$\alpha = \inf_{\|v\|_Y = \rho} I(v) > 0. \tag{5.2}$$

We show that there exists $u_0 \in Y$ such that $\|u_0\|_Y > \rho$ and $I(u_0) \le 0$. Choose $v_0 \in C_0^\infty(\Omega)$ such that $v_0 \ge 0$ and $W = \{x \in \Omega; v_0(x) \ge t_0\}$ has a positive measure, where t_0 is as in (f.2). We see that $F(x, v_0(x)) > 0$ for a.e. $x \in W$ from (f.2). Let $t > 1$ and define $W_t = \{x \in \Omega; tv_0(x) \ge t_0\}$, then $W \subset W_t$. By Lemma 5.1 (ii), there exists $\gamma \in L^{\alpha(\cdot)}(\Omega)(\subset L^1(\Omega))$ such that $\gamma(x) > 0$ a.e. $x \in \Omega$ and $F(x,t) \ge \gamma(x)t^\theta$ for $t \in [t_0, \infty)$. Thereby,

$$\int_{W_t} F(x, tv_0(x))dx \ge \int_{W_t} \gamma(x)t^\theta v_0(x)^\theta dx \ge t^\theta L(v_0),$$

where $L(v_0) = \int_W \gamma(x)v_0(x)^\theta dx > 0$. For $t \in [0, t_0]$, it follows from Lemma 5.1 (i) that

$$|F(x,t)| \le C_1'(1 + a(x)t^{q(x)}) \le C_1'(1 + a(x)t_0^{q^+} \vee t_0^{q^-}).$$

By (f.2), $F(x, st) \ge F(x,t)s^\theta$ for $t \in \mathbb{R} \setminus (-t_0, t_0)$ and $s > 1$. Indeed, if we define $h(s) = F(x, st)$, then

$$h'(s) = f(x, st)t = \frac{1}{s} f(x, st)st \ge \frac{\theta}{s} F(x, st) = \frac{\theta}{s} h(s).$$

Thus $h'(s)/h(s) \ge \theta/s$, so $\log h(s)/h(1) \ge \theta \log s$ for $s > 1$. This implies $h(s) \ge h(1)s^\theta$.

(A.3) implies that

$$A(x, s\boldsymbol{\xi}) + \frac{h_1(x)}{p(x)} \le s^{p(x)} \left(A(x, \boldsymbol{\xi}) + \frac{h_1(x)}{p(x)} \right) \quad \text{for } s > 1.$$

Indeed, if we define $k(s) = A(x, s\boldsymbol{\xi}) + h_1(x)/p(x)$, then we see that $k'(s) \le \frac{1}{s} p(x)k(s)$. Hence we obtain the inequality. Thus we see that, for $t > 1$, $\Phi(su) + \|h_1/p\|_{L^1(\Omega)} \le s^{p(x)}(\Phi(u) + \|h_1/p\|_{L^1(\Omega)}$ for $u \in Y$ and $s > 1$. Thereby we see that, for $t > 1$,

$$\begin{aligned} I(tv_0) &= \Phi(tv_0) - J(tv_0) \\ &\le \Phi(tv_0) - \int_{W_t} F(x, tv_0(x))dx - \int_{\Omega \setminus W_t} F(x, tv_0(x))dx \\ &\le t^{p^+}\Phi(v_0) + t^{p^+}\|h_1/p\|_{L^1(\Omega)} - \|h_1/p\|_{L^1(\Omega)} - t^\theta L(v_0) + C_1'|\Omega| + t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)}. \end{aligned}$$

Since $\theta > p^+$ and $L(v_0) > 0$, we can see that $I(tv_0) \to -\infty$ as $t \to \infty$. Hence there exists $t_1 > 1$ such that $\|t_1 v_0\|_Y > \rho$ and $I(t_1 v_0) \le 0$. Put $u_0 = t_1 v_0$.

If we define $\varphi(t) = tu_0$, then $\varphi \in G$, so $G \ne \emptyset$. Hence all the hypotheses of Proposition 5.2 hold. Therefore, $\beta = \inf\{\max I(\varphi([0,1]); \varphi \in G\}$ satisfies that $\beta \ge \alpha > 0$ and β is a critical value of I, that is, there exists $u_1 \in Y$ such that $I(u_1) = \beta$ and $I'(u_1) = 0$. Thus u_1 is a weak solution of (1.1). Since $I(u_1) = \beta \ge \alpha > 0 = I(0)$, u_1 is nontrivial weak solution of (1.1). This completes the proof of Theorem 4.3.

5.2 Proof of Theorem 4.5.

It follows from (f.4) that for $0 \le t \le 1$,

$$F(x,t) \ge \begin{cases} \int_{\delta'}^t f(x,s)ds & \text{if } t \ge \delta', \\ 0 & \text{if } t < \delta' \end{cases} \ge \begin{cases} \frac{c}{m}(t^m - (\delta')^m) & \text{if } t \ge \delta', \\ 0 & \text{if } t < \delta'. \end{cases}$$

Fix $t_1 \in (0,1)$ small enough and choose $\delta' \in (0,1)$ such that $(\delta')^m \le t_1$. If $(\delta')^m \le t$, then $F(x,t) \ge \frac{c}{m}(t^m - t)$ since $(\delta')^m \ge \delta'$. Choose $\varphi \in C_0^\infty(\Omega)$ so that $0 \le \varphi \le 1$ and $\varphi \not\equiv 0$. Put

$\Omega_{\delta'} = \{x \in \Omega; (\delta')^m \le t_1\varphi(x)\}$. Then we note that $|\Omega \setminus \Omega_{\delta'}| \to 0$ as $\delta' \to 0$, where $|A|$ denotes the measure of a measurable set A. Thus we have

$$\begin{aligned} J(t_1\varphi) &= \int_\Omega F(x, t_1\varphi(x))dx \\ &\ge \int_{\Omega_{\delta'}} F(x, t_1\varphi(x))dx \\ &\ge \frac{c}{m}\int_{\Omega_{\delta'}} ((t_1\varphi(x))^m - t_1\varphi(x))dx \\ &\ge \frac{c}{m}t_1^m\left(\int_\Omega \varphi(x)^m dx - \int_{\Omega\setminus\Omega_{\delta'}} \varphi(x)^m dx\right) - \frac{c}{m}t_1\int_{\Omega_{\delta'}}\varphi(x)dx \\ &\ge \frac{c}{m}t_1^m\left(\int_\Omega \varphi(x))^m dx - |\Omega \setminus \Omega_{\delta'}|\right) - \frac{c}{m}t_1|\Omega|. \end{aligned}$$

If we replace δ' with smaller one, if necessary, we may assume that $\int_\Omega \varphi(x)^m dx - |\Omega \setminus \Omega_{\delta'}| > 0$.

On the other hand, since $A(x, \boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$ and $A(x, \mathbf{0}) = 0$, we have $A(x, t_1\boldsymbol{\xi}) = A(x, t_1\boldsymbol{\xi} + (1 - t_1)\mathbf{0}) \le t_1 A(x, \boldsymbol{\xi})$. Thus

$$\Phi(t_1\varphi) = \int_\Omega A(x, t_1\nabla\varphi(x))dx \le t_1\Phi(\varphi).$$

Therefore, we have

$$I(t_1\varphi) = \Phi(t_1\varphi) - J(t_1\varphi) \le t_1\left(\Phi(\varphi) + \frac{c}{m}|\Omega|\right) - \frac{c}{m}t_1^m\left(\int_\Omega \varphi(x)^m dx - |\Omega \setminus \Omega_{\delta'}|\right).$$

Since $0 < m < 1$, if $t_1 > 0$ is small enough, then we see that $I(t_1\varphi) < 0$. By Proposition 3.12, I is bounded from below on $\overline{B_\rho(0)}$, where $B_\rho(0) = \{v \in Y; \|v\|_Y < \rho\}$, ρ is as in (3.13). Hence

$$-\infty < \underline{c} := \inf_{v\in\overline{B_\rho(0)}} I(v) < 0.$$

Let $0 < \varepsilon < \inf_{v\in\partial B_\rho(0)} I(v) - \inf_{v\in\overline{B_\rho(0)}} I(v)$. Here we note that $\inf_{v\in\partial B_\rho(0)} I(v) > 0$. Then there exists $u \in \overline{B_\rho(0)}$ such that

$$\inf_{v\in\overline{B_\rho(0)}} I(v) \le I(u) \le \inf_{v\in\overline{B_\rho(0)}} I(v) + \varepsilon^2.$$

Since $\inf_{v\in\overline{B_\rho(0)}} I(v) < 0$, we can choose $u \in \overline{B_\rho(0)}$ so that $I(u) < 0$. By applying the Ekeland variational principle (cf. Ekeland [12, Theorem 1.1]) in the complete metric space $\overline{B_\rho(0)}$, there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I(u_\varepsilon) \le I(u), \tag{5.3}$$

$$I(u_\varepsilon) \le I(v) + \varepsilon\|v - u_\varepsilon\|_Y \text{ for all } v \in \overline{B_\rho(0)}, \tag{5.4}$$

$$\|u - u_\varepsilon\|_Y \le \varepsilon. \tag{5.5}$$

Define a functional $\widehat{I} : \overline{B_\rho(0)} \to \mathbb{R}$ by $\widehat{I}(v) = I(v) + \varepsilon\|v - u_\varepsilon\|_Y$ for $v \in \overline{B_\rho(0)}$. Since $I(u_\varepsilon) \le I(u) < 0$ from (5.3) and $I(v) > 0$ for all $v \in \partial B_\rho(0)$, we have $u_\varepsilon \in B_\rho(0)$. Choose $\rho' > 0$ small enough so that $u_\varepsilon + w \in \overline{B_\rho(0)}$ for $w \in \overline{B_{\rho'}(0)}$. From (5.4), since $\widehat{I}(u_\varepsilon) \le \widehat{I}(u_\varepsilon + w)$ for all $w \in \overline{B_{\rho'}(0)}$, we have

$$\begin{aligned} &\frac{\langle I'(u_\varepsilon), w\rangle + \varepsilon\|w\|_Y}{\|w\|_Y} \\ &\quad = \frac{\langle I'(u_\varepsilon), tw\rangle + \varepsilon t\|w\|_Y - (\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon))}{t\|w\|_Y} + \frac{\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon)}{t\|w\|_Y}. \end{aligned}$$

Here we note that from (5.4),

$$\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon) = I(u_\varepsilon + tw) + \varepsilon\|tw\|_Y - I(u_\varepsilon) \geq 0$$

for $t \in (0,1)$. Hence

$$\frac{\langle I'(u_\varepsilon), w\rangle + \varepsilon\|w\|_Y}{\|w\|_Y} \geq \frac{\langle I'(u_\varepsilon), tw\rangle - (I(u_\varepsilon + tw) - I(u_\varepsilon))}{t\|w\|_Y} \to 0 \quad \text{as } t \to +0.$$

So $\langle I'(u_\varepsilon), w\rangle + \varepsilon\|w\|_Y \geq 0$ for all $w \in \overline{B_{\rho'}(0)}$, so $\langle I'(u_\varepsilon), w\rangle \geq -\varepsilon\|w\|_Y$. Replacing w with $-w$, we have $|\langle I'(u_\varepsilon), w\rangle| \leq \varepsilon\|w\|_Y$ for all $w \in \overline{B_{\rho'}(0)}$. Thus $\|I'(u_\varepsilon)\|_{Y^*} \leq \varepsilon$. Letting $\varepsilon \to 0$, we see that $I(u_\varepsilon) \to \underline{c}$ and $I'(u_\varepsilon) \to 0$ in Y^*. Since I satisfies the Palais–Smale condition in Y, there exist a subsequence $\{u_n\}$ of $\{u_\varepsilon\}$ and $u_2 \in \overline{B_\rho(0)}$ such that $u_n \to u_2$ in Y and $I'(u_2) = 0$. Therefore, u_2 is a weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 = I(0)$, u_2 is a nontrivial weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 < I(u_1)$, we have $u_1 \neq u_2$. This completes the proof of Theorem 4.5.

5.3 Proof of Theorem 4.6

We apply the following Symmetric Mountain-Pass Theorem due to the Rabinowitz [23, Theorem 9.12] (cf. Xie and Xiao [27, Proposition 2.1]).

Proposition 5.3. *Let V be an infinite-dimensional real Banach space. A functional $I : V \to \mathbb{R}$ is of C^1-class and satisfies the Palais–Smale condition. Furthermore, assume that*

(I.1) *$I(0) = 0$ and I is an even functional, that is, $I(-u) = I(u)$ for all $u \in V$.*

(I.2) *There exist positive constants α and ρ such that*

$$\inf_{u \in \partial B_\rho(0)} I(u) \geq \alpha.$$

(I.3) *For each finite-dimensional linear subspace $V_1 \subset V$, the set $\{u \in V_1; I(u) \geq 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

We apply Proposition 5.3 with $V = Y$. Note that the functional I defined by (3.6) is of class C^1 (Proposition 3.9) and satisfies the Palais–Smale condition (Proposition 3.14). From (A.4), (f.5) and (g.4), (I.1) is trivial. (I.2) follows from (3.13). Thus it suffices to derive (I.3).

Let $u \in Y$ with $\|u\|_Y > 1$. Since $\Phi(u) \leq c_1\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + C_1\|u\|_Y^{p^+}$ from Lemma 3.4 and $p^+ > 1$, we have

$$\Phi(u) \leq C_5\|u\|_Y^{p^+} \quad \text{for some constant } C_5 > 0. \tag{5.6}$$

Since $F(x,t)$ is an even function with respect to t, it follows from Lemma 5.1 (ii) that $F(x,t) \geq \gamma(x)|t|^\theta$ for $|t| \geq t_0$. Define $\Omega_{t_0} = \{x \in \Omega; |u(x)| \geq t_0\}$. Then

$$J(u) = \int_\Omega F(x, u(x))dx = \int_{\Omega_{t_0}} F(x, u(x))dx + \int_{\Omega\setminus\Omega_{t_0}} F(x, u(x))dx.$$

From (f.1),

$$\int_{\Omega\setminus\Omega_{t_0}} |F(x, u(x))|dx \leq C_1'|\Omega| + t_0^{q^+} \vee t_0^{q^-}\|a\|_{L^1(\Omega)}.$$

Hence we have

$$\begin{aligned} J(u) &\geq \int_{\Omega_{t_0}} \gamma(x)|u(x)|^\theta dx - C_6 \\ &= \int_\Omega \gamma(x)|u(x)|^\theta dx - \int_{\Omega\setminus\Omega_{t_0}} \gamma(x)|u(x)|^\theta dx - C_6 \\ &\geq \int_\Omega \gamma(x)|u(x)|^\theta dx - C_7, \end{aligned} \tag{5.7}$$

where C_7 is a constant. Similarly we have

$$K(u) \geq \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x - C_8, \tag{5.8}$$

where C_8 is a constant.

We note that

$$\left(\int_\Omega \gamma(x)|u(x)|^\theta dx + \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma\right)^{1/\theta}$$

is a norm in Y.

Let Y_1 be any finite-dimensional linear subspace of Y. Since Y_1 is of finite-dimensional, the above norm is equivalent to the norm $\|u\|_Y$ in Y_1, so there exists $C_9 > 0$ such that

$$C_9\|u\|_Y^\theta \leq \int_\Omega \gamma(x)|u(x)|^\theta dx + \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x.$$

Therefore, for $u \in Y_1$ with $\|u\|_Y > 1$, it follows from (5.6), (5.7) and (5.8) that

$$I(u) \leq C_5\|u\|_Y^{p^+} - C_9\|u\|_Y^\theta + C_7 + C_8.$$

If $u \in Y_1$ with $\|u\|_Y > 1$ satisfies $I(u) \geq 0$, then we have $C_9\|u\|_Y^\theta \leq C_5\|u\|_Y^{p^+} + C_7 + C_8$. Since $\theta > p^+$, the set $\{u \in Y_1; \|u\|_Y > 1, I(u) \geq 0\}$ is bounded, so $\{u \in Y_1; I(u) \geq 0\}$ is bounded. Since all the assumptions of Proposition 5.3 hold, I has an unbounded sequence of critical values, so problem (1.1) has infinitely many weak solutions. This completes the proof of Theorem 4.6.

References

[1] J. Aramaki, Existence of three weak solutions for a class of nonlinear operators involving $p(x)$-Laplacian with mixed boundary conditions, *Nonlinear Funct. Anal. Appl.* **26**(2021) No. 3, 531–551. `https://doi.org/10.22771/nfaa.2021.26.03.06`; Zbl 1481.35233.

[2] J. Aramaki, Mixed boundary value problem for a class of quasi-linear elliptic operators containing $p(\cdot)$-Laplacian in a variable exponent Sobolev space, *Adv. Math. Sci. Appl.* **31**(2022), No. 2, 207–239. MR4521438; Zbl 1511.35192.

[3] J. Aramaki, Kirchhoff-type problem with mixed boundary condition in a variable exponent Sobolev space, *Comm. Math. Res.* **39**(2023), No. 4, 539–574. `https://doi.org/10.4.208/cmr.2023-0014`; MR4691341; Zbl 07812211.

[4] J. Aramaki, Existence of nontrivial weak solutions for nonuniformly elliptic equation with mixed boundary condition in a variable exponent Sobolev space, *Electron. J. Qual. Theory Differ. Equ.* **2023**, No. 12, 1–22. `https://doi.org/10.14.232/ejqtde.2023.1.12`; MR4577528;Zbl 07742346.

[5] J. Aramaki, Existence of three weak solutions for a nonlinear problem with mixed boundary condition in a variable exponent Sobolev space, *J. Anal.* **32**(2024), No. 2, 733–755. https://doi.org/10.1007/s41478-023-00646-8; MR4716278.

[6] J. Aramaki, Existence of infinitely many weak solutions for non-uniformly elliptic equation with mixed boundary condition in a variable exponent Sobolev space, *Adv. Math. Sci. Appl.* **33**(2024), No. 1, 13–40.

[7] P. G. Ciarlet, G. Dinca, A Poincaré inequality in a Sobolev space with a variable exponent, *Chinese Ann. Math. Ser. B* **32**(2011), No. 3, 333–342. https://doi.org/10.1007/s11401-011-0648-1; MR2805404; Zbl 1305.46026.

[8] G. Dai, R. Hao, Existence of solutions for a $p(x)$-Kirchhoff-type equation, *J. Math. Anal. Appl.* **359**(2009), 275–284. https://doi.org/10.1016/j.jmaa.2009.05.031; MR2542174; Zbl 1172.35401.

[9] E. DiBenedetto, *Degenerate parabolic equations*, Springer, New York, 1993. https://doi.org/10.1007/978-1-4612-0895-2; MR1230384; Zbl 0794.35090.

[10] L. Diening, *Theoretical and numerical results for electrorheological fluids*, PhD thesis, University of Freiburg, 2002.

[11] D. E. Edmunds, J. Rákosník, Sobolev embeddings with variable exponent, *Studia Math.* **143**(2000), No. 3, 267–293. https://doi.org/10.4064/sm-143-3-267-293; MR1815935; Zbl 0974.46040.

[12] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47**(1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0; MR346619; Zbl 0286.49015.

[13] X. L. Fan, Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.* **312**, (2005), 464–477. https://doi.org/10.1016/j.jmaa.2005.03.057; MR2179089; Zbl 1154.35336.

[14] X. L. Fan, Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.* **339**(2008), 1395–1412. https://doi.org/10.1016/j.jmaa.2007.08.003; MR2377096; Zbl 1136.46025.

[15] X. L. Fan, Q. H Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52**(2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5; MR1954585; Zbl 1146.35353.

[16] X. L. Fan, Q. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302**(2005), 306–317. https://doi.org/10.1016/j.jmaa.2003.11.020; MR2107835; Zbl 1072.35138.

[17] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263**(2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617; MR1866056; Zbl 1028.46041.

[18] R. Glowinski, A. Marroco, Sur l'approximation, par éléments finis d'ordre un et al résolution, par penalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires, *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.* **9**(1975), 41–76. MR0388811; Zbl 0368.65053.

[19] T. C. Halsey, Electrorheological fluids, *Science* **258**(1992), 761–766. https://doi.org/10.1126/science.258.5083.761.

[20] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, *Czechoslovak Math. J.* **41**(1991), No. 116, 592–618. MR1134951; Zbl 0784.46029.

[21] R. A. Mashiyev, B. Cekic, M. Avci, Z. Yucedag, Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition, *Complex Var. Elliptic Equ.* **57**(2012), No. 5, 579–595. https://doi.org/10.1080/17476933.2012.667911; MR2955967; Zbl 1277.35173.

[22] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proceeding of the Royal Society A.* **462**(2006), 2625–2641. https://doi.org/10.1098/rspa.2005.1633; MR2253555; Zbl 1149.76692.

[23] P. H. Rabinowitz, *Minimax methods in critical point theory with application to differential equations*, CBMS Reg. Conf. Ser. in Math., Vol. 65, Am. Math. Soc., Providence 1986. https://doi.org/10.1090/cbms/065; MR0845785; Zbl 0609.58002.

[24] M. Růžička, *Electrotheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, Vol. 1784 Springer-Verlag, Berlin 2000. MR1810360; Zbl 0962.76001.

[25] F. D. Thelin, Local regularity properties for the solutions of nonlinear partial differential equation, *Nonlinear Anal.* **6**(1982), 839–844. https://doi.org/10.1016/0362-546X(82)90068-2; MR0671725; Zbl 0493.35021.

[26] M. Willem, *Minimax theorems*, Birkhäuser, Boston, Basel, Berlin 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 085649001.

[27] Q. Xie, H. Xiao, Infinitely many solutions for the discrete Schrödinger equations with a nonlocal term, *Bound. Value Probl.* **2022**, Paper No. 2, 12 pp. https://doi.org/10.1186/s13661-022-01583-4; MR4368381; Zbl 1500.39010.

[28] J. Yao, Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators, *Nonlinear Anal.* **68**(2008), 1271–1283. https://doi.org/10.1016/j.na.2006.12.020; MR2381670; Zbl 1158.35046.

[29] Z. Yücedağ, Solutions of nonlinear problems involving $p(x)$-Laplacian operator, *Adv. Nonlinear Anal.* **4**(2015), No. 4, 1–9. https://doi.org/10.1515/anona-2015-0044; MR3420320; Zbl 1328.35058.

[30] D. Zhao, W. J. Qing, X. L. Fan, On generalized Orlicz space $L^{p(x)}(\Omega)$ (in Chinese), *J. Gansu Sci.* **9**(1997), No. 2, 1–7.

[31] V. V. Zhikov, Averaging of functionals of the calculus of variation and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.* **29**(1987), 33–66. MR0864171; Zbl 0599.49031.