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Multiple solutions for a class of Kirchhoff-type equation with critical growth

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Abstract. In this paper, we study the multiplicity of solutions to a class of Kirchhoff-type equation with critical growth

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=\lambda h(x)f(u)+g(x)u^5 \quad \text{in } \mathbb{R}^3,$$

where a, $b>0$, λ is a positive parameter and f is a continuous nonlinearity with subcritical growth. Under suitable conditions on the potentials $V(x)$, $h(x)$ and $g(x)$, we prove the multiplicity results and investigate the relation between the number of solutions with the topology of the set where g attains its maximum value for small values of the parameter λ. The proofs are based on Nehari manifold and Lusternik–Schnirelmann theory.

Keywords: Kirchhoff-type problem, critical growth, variational method.

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1 Introduction

Consider the following Kirchhoff-type problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=\lambda h(x)f(u)+g(x)u^5 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where a, $b>0$ are constants and $\lambda>0$ is a parameter. The Kirchhoff-type problem is primarily introduced in [10] to generalize the classical D'Alembert wave equation for free vibrations of elastic strings. More precisely, the original equation is

$$h\rho\frac{\partial^2 u}{\partial t^2}-\left(P_0+\frac{Eh}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}+\delta\frac{\partial u}{\partial t}+f(x,u)=0 \tag{1.2}$$

for $t\geq 0$ and $0<x<L$, where $u=u(t,x)$ is the lateral displacement at the time t and at the space coordinate x, L the length of the string, h the cross-section area, E the Young modulus

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of the material, ρ the mass density, P_0 the initial axial tension, δ the resistance modulus and f the external force. When $\delta = f = 0$, equation (1.2) is introduced by Kirchhoff [10]. For more physical and mathematical background on Kirchhoff-type problems, we refer the readers to [2,7] and the references therein.

If we set $V = 0$ and replace $\mathbb{R}^3$ by a smooth bounded domain $\Omega \subset \mathbb{R}^N (N \geq 3)$, then problem (1.1) becomes a special case of the following Kirchhoff Dirichlet problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right) \Delta u = \hat{f}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) is often referred to be nonlocal because of the presence of the term $\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u$ which implies that (1.3) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of problem (1.3) particularly interesting. After Lions [12] proposed an abstract functional analysis framework, problem (1.3) had attracted much attention, see, for example, [6,17–19] and the references therein. In [19], Qin et al. considered (1.3) in the case where $\hat{f}(x,u) := Q(x)u^5 + \lambda|u|^{p-1}u$ $(3 < p < 5)$, and proved the existence of one ground state solution by using variational methods that are constrained to the Nehari manifold. The relation between the number of maxima of Q and the number of positive solutions for the problem was also investigated. In [17], Naimen generalized the result of Brézis and Nirenberg ([5]) to problem (1.3) for the case when $\hat{f}(x,u) := \lambda f(x,u) + |u|^{2^*-2}u$, a, $b \geq 0$ and $a + b > 0$. Some existence results as well as nonexistence results were obtained. In [18], the authors further studied the high dimensional case $(N \geq 5)$, and proved the multiplicity of positive solutions of problem (1.3) when $\hat{f}(x,u) := \lambda u^p + u^{2^*-1}$ with $q \in [1, 2^* - 1)$. By combining the variational method and Lusternik–Schnirelmann theory, Cai et al. [6] discussed problem (1.3), where $N = 3$ and $\hat{f}(x,u) := |u|^{4-\varepsilon}u - \lambda u$ with $\varepsilon \in (0,2)$ and $\lambda \geq 0$, and obtained the existence of multiple positive solutions.

Recently, many researchers focused on the existence, multiplicity and asymptotic behavior of solutions of the following problem

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = \hat{f}(x,u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.4}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and $\hat{f} \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, see [8,11,23–27] and the references therein. In [25], Zhang studied problem (1.4) in the case where $V = 1$ and $\hat{f}(x,u) = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u + u^5$ with $p, q \in (4,6)$. Besides some other conditions, he assumed that $a, b \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} a(x) = a_\infty$, $\lim_{|x|\to\infty} b(x) = 0$ and $a(x) \geq a_\infty - Ce^{-a_0|x|}$ for some $a_0 > 0$ and $x \in \mathbb{R}^3$, and proved the existence of one ground state solution for each $\lambda > 0$. It was also proven the existence of two nontrivial solutions for $\lambda > 0$ small. Fan [8] discussed problem (1.4) when $V = 1$ and $\hat{f}(x,u) = \lambda f(x)u^{p-2} + g(x)u^5$ with $(4 < p < 6)$. With the help of Nehari manifold and Lusternik–Schnirelmann theory, he obtained a relationship between the number of positive solutions and the topology of the global maximum set of g. Later, by using a technique introduced by Adachi and Tanaka [1], Zhang et al. [27] obtained the existence of two nontrivial solutions for problem (1.4) when $\hat{f}(x,u) := \lambda f(u) + g(x)u^5$ with f belongs to $C^1(\mathbb{R}, \mathbb{R})$, V has a positive lower bound and satisfies the condition

$$\exists r > 0 \text{ such that } \lim_{|y|\to\infty} \operatorname{meas}\left\{x \in \mathbb{R}^3 : |x - y| < r, V(x) \leq M\right\} = 0, \qquad \forall M > 0.$$

Following [27], Zhang et al. [28] studied the multiplicity of solutions for the ciritical fractional Schrödinger equation with a small superlinear term of the form $(-\Delta)^s u + V(x)u = \lambda f(x,u) + g(x)|u|^{2_s^*-2}u$ in $\mathbb{R}^N$, where $N \geq 3$, $s \in (0,1)$ and $2_s^* = \frac{2N}{N-2s}$ is the critical exponent. Li et al. [11] studied the existence and concentration of positive solutions for the following nonlinear Kirchhoff-type problem

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = P(x)f(u) + Q(x)u^5 \quad \text{in } \mathbb{R}^3,$$

where a, $b > 0$, $\varepsilon > 0$ is a parameter and f is a continuous subcritical nonlinearity. As $\varepsilon \to 0$, they explored the asymptotic behavior of the semiclassical solutions. See also [26, 30] for related results.

Motivated by the works mentioned above, in this paper, we consider the multiplicity of solutions for the critical Kirchhoff-type problem (1.1) under more general conditions. Precisely, we make the following hypotheses:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) := V_0 > 0$ and $\lim_{|x|\to\infty} V(x) = V_\infty > 0$.

(h) $h \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} h(x) := h_0 > 0$ and $\lim_{|x|\to\infty} h(x) = h_\infty > 0$.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} \frac{f(s)}{s} = \lim_{|s|\to\infty} \frac{f(s)}{s^5} = 0$.

(f_2) $\frac{f(s)}{s^3}$ is positive for $s \neq 0$, nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$.

(f_3) $\lim_{s\to+\infty} \frac{F(s)}{|s|^4} = +\infty$, where $F(s) = \int_0^s f(t)dt$.

(g_1) $g \in C(\mathbb{R}^3, \mathbb{R})$, $g_0 := \inf_{x\in\mathbb{R}^3} g(x) > 0$, $g_M := \sup_{x\in\mathbb{R}^3} g(x) < +\infty$ and $g_\infty := \liminf_{|x|\to\infty} g(x) < g_M$.

(g_2) There exists $\rho_0 > 0$ such that $g(x) = g_M$ for $\rho_0 < |x| < 2\rho_0$. Moreover, $g(0) < g_M$.

For dealing with the multiplicity of solutions to problem (1.1), we recall the Lusternik–Schnirelmann category theory. Suppose that Y is a closed subset of a topological space X, we denoted by $\text{cat}_X(Y)$ the Lusternik–Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y; see [4] for more details. Denote

$$\Lambda := \left\{y \in \mathbb{R}^3 : g(y) = g_M\right\} \quad \text{and} \quad \Lambda_d := \left\{x \in \mathbb{R}^3 : \text{dist}(x, \Lambda) < d\right\} \quad \text{for } d > 0.$$

We assume that

(g_3) The set Λ is nonempty and bounded, there exists $\rho \geq 1$ such that $g(x) - g(y) = O(|x-y|^\rho)$ as $x \to y$ uniformly for $y \in \Lambda$.

The main results of this paper are the following.

Theorem 1.1. *Assume that (V), (h), (f_1)–(f_3) and (g_1)–(g_2) are satisfied. Then there exists $\lambda_0 > 0$ such that problem (1.1) has at least two nontrivial solutions for $\lambda \in (0, \lambda_0)$.*

Theorem 1.2. *Assume that (V), (h), (f_1)–(f_3), (g_1) and (g_3) are satisfied. Then for any $d > 0$, there exists $\lambda_d > 0$ such that, for any $\lambda \in (0, \lambda_d)$, problem (1.1) has at least $\text{cat}_{\Lambda_d}(\Lambda)$ nontrivial solutions.*

Remark 1.3. By assumption (g_3), there exist $C, r > 0$ such that for any $y \in \Lambda$,

$$|g(x) - g(y)| \le C|x - y|^\rho, \qquad \forall x \in B_r(y),$$

where $B_r(y)$ denotes the ball in $\mathbb{R}^3$ with radius r and center y.

Remark 1.4. We point out that, in some special cases, Theorem 1.2 permits to find an arbitrarily large number of solutions of problem (1.1). For example, suppose that (V), (h), (f_1)–(f_3) hold, $g \in C(\mathbb{R}^3, (0, +\infty))$ satisfying $0 < g_0 \le g_\infty < g_M$, and there exist k points $x_1, x_2, \ldots, x_k$ in $\mathbb{R}^3$ such that $g(x_i)$ are strict local maxima satisfying $g(x_i) = g_M = \max_{x\in\mathbb{R}^3} g(x)$, and

$$|g(x) - g(x_i)| = O(|x - x_i|^\rho) \quad \text{as } x \to x_i$$

for each $i = 1, 2, \ldots, k$ and some $\rho \ge 1$. Then it is easy to check that there exists $d = d(k) > 0$ such that $\operatorname{cat}_{\Lambda_d}(\Lambda) \ge k$. By Theorem 1.2, problem (1.1) has at least k solutions for any $\lambda \in (0, \lambda_d)$.

The proofs of Theorem 1.1 and Theorem 1.2 are based on variational methods. Since f is only continuous, we can not use the Nehari manifold arguments developed in [9, 14, 16] in which the condition $f \in C^1$ is required and to overcome this difficulty, we apply some variants of critical point theorems due to Szulkin and Weth [20]. Moreover, there are two main difficulties to prove our result. First, the lack of compactness which caused by the unbounded domain and the critical growth terms makes the bounded (PS) sequences could not converge. Second, the appearance of the nonlocal term, it would be natural to consider how the interaction between the nonlocal term and the critical nonlinear term will effect the existence and multiplicity of solutions of problem (1.1). To overcome these difficulties, we adapt a technique introduced by Benci and Cerami [4] and use the Lusternik–Schnirelmann category.

The paper is organized as follows. In Section 2, we present some technique lemmas and make the estimations for the functionals associated to problem (1.1). In Sections 3 and 4, we show the multiplicity results and complete the proofs of Theorems 1.1 and 1.2, respectively.

Throughout the paper, we make use of the following **notations**. $H^1(\mathbb{R}^3)$ is the Hilbert space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx$. $L^s(\mathbb{R}^3)$, $1 \le s \le +\infty$, denotes the usual Lebesgue space with the norm $\|\cdot\|_s$. $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^3}|\nabla u|^2 dx$. S denotes the best Sobolev constant $S := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \|u\|^2_{\mathcal{D}^{1,2}} \setminus \|u\|_6^2$. Finally, $C, C_1, C_2, \ldots$ denote different positive constants whose exact value is inessential.

2 Preliminaries

Let $E = H^1(\mathbb{R}^3)$ and $\|u\| = \left(\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)dx\right)^{1/2}$. Then, by (V), $\|\cdot\|$ is an equivalent norm on E. We defined the functional on E by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3}\left(\lambda h(x)F(u) + \frac{1}{6}g(x)|u|^6\right)dx.$$

It follows from (f_1) that for any $\varepsilon > 0$, $p \in (2, 6)$, there exists $C_\varepsilon > 0$ such that

$$\max\{|F(u)|, |f(u)u|\} \le \varepsilon|u|^2 + C_\varepsilon|u|^6, \qquad \forall u \in \mathbb{R}, \tag{2.1}$$

$$\max\{|F(u)|, |f(u)u|\} \le \varepsilon(|u|^2 + |u|^6) + C_\varepsilon |u|^p, \qquad \forall u \in \mathbb{R}. \tag{2.2}$$

By (f_2), we derive that

$$\frac{1}{4} f(u)u \ge F(u) \ge 0, \qquad \forall u \in \mathbb{R} \tag{2.3}$$

and

$$\frac{1}{4} f(t)t - F(t) \text{ is nondecreasing in } t > 0 \text{ and nonincreasing in } t < 0. \tag{2.4}$$

Indeed, for $0 \le s \le t$, we have

$$\begin{aligned}
\left(\frac{1}{4} f(t)t - F(t)\right) - \left(\frac{1}{4} f(s)s - F(s)\right) &= \frac{1}{4}\left(f(t)t - f(s)s\right) - \left(F(t) - F(s)\right) \\
&= \int_0^t \frac{f(t)}{t^3}\tau^3 d\tau - \int_0^s \frac{f(s)}{s^3}\tau^3 d\tau - \int_s^t f(\tau) d\tau \\
&= \int_0^s \left(\frac{f(t)}{t^3} - \frac{f(s)}{s^3}\right)\tau^3 d\tau + \int_s^t \left(\frac{f(t)}{t^3} - \frac{f(\tau)}{\tau^3}\right)\tau^3 d\tau \\
&\ge 0.
\end{aligned}$$

Arguing similarly for the case $t \le s \le 0$.

In order to find the critical points of I_λ, we consider the Nehari manifold

$$\mathcal{M}_\lambda = \left\{u \in E \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\right\}.$$

Obviously, $\mathcal{M}_\lambda$ contains all nontrivial critical points of I_λ. Since it is not assumed that f is differentiable, $\mathcal{M}_\lambda$ may not be a C^1-manifold. To overcome the non-differentiability of $\mathcal{M}_\lambda$, we adapt a technique developed in Szulkin and Weth [20].

Lemma 2.1. *Under conditions (V), (h), (f_1)–(f_2) and (g_1), for $\lambda \in (0,1)$, we have*

- *(i) for each $u \in E \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{M}_\lambda$. Moreover, the point t_u is a maximum for $t \to I_\lambda(tu)$;*
- *(ii) the set $\mathcal{M}_\lambda$ is bounded away from 0;*
- *(iii) let $S_1 = \{u \in E : \|u\| = 1\}$, then there exists $\alpha > 0$ such that $t_u \ge \alpha$ for each $u \in S_1$ and, for each compact subset $K \subset S_1$, there exists a constant $C_K > 0$ such that $t_u \le C_K$ for all $u \in K$;*
- *(iv) the mapping m_λ is a homeomorphism between S_1 and $\mathcal{M}_\lambda$, and for every $u \in \mathcal{M}_\lambda$, $m_\lambda^{-1}(u) = \frac{u}{\|u\|} \in S_1$.*

Proof. (i) For each $u \in E \setminus \{0\}$ and $t > 0$, set $g(t) = I_\lambda(tu)$. It is easy to see that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Thus g has a positive maximum at $t = t_u > 0$ such that $g'(t_u) = 0$ and $t_u u \in \mathcal{M}_\lambda$. Noticing

$$g'(t) = t\left[\|u\|^2 + t^2\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2\right] - t^3\left[\lambda \int_{\mathbb{R}^3} h(x)\frac{f(tu)}{(tu)^3}u^4 dx + t^2 \int_{\mathbb{R}^3} g(x)|u|^6 dx\right],$$

we have that t_u is unique. Indeed, suppose $t'_u > t_u > 0$ such that $t'_u u$, $t_u u \in \mathcal{M}_\lambda$. Then we deduce

$$\frac{\|u\|^2}{t_u^2} + b\|\nabla u\|_2^4 = \lambda \int_{\mathbb{R}^3} h(x)\frac{f(t_u u)}{(t_u u)^3}u^4 dx + t_u^2 \int_{\mathbb{R}^3} g(x)|u|^6 dx,$$

$$\frac{\|u\|^2}{t_u'^2}+b\|\nabla u\|_2^4=\lambda\int_{\mathbb{R}^3}h(x)\frac{f(t_u'u)}{(t_u'u)^3}u^4dx+t_u'^2\int_{\mathbb{R}^3}g(x)|u|^6dx,$$

and hence,

$$\left(\frac{1}{t_u^2}-\frac{1}{t_u'^2}\right)\|u\|^2=\lambda\int_{\mathbb{R}^3}h(x)\left(\frac{f(t_uu)}{(t_uu)^3}-\frac{f(t_u'u)}{(t_u'u)^3}\right)u^4dx+\left(t_u^2-t_u'^2\right)\int_{\mathbb{R}^3}g(x)|u|^6dx,$$

which is impossible in view of (f_2) and $t_u'>t_u>0$.

(ii) By using (2.1), (h) and (g_1), we deduce that for any $u\in\mathcal{M}_\lambda$,

$$\begin{aligned}\|u\|^2&\le\int_{\mathbb{R}^3}\left(\lambda h(x)f(u)u+g(x)|u|^6\right)dx\\&\le C\varepsilon\int_{\mathbb{R}^3}|u|^2dx+(C_1C_\varepsilon+g_M)\int_{\mathbb{R}^3}|u|^6dx\\&\le\frac{C\varepsilon}{V_0}\|u\|^2+\frac{C_1C_\varepsilon+g_M}{(aS)^3}\|u\|^6,\end{aligned}$$

which implies that $\|u\|^2\ge C_2$ for some $C_2>0$.

(iii) For each $u\in S_1$, there exists $t_u>0$ such that $t_uu\in\mathcal{M}_\lambda$. By (ii), we have

$$t_u=\|t_uu\|\ge\alpha.$$

Now we prove that $t_u\le C_K$ for all $u\in K\subset S_1$. Arguing indirectly, assume that there exists $\{u_n\}\subset K\subset S_1$ such that $t_{u_n}\to\infty$. Since K is compact, we have $u_n\to u\in K$ and $\int_{\mathbb{R}^3}|u_n|^6dx\to\int_{\mathbb{R}^3}|u|^6dx>0$. Then,

$$I_\lambda(t_{u_n}u_n)\le\frac{t_{u_n}^2}{2}\|u_n\|^2+\frac{bt_{u_n}^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2-\frac{t_{u_n}^6}{6}\int_{\mathbb{R}^3}g_0|u_n|^6dx\to-\infty$$

as $n\to\infty$, which leads to a contradiction because (2.3) implies that, for all $u\in\mathcal{M}_\lambda$,

$$\begin{aligned}I_\lambda(u)&=I_\lambda(u)-\frac{1}{4}\langle I_\lambda'(u),u\rangle,\\&=\frac{1}{4}\|u\|^2+\lambda\int_{\mathbb{R}^3}h(x)\left(\frac{1}{4}f(u)u-F(u)\right)dx+\frac{1}{12}\int_{\mathbb{R}^3}g(x)|u|^6dx\\&\ge0.\end{aligned}$$

(iv) Let us define the maps $\hat{m}_\lambda:E\setminus\{0\}\to\mathcal{M}_\lambda$ and $m_\lambda:S_1\to\mathcal{M}_\lambda$ by setting

$$\hat{m}_\lambda(u)=t_uu\quad\text{and}\quad m_\lambda=\hat{m}_\lambda|_{S_1}.\tag{2.5}$$

By virtue of (i)–(iii) and [20, Proposition 3.1], we deduce that m_λ is a homemorphism between S_1 and $\mathcal{M}_\lambda$, and the inverse of m_λ is given by $m_\lambda^{-1}(u)=\frac{u}{\|u\|}$. □

Now we define the functionals $\hat{J}_\lambda:E\setminus\{0\}\to\mathbb{R}$ and $J_\lambda:S\to\mathbb{R}$, as

$$\hat{J}_\lambda(u)=I_\lambda(\hat{m}_\lambda(u))\quad\text{and}\quad J_\lambda(u)=\hat{J}_\lambda|_{S_1},$$

where $\hat{m}_\lambda(u)=t_uu$ is given in (2.5). As in [20], we have the following conclusion.

Lemma 2.2. *Under the conditions of Lemma 2.1, for $\lambda\in(0,1)$, we have* *(i)* $J_\lambda \in C^1(S_1, \mathbb{R})$ *and for each* $v \in T_u(S_1) := \{v \in E : \langle u, v\rangle = 0\}$,

$$\langle J'_\lambda(u), v\rangle = \|m_\lambda(u)\| \langle I'_\lambda(m_\lambda(u)), v\rangle;$$

(ii) $\{u_n\}$ *is a Palais–Smale sequence for* J_λ *if and only if* $\{m_\lambda(u_n)\}$ *is a Palais–Smale sequence for* I_λ. *If* $\{u_n\} \subset \mathcal{M}_\lambda$ *is a bounded Palais–Smale sequence for* I_λ, *then* $\{m_\lambda^{-1}(u_n)\}$ *is a Palais–Smale sequence for* J_λ;

(iii) $u \in S_1$ *is a critical point of* J_λ *if and only if* $m_\lambda(u)$ *is a nontrivial critical point of* I_λ. *Moreover, the corresponding values coincide and* $\inf_{S_1} J_\lambda = \inf_{\mathcal{M}_\lambda} I_\lambda$.

Taking

$$c^* := \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}\right)^2$$

and $m_\lambda^\infty := \inf_{u \in \mathcal{M}_\lambda^\infty} I_\lambda^\infty(u)$, where

$$I_\lambda^\infty(u) = \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} \left(\lambda h_\infty F(u) + \frac{1}{6} g_\infty |u|^6\right) dx,$$

$\mathcal{M}_\lambda^\infty = \{u \in E \setminus \{0\} : \langle I_\lambda^{\infty\prime}(u), u\rangle = 0\}$ and $\|u\|_{V_\infty} = \left(\int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + V_\infty uv) dx\right)^{\frac{1}{2}}$. We have the following local compactness result for I_λ.

Lemma 2.3. *Assume that conditions* (V), (h), (f_1)–(f_2) *and* (g_1) *are satisfied. Let* $\lambda > 0$ *and* $\{u_n\} \subset E$ *be a sequence such that* $I_\lambda(u_n) \to c_\lambda \in (-\infty, \min\{c^*, m_\lambda^\infty\})$ *and* $I'_\lambda(u_n) \to 0$ *as* $n \to \infty$. *Then* (u_n) *has a strongly convergent subsequence.*

Proof. By (V) and (2.3), we have

$$c_\lambda + o(1) + o(1)\|u_n\| = I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \geq \frac{1}{4}\|u_n\|^2,$$

which implies that $\{u_n\}_{n\in\mathbb{N}} \subset E$ is bounded. Going if necessary to a subsequence, we may assume that there is $u \in E$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } E, \quad u_n(x) \to u(x) \ \text{ a.e. } x \in \mathbb{R}^3, \\ u_n &\to u && \text{in } L^s(\Omega) \ \ (2 < s < 6), \\ |u_n(x)| &\leq w(x) && \text{for some } w \in L^s(\Omega). \end{aligned} \tag{2.6}$$

Take $A = \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx$. We define the functionals G, H, Φ, Ψ on E by

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|^2 + \frac{bA}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} \left(\lambda h(x)F(u) + \frac{1}{6}g(x)u^6\right) dx, \\ H(u) &= \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{bA}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} \left(\lambda h_\infty F(u) + \frac{1}{6}g(x)u^6\right) dx, \\ \Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{bA}{4}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} \left(\lambda h(x)F(u) + \frac{1}{6}g(x)u^6\right) dx, \\ \Psi(u) &= \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{bA}{4}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} \left(\lambda h_\infty F(u) + \frac{1}{6}g(x)u^6\right) dx. \end{aligned}$$

We claim that $G'(u) = 0$, i.e., $\langle G'(u), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Assume that $1 \leq p, q, r, s < +\infty$, Ω is a bounded domain and $h \in C(\Omega \times \mathbb{R})$ satisfying $|h(x,u)| \leq C(|u|^{p/r} + |u|^{q/s})$, then, according to [22, Theorem A.4], the operator

$$A : L^p(\Omega) \bigcap L^q(\Omega) \longrightarrow L^r(\Omega) + L^s(\Omega) : u \to h(x,u)$$

is continuous, where $L^p(\Omega) \bigcap L^q(\Omega)$ is the space endowed with the norm $|u|_{p\wedge q} = \|u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}$ and $L^r(\Omega) + L^s(\Omega)$ endowed with the norm

$$|u|_{r\vee s} = \inf\left\{\|v\|_{L^r(\Omega)} + \|w\|_{L^s(\Omega)} : u = v + w, v \in L^r(\Omega), w \in L^s(\Omega)\right\}.$$

Now set $p = r = 2$, $q \in (5,6)$, $s = q/5$ and $h(x,u) = \lambda h(x) f(u) u + g(x) u^5$. By (h), (g) and (f_1), we have

$$|h(x,u)| \leq C(|u|^{\frac{2}{2}} + |u|^{\frac{q}{s}}), \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Since $\varphi \in C_0^\infty(\mathbb{R}^3)$ has a compact support Ω_0, $u_n \rightharpoonup u$ in E implies that $u_n \to u$ in $L^2(\Omega_0) \bigcap L^q(\Omega_0)$. So, by virtue of [22, Theorem A.4],

$$h(x, u_n) \to h(x,u) \quad \text{in } L^2(\Omega_0) + L^s(\Omega_0).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^3} |(h(x,u_n) - h(x,u))\varphi| dx &= \int_{\Omega_0} |(h(x,u_n) - h(x,u))\varphi| dx \\ &\leq |h(x,u_n) - h(x,u)|_{2\vee s} |\varphi|_{2\wedge s'} \xrightarrow{n} 0,\end{aligned}$$

where $1/s + 1/s' = 1$. Combining this and (2.6), we get that $o(1) = \langle I'_\lambda(u_n), \varphi\rangle = \langle G'(u), \varphi\rangle + o(1)$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Thus $G'(u) = 0$.

Let $v_n := u_n - u$. It follows from the Brézis–Lieb lemma, [31, Lemma 2.2] and (f_1) that

$$\begin{aligned}\|u_n\|^2 - \|v_n\|^2 - \|u\|^2 &= o(1), \\ \int_{\mathbb{R}^3} g(x)(u_n^6 - u^6 - v_n^6) dx &= o(1), \\ \int_{\mathbb{R}^3} h(x)\left(F(u_n) - F(u) - F(v_n)\right) dx &= o(1).\end{aligned} \tag{2.7}$$

Noting u is a critical point of G, arguing as in [29, Lemma 2.3], we can conclude that u is locally bounded. Hence, for each $\xi \in E$, by [22, Lemma 8.9], we get

$$\left|\int_{\mathbb{R}^3} g(x)(u_n^5 - u^5 - v_n^5)\xi dx\right| = o(1)\|\xi\|, \tag{2.8}$$

and, similar to [22, Lemma 8.1],

$$\left|\int_{\mathbb{R}^3} h(x)(f(u_n) - f(u) - f(v_n))\xi dx\right| = o(1)\|\xi\|. \tag{2.9}$$

Since $v_n \rightharpoonup 0$ in E, by (V), (h) and (2.2), we deduce that

$$\int_{\mathbb{R}^3} (V(x) - V_\infty) v_n^2 dx \to 0, \qquad \int_{\mathbb{R}^3} (h(x) - h_\infty) F(v_n) dx \to 0, \tag{2.10}$$

and

$$\int_{\mathbb{R}^3} (V(x) - V_\infty) v_n \xi dx \to 0, \qquad \int_{\mathbb{R}^3} (h(x) - h_\infty) f(v_n) \xi dx, \qquad \forall \xi \in E \tag{2.11}$$

as $n \to \infty$. Hence we have

$$\begin{aligned}
c_\lambda + o(1) &= I_\lambda(u_n) \\
&= \frac{1}{2}(\|u\|^2 + \|v_n\|^2) + \frac{bA}{4}\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v_n|^2)dx \\
&\quad - \int_{\mathbb{R}^3} \lambda h(x)(F(u) + F(v_n))dx - \int_{\mathbb{R}^3} \frac{1}{6}g(x)(u^6 + v_n^6)dx + o(1) \\
&= \frac{1}{2}(\|u\|^2 + \|v_n\|^2_{V_\infty}) + \frac{bA}{4}\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v_n|^2)dx \\
&\quad - \int_{\mathbb{R}^3} \lambda(h(x)F(u) + h_\infty F(v_n))dx - \int_{\mathbb{R}^3} \frac{1}{6}g(x)(u^6 + v_n^6)dx + o(1) \\
&= \Phi(u) + \Psi(v_n) + o(1),
\end{aligned}$$

by (2.10) and (2.7). Moreover, noting $G'(u) = 0$, by (h), (g_1) and (2.3) we have

$$\begin{aligned}
\Phi(u) &= \Phi(u) - \frac{1}{4}\langle G'(u), u\rangle \\
&= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3}\left[\lambda h(x)\left(\frac{1}{4}f(u)u - F(u)\right) + \frac{1}{12}g(x)u^6\right]dx \\
&\geq 0,
\end{aligned}$$

and hence

$$c_\lambda + o(1) \geq \Psi(v_n). \tag{2.12}$$

Combining (2.8), (2.9) and (2.11), we obtain that

$$\begin{aligned}
o(1) &= \langle I'_\lambda(u_n), \xi\rangle - \langle G'(u), \xi\rangle \\
&= (v_n, \xi) + bA\int_{\mathbb{R}^3}\nabla v_n \nabla \xi dx - \int_{\mathbb{R}^3}(\lambda h(x)f(v_n)\xi + g(x)v_n^5\xi)dx + o(1) \\
&= \int_{\mathbb{R}^3}(a\nabla v_n\nabla\xi + V_\infty v_n\xi)dx + bA\int_{\mathbb{R}^3}\nabla v_n\nabla\xi dx - \int_{\mathbb{R}^3}(\lambda h_\infty f(v_n)\xi + g(x)v_n^5\xi)dx + o(1) \\
&= \langle H'(v_n), \xi\rangle + o(1), \qquad \forall \xi \in E,
\end{aligned}$$

which implies that

$$H'(v_n) = o(1). \tag{2.13}$$

Next we prove that $v_n \to 0$ in E. According to [31, Lemma 2.1], for some subsequence of $\{v_n\}$, either "vanishing" or "nonvanishing" holds. If "nonvanishing" occurs, we can find $(y_n) \subset \mathbb{R}^3$ with $y_n \xrightarrow{n} \infty$ such that, for $w_n(x) := v_n(x + y_n)$, there is $w \in E \setminus \{0\}$ satisfying

$$\begin{aligned}
w_n &\rightharpoonup w && \text{in } E, \\
w_n &\to w && \text{in } L^s_{loc}(\mathbb{R}^3) \ \ (2 \leq s < 6), \\
w_n(x) &\to w(x) && \text{a.e. } x \in \mathbb{R}^3.
\end{aligned} \tag{2.14}$$

We claim that

$$L'(w) = 0, \tag{2.15}$$

where

$$L(u) = \frac{1}{2}\|u\|^2_{V_\infty} + \frac{bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left(\lambda h_\infty F(u) + \frac{1}{6}g_\infty u^6\right)dx.$$

Indeed, for every $\xi \in E$, set $\xi_n(x) = \xi(x - y_n)$. We have $\|\xi_n\|_{H^1} = \|\xi\|_{H^1}$, and hence, by (2.13) and (2.14),

$$\begin{aligned}
&|\langle H'(v_n), \xi_n\rangle| \\
&= \left| \int_{\mathbb{R}^3} (a\nabla v_n \cdot \nabla \xi_n + V_\infty v_n \xi_n + bA\nabla v_n \cdot \nabla \xi_n - \lambda h_\infty f(v_n)\xi_n - g(x) v_n^5 \xi_n)\, dx \right| \\
&= \left| \int_{\mathbb{R}^3} (a\nabla w_n \cdot \nabla \xi + V_\infty w_n \xi + bA\nabla w_n \cdot \nabla \xi - \lambda h_\infty f(w_n)\xi - g(x+y_n) w_n^5 \xi)\, dx \right| \\
&= \left| \int_{\mathbb{R}^3} (a\nabla w \cdot \nabla \xi + V_\infty w \xi + bA\nabla w \cdot \nabla \xi)\, dx - \int_{supp\xi} (\lambda h_\infty f(w)\xi + g_\infty w^5 \xi)\, dx \right| + o(1) \\
&= |\langle L'(w), \xi\rangle| + o(1),
\end{aligned}$$

and

$$|\langle H'(v_n), \xi_n\rangle| \le \|H'(v_n)\| \|\xi_n\| \le C\|H'(v_n)\| \|\xi_n\|_{H^1} \le C\|H'(v_n)\| \|\xi\|_{H^1} \xrightarrow{n} 0.$$

So (2.15) holds. From (2.15), we see that

$$\|w\|^2_{V_\infty} + bA \int_{\mathbb{R}^3} |\nabla w|^2 dx = \int_{\mathbb{R}^3} (\lambda h_\infty f(w) w + g_\infty |w|^6) dx. \tag{2.16}$$

Since $w \neq 0$, there exists a unique $t > 0$ such that $tw \in \mathcal{M}^\infty_\lambda$, i.e.,

$$t^2 \|w\|^2_{V_\infty} + bt^4 \left(\int_{\mathbb{R}^3} |\nabla w|^2 dx \right)^2 = \int_{\mathbb{R}^3} (\lambda h_\infty f(tw) tw + t^6 g_\infty |w|^6) dx. \tag{2.17}$$

We claim that $t \le 1$. For otherwise $t > 1$, then it follows from (2.17), (2.16), (f_2) and the fact $A \ge \int_{\mathbb{R}^3} |\nabla w|^2 dx$ that

$$\begin{aligned}
t^2 \|w\|^2_{V_\infty} + bt^4 \left(\int_{\mathbb{R}^3} |\nabla w|^2 dx \right)^2 &< t^4 \left[\|w\|^2_{V_\infty} + b \left(\int_{\mathbb{R}^3} |\nabla w|^2 dx \right)^2 \right] \\
&\le t^4 \left(\|w\|^2_{V_\infty} + bA \int_{\mathbb{R}^3} |\nabla w|^2 dx \right) \\
&= t^4 \int_{\mathbb{R}^3} (\lambda h_\infty f(w) w + g_\infty |w|^6) dx \\
&\le \int_{\mathbb{R}^3} (\lambda h_\infty f(w) t^4 w + t^6 g_\infty |w|^6) dx \\
&\le \int_{\mathbb{R}^3} \left(\lambda h_\infty \frac{f(tw)}{(tw)^3} t^4 w^4 + t^6 g_\infty |w|^6 \right) dx \\
&= \int_{\mathbb{R}^3} (\lambda h_\infty f(tw) tw + t^6 g_\infty |w|^6) dx \\
&= t^2 \|w\|^2_{V_\infty} + bt^4 \left(\int_{\mathbb{R}^3} |\nabla w|^2 dx \right)^2,
\end{aligned}$$

a contradiction. Thus $t \le 1$. Combining this with (2.4), (2.12), (2.13) and Fatou's lemma, we deduce that

$$\begin{aligned}
c_\lambda + o(1) &\ge \Psi(v_n) - \frac{1}{4} \langle H'(v_n), v_n\rangle \\
&= \frac{1}{4} \|v_n\|^2_{V_\infty} + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4} f(v_n) v_n - F(v_n) \right) dx + \frac{1}{12} \int_{\mathbb{R}^3} g(x) |v_n|^6 dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\|w_n\|_{V_\infty}^2 + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4} f(w_n)w_n - F(w_n)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g(x+y_n)|w_n|^6 dx \\
&\geq \frac{1}{4}\|w\|_{V_\infty}^2 + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4} f(w)w - F(w)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g_\infty |w|^6 dx + o(1) \\
&\geq \frac{1}{4}\|tw\|_{V_\infty}^2 + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4} f(tw)tw - F(tw)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g_\infty |tw|^6 dx + o(1) \\
&= I_\lambda^\infty(tw) - \frac{1}{4}\langle I_\lambda^{\infty\prime}(tw), tw\rangle + o(1) \\
&= I_\lambda^\infty(tw) + o(1) \\
&\geq m_\lambda^\infty + o(1),
\end{aligned}
$$

which contradicts $c_\lambda < m_\lambda^\infty$.

Thus, “nonvanishing” cannot occur, and then we have only the “vanishing” case. In this case, $v_n \to 0$ in $L^s(\mathbb{R}^3)$ $(2 < s < 6)$, and hence, by (2.2), we see that

$$
\int_{\mathbb{R}^3} h(x)F(v_n)dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} h(x)f(v_n)v_n dx \to 0
$$

as $n \to \infty$. Combining this and (2.12)–(2.13), we obtain

$$
c_\lambda + o(1) \geq \Psi(v_n) = \frac{1}{2}\|v_n\|_{V_\infty}^2 + \frac{bA}{4}\int_{\mathbb{R}^3} |v_n|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} g(x)|v_n|^6 dx + o(1), \tag{2.18}
$$

$$
o(1) = \langle H'(v_n), v_n\rangle = \|v_n\|_{V_\infty}^2 + bA\int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \int_{\mathbb{R}^3} g(x)|v_n|^6 dx + o(1). \tag{2.19}
$$

Set $l = \lim_{n\to\infty}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}}$. If $l > 0$, then using (2.19) and the fact $g(x) \leq g_M$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} g(x)|v_n|^6 dx &\geq \int_{\mathbb{R}^3} a|\nabla v_n|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 + o(1) \\
&\geq aS\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{1}{3}} + bS^2\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{2}{3}} + o(1) \\
&\geq \frac{aS}{g_M^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{g_M^{\frac{2}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{2}{3}} + o(1),
\end{aligned}
$$

which implies that $l \geq \frac{bS^2 + \sqrt{(bS^2)^2 + 4aSg_M}}{2g_M^{\frac{2}{3}}}$. Combining this and (2.18), (2.19), we deduce that

$$
\begin{aligned}
c_\lambda + o(1) &\geq \Psi(v_n) - \frac{1}{6}\langle H'(v_n), v_n\rangle \\
&\geq \frac{a}{3}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 \\
&\geq \frac{aS}{3g_M^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{12g_M^{\frac{2}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{2}{3}} \\
&\geq \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}\right)^2 + o(1) \\
&= c^* + o(1),
\end{aligned}
$$

which is a contradiction. Thus $l = 0$, which, together with (2.19), yields that $v_n \to 0$ in E. Therefore $u_n \to u$ in E and the proof is complete. □

Lemma 2.4. *Under the conditions of Lemma 2.3, then there exists $\lambda_1 > 0$ such that $c^* < m_\lambda^\infty$ for $\lambda \in (0, \lambda_1)$.*

Proof. Suppose by contradiction that there is $\lambda_n \to 0$ such that $m_{\lambda_n}^\infty \le c^*$ for all n. In view of [13], $m_{\lambda_n}^\infty$ is attained by a positive solution $u_n \in M_{\lambda_n}^\infty$ such that $I_{\lambda_n}^\infty(u_n) = m_{\lambda_n}^\infty$. We claim that there exist C_3, $C_4 > 0$ (independent of λ) such that $C_3 \le \|u_n\|_{V_\infty} \le C_4$ for all n. Indeed, by (2.3), we have

$$c^* \ge m_{\lambda_n}^\infty = I_{\lambda_n}^\infty(u_n) - \frac{1}{4}\langle (I_{\lambda_n}^\infty)'(u_n), u_n\rangle \ge \frac{1}{4}\|u_n\|_{V_\infty}^2$$

for all n, that is, $\|u_n\|_{V_\infty}^2 \le 4c^*$ for all n. On the other hand, since $u_n \in M_{\lambda_n}^\infty$, by condition (g_1) and (2.1), we obtain for $\varepsilon \in (0, \frac{V_\infty}{2h_\infty})$,

$$\|u_n\|_{V_\infty}^2 \le \lambda_n h_\infty \varepsilon \int_{\mathbb{R}^3} |u_n|^2 dx + (\lambda_n h_\infty C_\varepsilon + g_\infty) \int_{\mathbb{R}^3} |u_n|^6 dx$$

and then,

$$\frac{1}{2}\|u_n\|_{V_\infty}^2 \le (h_\infty C_\varepsilon + g_\infty)(aS)^{-3}\|u_n\|_{V_\infty}^6$$

for large n, which implies that

$$\|u_n\|_{V_\infty}^2 \ge \frac{(aS)^{\frac{3}{2}}}{\sqrt{2(h_\infty C_\varepsilon + g_\infty)}} \tag{2.20}$$

for n large. Then, noting $\lambda_n \to 0$, we deduce that $\lambda_n \int_{\mathbb{R}^3} h(x)F(u_n)dx = o(1)$ and $\lambda_n \int_{\mathbb{R}^3} h(x)f(u_n)u_n dx = o(1)$. Hence

$$\begin{aligned} m_{\lambda_n}^\infty &= \frac{1}{2}\|u_n\|_{V_\infty}^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} g_\infty |u_n|^6 dx + o(1), \\ 0 &= \|u_n\|_{V_\infty}^2 + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} g_\infty |u_n|^6 dx + o(1). \end{aligned} \tag{2.21}$$

Set $\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = D$. One has $D > 0$. Indeed, if $D = 0$, then $\int_{\mathbb{R}^3} |u_n|^6 dx \to 0$ as $n \to \infty$, and thus, by (2.21), $\|u_n\|_{V_\infty}^2 \to 0$. This gives a contradiction to (2.20). It follows from (2.21) and the definition of S that $D \ge \frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_\infty}}{2g_\infty}$. Hence

$$\begin{aligned} m_{\lambda_n}^\infty &= I_{\lambda_n}^\infty(u_n) - \frac{1}{6}\langle (I_{\lambda_n}^\infty)'(u_n), u_n\rangle \\ &\ge \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 + o(1) \\ &\ge \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_\infty}}{2g_\infty}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_\infty}}{2g_\infty}\right)^2 + o(1), \end{aligned}$$

a contradiction with $m_{\lambda_n}^\infty \le c^*$ and $g_\infty < g_M$. □

Corollary 2.5. *Under the conditions of Lemma 2.3, for each $\lambda \in (0, \lambda_1)$, we have J_λ satisfies the Palais–Smale condition for $c_\lambda < c^*$.*

Proof. Let $\{u_n\} \subset S_1$ be a Palais–Smale sequence for J_λ. By Lemma 2.2, $\{m_\lambda(u_n)\} \subset \mathcal{M}_\lambda$ is a Palais–Smale sequence for I_λ, and then using Lemma 2.3, we deduce that $w_n := m_\lambda(u_n) \to w$ in E after passing to a subsequence. Since the mapping m_λ is a homeomorphism between S_1 and $\mathcal{M}_\lambda$, we see that m_λ^{-1} is continuous. Hence $u_n = m_\lambda^{-1}(w_n) \to m_\lambda^{-1}(w)$ in E. The proof is complete. □

For $\varepsilon > 0$ and $y \in \Lambda$, let

$$u_{\varepsilon,y}(x) = \frac{\psi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}},$$

where $\psi \in C_0^\infty(B_{2r_0}(y))$ such that $\psi(x) = 1$ for $|x-y| \le r_0$, $0 \le \psi(x) \le 1$ and $|\nabla\psi| \le 2$. It is well known that S is attained by the function $\frac{\varepsilon^{1/4}}{(\varepsilon+|x|^2)^{1/2}}$. For $\varepsilon > 0$ small, we have (see [22]):

$$\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3} |u_{\varepsilon,y}|^6 dx = K_2 + O(\varepsilon^{\frac{3}{2}}) \tag{2.22}$$

and

$$\int_{\mathbb{R}^3} |u_{\varepsilon,y}|^s dx = \begin{cases} O(\varepsilon^{\frac{6-s}{4}}), & s \in (3,6), \\ O(\varepsilon^{\frac{3}{4}}|\ln\varepsilon|), & s = 3, \\ O(\varepsilon^{\frac{s}{4}}), & s \in [2,3), \end{cases} \tag{2.23}$$

where K_1, K_2 are positive constants and $S = K_1/K_2^{1/3}$.

Lemma 2.6. *Assume that conditions (V), (h), (f_1), (f_3) and (g_3) are satisfied. Then there exist C_0, $\varepsilon_0 > 0$ independent of $y \in \Lambda$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\sup_{t\ge0} I_\lambda(tu_{\varepsilon,y}) \le c^* - C_0\varepsilon^{\frac{1}{2}}$.*

Proof. For $y \in \Lambda$, we get

$$\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx = \int_{\mathbb{R}^3} (g(x) - g(y))|u_{\varepsilon,y}|^6 dx + \int_{\mathbb{R}^3} g_M|u_{\varepsilon,y}|^6 dx. \tag{2.24}$$

By (g_3), there exist $r_1 \in (0, 2r_0)$ and $C > 0$ such that $|g(x) - g(y)| \le C|x-y|^\rho$ for $|x-y| < r_1$ and for $y \in \Lambda$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |g(x) - g(y)||u_{\varepsilon,y}|^6 dx &= \int_{|x-y|\le 2r_0} |g(x) - g(y)||u_{\varepsilon,y}|^6 dx \\
&\le \int_{|x-y|<r_1} C|x-y|^\rho \frac{\varepsilon^{\frac{3}{2}}}{(\varepsilon + |x-y|^2)^3} dx \\
&\quad + \int_{r_1 \le |x-y| \le 2r_0} \frac{2g_M\varepsilon^{\frac{3}{2}}}{(\varepsilon + |x-y|^2)^3} dx \\
&\le C\int_0^{r_1} \frac{\varepsilon^{\frac{3}{2}} r^{2+\rho}}{(\varepsilon + r^2)^3} dr + C\int_{r_1}^{2r_0} \frac{\varepsilon^{\frac{3}{2}} r^2}{(\varepsilon + r^2)^3} dr \\
&\le C\varepsilon^{\frac{\rho}{2}} \int_0^{\frac{r_1}{\sqrt{\varepsilon}}} \frac{r^{2+\rho}}{(1+r^2)^3} dr + C\int_{\frac{r_1}{\sqrt{\varepsilon}}}^{\frac{2r_0}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^3} dr \\
&\le C_5 h(\varepsilon),
\end{aligned} \tag{2.25}$$

where

$$h(\varepsilon) = \begin{cases} \varepsilon^{\frac{\rho}{2}}, & 1 \le \rho < 3, \\ \varepsilon^{\frac{3}{2}}|\ln\varepsilon|, & \rho = 3, \\ \varepsilon^{\frac{3}{2}}, & \rho > 3. \end{cases}$$

From (2.24) and (2.25), we obtain that

$$\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx = g_M \int_{\mathbb{R}^3} |u_{\varepsilon,y}|^6 dx + O(h(\varepsilon)). \tag{2.26}$$

It follows from (2.22) and (2.23) that there exists $\varepsilon_1 > 0$ (independent of $y \in \Lambda$) such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx \le \frac{3K_1}{2}, \qquad \|u_{\varepsilon,y}\|^2 \le \frac{3aK_1}{2}, \qquad \int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx \ge \frac{g_0 K_2}{2}.$$

Then, using (2.3),

$$\begin{aligned} I_\lambda(tu_{\varepsilon,y}) &\le \frac{t^2}{2}\|u_{\varepsilon,y}\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx \\ &\le \frac{3aK_1}{4}t^2 + \frac{9bK_1^2}{16}t^4 - \frac{g_0K_2}{12}t^6, \end{aligned}$$

which implies that there are $t_1 > 0$ small and $t_2 > 0$ large (independent of ε) such that

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty)} I_\lambda(tu_{\varepsilon,y}) \le \frac{c^*}{2}. \tag{2.27}$$

Set $B_\varepsilon = \frac{\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx}{\left(\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx\right)^{1/3}}$. By (2.26) and (2.22), we have

$$B_\varepsilon = \frac{K_1 + O(\varepsilon^{\frac{1}{2}})}{(g_M K_2 + O(h(\varepsilon)))^{\frac{1}{3}}} \le \frac{S}{g_M^{\frac{1}{3}}} + O(\varepsilon^{\frac{1}{2}}). \tag{2.28}$$

Take $k(t) = \frac{a}{2}t^2\|\nabla u_{\varepsilon,y}\|_2^2 + \frac{b}{4}t^4\|\nabla u_{\varepsilon,y}\|_2^4 - \frac{t^6}{6}\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx$. Then

$$k'(t) = t\left(a\|\nabla u_{\varepsilon,y}\|_2^2 + bt^2\|\nabla u_{\varepsilon,y}\|_2^4 - t^4\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx\right),$$

and k attains its maximum at

$$t_0 = \left(\frac{b\|\nabla u_{\varepsilon,y}\|_2^4 + \sqrt{b^2\|\nabla u_{\varepsilon,y}\|_2^8 + 4a\|\nabla u_{\varepsilon,y}\|_2^2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}}{2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}\right)^{\frac{1}{2}}.$$

A direct calculation shows that

$$\begin{aligned} \max_{t\ge 0} k(t) &= k(t_0) \\ &= \frac{a\|\nabla u_{\varepsilon,y}\|_2^2\left(b\|\nabla u_{\varepsilon,y}\|_2^4 + \sqrt{b^2\|\nabla u_{\varepsilon,y}\|_2^8 + 4a\|\nabla u_{\varepsilon,y}\|_2^2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}\right)}{6\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx} \\ &\quad + \frac{b\|\nabla u_{\varepsilon,y}\|_2^4\left(b\|\nabla u_{\varepsilon,y}\|_2^4 + \sqrt{b^2\|\nabla u_{\varepsilon,y}\|_2^8 + 4a\|\nabla u_{\varepsilon,y}\|_2^2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}\right)^2}{12\left(2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx\right)^2} \\ &= \frac{a}{3}\left(\frac{bB_\varepsilon^3 + \sqrt{b^2B_\varepsilon^6 + 4aB_\varepsilon^3}}{2}\right) + \frac{b}{12}\left(\frac{bB_\varepsilon^3 + \sqrt{b^2B_\varepsilon^6 + 4aB_\varepsilon^3}}{2}\right)^2 \\ &\le \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4ag_MS^3}}{2g_M}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4ag_MS^3}}{2g_M}\right)^2 + C\varepsilon^{\frac{1}{2}} \\ &= c^* + C\varepsilon^{\frac{1}{2}} \end{aligned}$$

by (2.28). Hence, there exist $C_0 > 0$ and $\varepsilon_2 \in (0, \varepsilon_1)$ (independent of $y \in \Lambda$) such that for $\varepsilon \in (0, \varepsilon_2)$,

$$\begin{aligned}\sup_{t\in[t_1,t_2]} I_\lambda(tu_{\varepsilon,y}) &\le \sup_{t\ge 0} k(t) + \frac{t_2^2}{2}\int_{\mathbb{R}^3} V(x)|u_{\varepsilon,y}|^2dx - \inf_{t\in[t_1,t_2]} \lambda \int_{\mathbb{R}^3} h(x)F(tu_{\varepsilon,y})dx \\ &\le c^* + C_0\varepsilon^{\frac{1}{2}} - \inf_{t\in[t_1,t_2]} \lambda \int_{\mathbb{R}^3} h_0F(tu_{\varepsilon,y})dx. \end{aligned} \tag{2.29}$$

From (f_3), for any $L > 0$, there is $R_L > 0$ such that $F(u) \ge L|u|^4$ for all $u \ge R_L$. Now choosing $\varepsilon_0 \in \big(0, \min\big\{\varepsilon_2, r_0^2, (\frac{t_1}{\sqrt{2}R_L})^4\big\}\big)$, we have for $\varepsilon \in (0, \varepsilon_0)$,

$$u_{\varepsilon,y}(x) = \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}} \ge \frac{1}{\sqrt{2}\varepsilon^{\frac{1}{4}}}, \qquad \forall |x-y| \le \sqrt{\varepsilon},$$

and then,

$$\begin{aligned}\inf_{t\in[t_1,t_2]} \int_{\mathbb{R}^3} F(tu_{\varepsilon,y})dx &\ge \inf_{t\in[t_1,t_2]} \int_{|x-y|\le\sqrt{\varepsilon}} F(tu_{\varepsilon,y})dx \\ &\ge \frac{Lt_1^4}{4\varepsilon}\int_{|x-y|\le\sqrt{\varepsilon}} dx \\ &= \frac{1}{4}Lt_1^4\varepsilon^{\frac{1}{2}}\int_{|x|\le 1} dx,\end{aligned}$$

which, together with (2.29), shows that

$$\sup_{t\in[t_1,t_2]} I_\lambda(tu_{\varepsilon,y}) \le c^* + C_0\varepsilon^{\frac{1}{2}} - \frac{\lambda h_0}{4}Lt_1^4\varepsilon^{\frac{1}{2}}\int_{|x|\le 1} dx.$$

Choosing $L > 0$ large enough, we derive that there exists $\varepsilon_0 \in (0, \varepsilon_2)$ uniformly in y such that for $\varepsilon \in (0, \varepsilon_0)$, $\sup_{t\in[t_1,t_2]} I_\lambda(tu_{\varepsilon,y}) \le c^* - C_0\varepsilon^{\frac{1}{2}}$. Combining this and (2.27), we get the conclusion. □

3 Proof of Theorem 1.1

In this section, we suppose all the conditions of Theorem 1.1 are satisfied. Define

$$\hat{I}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} g(x)|u|^6 dx$$

for $u \in E$. The following lemma plays a key role in proving Theorem 1.1.

Lemma 3.1. *There exists $\lambda_0 \in (0, \lambda_1)$ such that if $\lambda \in (0, \lambda_0)$, then $\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^6 dx \ne 0$ for all $u \in S_1$ with $J_\lambda(u) < c^*$.*

Proof. We adapt an argument in [27]. Assume by contradiction that there exist $\lambda_n \downarrow 0$ and $\{u_n\} \subset S_1$ such that $J_{\lambda_n}(u_n) < c^*$ and $\int_{\mathbb{R}^3} \frac{x}{|x|}|u_n|^6 dx = 0$. By Lemma 2.1, there exists $t_n > 0$ such that $v_n := t_nu_n \in \mathcal{M}_{\lambda_n}$. Then one has $I_{\lambda_n}(v_n) = J_{\lambda_n}(u_n) < c^*$ and $\int_{\mathbb{R}^3} \frac{x}{|x|}|v_n|^6 dx = 0$. Since $\{v_n\} \subset \mathcal{M}_\lambda$, it follows from Lemma 2.1 (ii) that $\{v_n\}$ is bounded, and then, by $\lambda_n \to 0$,

$$\lambda_n \int_{\mathbb{R}^3} h(x)F(v_n)dx = o(1) \quad \text{and} \quad \lambda_n \int_{\mathbb{R}^3} h(x)f(v_n)v_ndx = o(1).$$

Therefore,

$$c^* \geq \hat{I}(v_n) + o(1) \quad \text{and} \quad \langle \hat{I}'(v_n), v_n \rangle = o(1). \tag{3.1}$$

Take $l = \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}}$. By the proof of (ii) of Lemma 2.1, one infers that $\|v_n\|_6 \geq C$ for each $n \in \mathbb{N}$ and some constant $C > 0$. Therefore, $l > 0$, which, jointly with (3.1) and the fact $g(x) \leq g_M$ for all $x \in \mathbb{R}^3$, we deduce that $l \geq \left(bS^2 + \sqrt{(bS^2)^2 + 4aSg_M}\right)/2g_M^{\frac{2}{3}}$, and hence

$$\begin{aligned}
c^* + o(1) \geq & \hat{I}(v_n) - \frac{1}{6}\langle \hat{I}'(v_n), v_n \rangle \\
= & \frac{a}{3}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{3}\int_{\mathbb{R}^3} V(x)|v_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 \\
\geq & \frac{aS}{3}\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{12}\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{2}{3}} \\
\geq & \frac{aS}{3g_M^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{12g_M^{\frac{2}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{2}{3}} \\
\geq & \frac{aS}{3g_M^{\frac{1}{3}}} l + \frac{bS^2}{12g_M^{\frac{2}{3}}} l^2 + o(1) \\
\geq & c^* + o(1),
\end{aligned}$$

which implies that $\int_{\mathbb{R}^3} V(x)|v_n|^2 dx \to 0$, $\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to \frac{bS^3+\sqrt{(bS^3)^2+4aS^3g_M}}{2g_M}$ and

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} = \lim_{n\to\infty}\left(\int_{\mathbb{R}^3} g_M|v_n|^6\right)^{\frac{1}{3}} = \frac{bS^2 + \sqrt{(bS^2)^2 + 4aSg_M}}{2g_M^{\frac{2}{3}}}. \tag{3.2}$$

Set $w_n = v_n/|v_n|_6$. Then $\int_{\mathbb{R}^3} |\nabla w_n|^2 dx \to S$ and $\int_{\mathbb{R}^3} |w_n|^6 dx = 1$. From [22, Theorem 1.41], there exist $w \in E$, $\{z_n\} \subset \mathbb{R}^3$ and $\mu_n \in (0, +\infty)$ such that $\|\mu_n^{\frac{1}{2}} w_n(\mu_n x + z_n) - w\|_{D^{1,2}} \to 0$ up to a subsequence, i.e.,

$$\left\| w_n - \frac{1}{\mu_n^{\frac{1}{2}}} w\left(\frac{x - z_n}{\mu_n}\right) \right\|_{D^{1,2}} \xrightarrow{n} 0. \tag{3.3}$$

Hence $\int_{\mathbb{R}^3} |\nabla w|^2 dx = S$ and $\int_{\mathbb{R}^3} |w|^6 dx = 1$, i.e., S is achieved by w. From [21], the minimizers of S are of the form $\frac{c_0}{(1+h_0(x-x_0)^2)^{\frac{1}{2}}}$, where $c_0 \neq 0$, $h_0 > 0$ and $x_0 \in \mathbb{R}^3$. Thus

$$\left\| w_n - \frac{c_0 \mu_n^{\frac{1}{2}}}{(\mu_n^2 + h_0^2| \cdot - z_n - x_0\mu_n|^2)^{\frac{1}{2}}} \right\|_{D^{1,2}} \xrightarrow{n} 0. \tag{3.4}$$

Observing $g(0) < g_M$, we see that there is $\delta > 0$ such that $g(x) \leq \frac{g(0)+g_M}{2}$ for $|x| \leq \delta$. We distinguish two cases.

Case 1. $\mu_n \to \mu_0 \in (0, +\infty]$ as $n \to \infty$.

Since $\int_{\mathbb{R}^3} V(x)|v_n|^2 dx \to 0$ and $V(x) \geq V_0 (> 0)$, one has $\int_{\mathbb{R}^3} |v_n(x)|^2 dx \to 0$, and hence $\int_{\mathbb{R}^3} |w_n(x)|^2 dx \to 0$ $(n \to \infty)$. Setting $x = \mu_n y + z_n$, it follows that

$$\mu_n^2 \int_{\mathbb{R}^3} |\mu_n^{\frac{1}{2}} w_n(\mu_n y + z_n)|^2 dy \to 0.$$

This, together with the fact $\mu_n \to \mu_0 \in (0, +\infty]$ and (3.3), gives $\int_{\mathbb{R}^3} |w|^2 dx = 0$, which is a contradiction.

Case 2. $\mu_n \to 0$ as $n \to \infty$. We will further consider two cases.

(i) $|z_n| \le \delta$ for large n.

Letting $z_n \to z_0$ $(n \to \infty)$, then $|z_0| \le \delta$ and $g(z_0) \le \frac{g(0)+g_M}{2}$. It follows from (3.2) and (3.3) that

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} (g_M - g(x))|w_n|^6 dx \\ &= \int_{\mathbb{R}^3} (g_M - g(x)) \left| \frac{1}{\mu_n^2} w \left(\frac{x - z_n}{\mu_n} \right) \right|^6 dx + o(1) \\ &= \int_{\mathbb{R}^3} (g_M - g(\mu_n x + z_n))|w|^6 dx + o(1). \end{aligned}$$

Using the Lebesgue dominated convergence theorem, we have

$$0 = \int_{\mathbb{R}^3} (g_M - g(z_0))|w|^6 dx \ge \frac{g_M - g(0)}{2} \int_{\mathbb{R}^3} |w|^6 dx = \frac{g_M - g(0)}{2} > 0,$$

a contradiction.

(ii) There is a subsequence $\{z_{n_k}\} \subset \{z_n\}$ such that $|z_{n_k}| \ge \delta$ for all k. Without loss of generality, we assume that $|z_n| \ge \delta$ for all n. Since $\int_{\mathbb{R}^3} \frac{x}{|x|} |v_n|^6 dx = 0$, one has $\int_{\mathbb{R}^3} \frac{x}{|x|} |w_n|^6 dx = 0$. Hence, by (3.4),

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} \frac{x}{|x|} \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0 \mu_n|^2\right)^3} dx \\ &= \int_{\mathbb{R}^3} \left(\frac{x}{|x|} - \frac{z_n + x_0 \mu_n}{|z_n + x_0 \mu_n|} \right) \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0 \mu_n|^2\right)^3} dx \\ &\quad + \frac{z_n + x_0 \mu_n}{|z_n + x_0 \mu_n|} \int_{\mathbb{R}^3} \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0 \mu_n|^2\right)^3} dx. \end{aligned} \tag{3.5}$$

Since $\mu_n \to 0$ and $|z_n| \ge \delta$ for all n, we have

$$|z_n + x_0 \mu_n| \ge |z_n| - \mu_n |x_0| \ge \frac{\delta}{2}$$

for large n. Combining this and the fact

$$\left| \frac{x}{|x|} - \frac{z}{|z|} \right| \le \frac{|x(|z| - |x|) + |x|(x - z)|}{|x||z|} \le \frac{2|x - z|}{|z|}$$

for all $x, z \in \mathbb{R}^3 \setminus \{0\}$, we deduce that

$$\begin{aligned} &\int_{|x - (z_n + x_0 \mu_n)| \le \mu_n} \left| \frac{x}{|x|} - \frac{z_n + x_0 \mu_n}{|z_n + x_0 \mu_n|} \right| \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0 \mu_n|^2\right)^3} dx \\ &\le \int_{|x - (z_n + x_0 \mu_n)| \le \mu_n} \frac{2|x - (z_n + x_0 \mu_n)|}{|z_n + x_0 \mu_n|} \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0 \mu_n|^2\right)^3} dx \\ &\le \frac{4\mu_n}{\delta} \int_{\mathbb{R}^3} \frac{c_0^6}{\left(1 + h_0^2 |x|^2\right)^3} dx \\ &\le C_6 \mu_n \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\int_{|x-(z_n+x_0\mu_n)|\geq\mu_n}\left|\frac{x}{|x|}-\frac{z_n+x_0\mu_n}{|z_n+x_0\mu_n|}\right|\frac{c_0^6\mu_n^3}{\left(\mu_n^2+h_0^2|x-z_n-x_0\mu_n|^2\right)^3}dx\\
&\leq\frac{2}{|z_n+x_0\mu_n|}\int_{|x-(z_n+x_0\mu_n)|\geq\mu_n}\frac{c_0^6\mu_n^3|x-(z_n+x_0\mu_n)|}{\left(\mu_n^2+h_0^2|x-z_n-x_0\mu_n|^2\right)^3}dx\\
&\leq\frac{4\mu_n}{\delta}\int_{|x|\geq 1}\frac{c_0^6|x|}{\left(1+h_0^2|x|^2\right)^3}dx\\
&\leq C_7\mu_n. \qquad (3.7)
\end{aligned}$$

Hence we obtain, by (3.5)–(3.7),

$$\begin{aligned}
0&=\lim_{n\to\infty}\left|\frac{z_n+x_0\mu_n}{|z_n+x_0\mu_n|}\int_{\mathbb{R}^3}\frac{c_0^6\mu_n^3}{\left(\mu_n^2+h_0^2|x-z_n-x_0\mu_n|^2\right)^3}dx\right|\\
&=\int_{\mathbb{R}^3}\frac{c_0^6}{\left(1+h_0^2|x|^2\right)^3}dx>0,
\end{aligned}$$

which is a contradiction. □

To prove Theorem 1.1, we recall a multiplicity result for critical points involving Ljusternik–Schnirelman category, which has been widely used in dealing with semilinear elliptic equations.

Lemma 3.2 (see Proposition 2.4 in [1]). *Let M be a Hilbert manifold and $I\in C^1(M,\mathbb{R})$. If there exist $c_0\in\mathbb{R}$ and $k\in\mathbb{N}$ such that $I(u)$ satisfies the (PS) condition for $c\leq c_0$ and* $\text{cat}(\{u\in M: I(u)\leq c_0\})\geq k$*, then $I(u)$ admits at least k critical points in $\{u\in M: I(u)\leq c_0\}$.*

Lemma 3.3 (see Theorem 2.5 in [1]). *Let X be a topological space. Assume that there exist two continuous mappings*

$$F:\mathbb{S}^2=\left\{y\in\mathbb{R}^3:|y|=1\right\}\to X,\qquad G:X\to\mathbb{S}^2$$

such that $G\circ F$ is homotopic to identity, that is, there is a continuous mapping $\zeta:[0,1]\times\mathbb{S}^2\to\mathbb{S}^2$ such that $\zeta(0,x)=(G\circ F)(x)$ for $x\in\mathbb{S}^2$ and $\zeta(1,x)=x$ for $x\in\mathbb{S}^2$. Then $cat(X)\geq 2$.

Proof of Theorem 1.1. Let $\lambda\in(0,\lambda_0)$ with λ_0 given in Lemma 3.1. Take $y=\frac{3}{2}\rho_0 z$, where ρ_0 is the constant given in (g_2) and $z\in\mathbb{S}^2$. Let $r_1<\frac{1}{4}\rho_0$. By (g_2), one has $g(x)=g_M$ for $|x-\frac{3}{2}\rho_0 z|\leq 2r_1$. Noting

$$u_{\varepsilon,y}(x)=\frac{\psi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x-\frac{3}{2}\rho_0 z|^2)^{\frac{1}{2}}},$$

where $\psi\in C_0^\infty(B_{2r_1}(\frac{3}{2}\rho_0 z))$ such that $\psi(x)=1$ for $|x-\frac{3}{2}\rho_0 z|\leq r_1$ and $0\leq\psi(x)\leq 1$, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3}g(x)|u_{\varepsilon,y}|^6dx&=\int_{|x-\frac{3}{2}\rho_0 z|<2r_1}g(x)|u_{\varepsilon,y}|^6dx\\
&=g_M\int_{|x-\frac{3}{2}\rho_0 z|<2r_1}|u_{\varepsilon,y}|^6dx\\
&=g_M\int_{\mathbb{R}^3}|u_{\varepsilon,y}|^6dx.
\end{aligned}$$

Arguing as in the proof of Lemma 2.6, we still conclude that $\sup_{t\geq 0} I_\lambda(tu_{\varepsilon,y}) \leq c^* - C_0\varepsilon^{\frac{1}{2}}$. Set $h(t) = I_\lambda(\frac{tu_{\varepsilon,y}}{\|u_{\varepsilon,y}\|})$, where $t > 0$ and $\varepsilon \in (0, \varepsilon_0)$ with ε_0 given in Lemma 2.6. It follows from Lemma 2.1 that $h(t)$ attains its maximum at a unique point t_y and $t_y u_{\varepsilon,y} \in \mathcal{M}_\lambda$. By Lemma 2.6,

$$J_\lambda\left(\frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}\right) = I_\lambda(t_y u_{\varepsilon,y}) = \sup_{t\geq 0} I_\lambda(tu_{\varepsilon,y}) \leq c^* - C_0\varepsilon^{\frac{1}{2}} < c^*.$$

Define $F : \mathbb{S}^2 \to S_1$ by $F(z) = \frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}$. Then

$$F : \mathbb{S}^2 \to \{u \in S_1 : J_\lambda(u) < c^*\}.$$

Let $G : \{u \in S_1 : J_\lambda(u) < c^*\} \to \mathbb{S}^2$ by $G(u) = \frac{\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^6 dx}{\left|\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^6 dx\right|}$. Then G is well defined and continuous by virtue of Lemma 3.1. Define $\zeta(\theta, z) : [0,1] \times \mathbb{S}^2 \to \mathbb{S}^2$ such that $\zeta(\theta, z) = G(\frac{u_{(1-\theta)\varepsilon,y}}{\|u_{(1-\theta)\varepsilon,y}\|})$ for $\theta \in [0,1)$ and $\zeta(1,z) = z$. It follows from (2.22) that

$$\begin{aligned}
&\lim_{\theta\to 1^-} \int_{\mathbb{R}^3} \frac{x}{|x|}|u_{(1-\theta)\varepsilon,y}|^6 dx \\
&\quad = \lim_{\theta\to 1^-} \int_{\mathbb{R}^3} \left(\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right)|u_{(1-\theta)\varepsilon,y}|^6 dx + \lim_{\theta\to 1^-} \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\int_{\mathbb{R}^3} |u_{(1-\theta)\varepsilon,y}|^6 dx \\
&\quad = \lim_{\theta\to 1^-} \int_{\mathbb{R}^3} \left(\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right)|u_{(1-\theta)\varepsilon,y}|^6 dx + K_2 z.
\end{aligned} \tag{3.8}$$

Since $|u_{(1-\theta)\varepsilon,y}|^6 \leq \frac{((1-\theta)\varepsilon)^{\frac{3}{2}}}{\left((1-\theta)\varepsilon + |x - \frac{3}{2}\rho_0 z|^2\right)^3}$ and since $\left|\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right| \leq \frac{2|x-\frac{3}{2}\rho_0 z|}{|\frac{3}{2}\rho_0 z|}$, we deduce that

$$\begin{aligned}
&\int_{|x-\frac{2}{3}\rho_0 z|\leq\sqrt{(1-\theta)\varepsilon}} \left|\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right| |u_{(1-\theta)\varepsilon,y}|^6 dx \\
&\quad \leq \int_{|x-\frac{2}{3}\rho_0 z|\leq\sqrt{(1-\theta)\varepsilon}} \frac{4\,|x-\frac{2}{3}\rho_0 z|}{3\rho_0} \frac{1}{((1-\theta)\varepsilon)^{\frac{3}{2}}\left(1 + \left|\frac{x-\frac{2}{3}\rho_0 z}{\sqrt{(1-\theta)\varepsilon}}\right|^2\right)^3} dx \\
&\quad \leq \frac{4\sqrt{(1-\theta)\varepsilon}}{3\rho_0} \int_{|x|\leq 1} \frac{1}{(1+|x|^2)^3} dx \\
&\quad \leq C_8\sqrt{(1-\theta)\varepsilon}
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\int_{|x-\frac{2}{3}\rho_0 z|\geq\sqrt{(1-\theta)\varepsilon}} \left|\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right| |u_{(1-\theta)\varepsilon,y}|^6 dx \\
&\quad \leq \frac{4}{3\rho_0} \int_{|x-\frac{2}{3}\rho_0 z|\geq\sqrt{(1-\theta)\varepsilon}} \frac{|x-\frac{3}{2}\rho_0 z|}{((1-\theta)\varepsilon)^{\frac{3}{2}}\left(1 + \left|\frac{x-\frac{2}{3}\rho_0 z}{\sqrt{(1-\theta)\varepsilon}}\right|^2\right)^3} dx \\
&\quad \leq \frac{4\sqrt{(1-\theta)\varepsilon}}{3\rho_0} \int_{|x|\geq 1} \frac{|x|}{(1+|x|^2)^3} dx \\
&\quad \leq C_9\sqrt{(1-\theta)\varepsilon}.
\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10), we have

$$\lim_{\theta\to 1^-}\int_{\mathbb{R}^3}\frac{x}{|x|}|u_{(1-\theta)\varepsilon,y}|^6dx = K_2 z,$$

which, together with the continuous of G, gives that $\zeta \in C([0,1]\times\mathbb{S}^2,\mathbb{S}^2)$. Noting $\zeta(0,z) = G(\frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}) = G\circ F(z)$ and $\zeta(1,z) = z$ for $z\in\mathbb{S}^2$, one has $G\circ F:\mathbb{S}^2\to\mathbb{S}^2$, $z\to G\circ F(z)$ is homotopic to the identity. Thus, by Lemma 3.3,

$$\operatorname{cat}\left\{u\in S_1 : J_\lambda(u)\le c^* - C_0\varepsilon^{\frac{1}{2}}\right\}\ge 2.$$

Therefore, using Corollary 2.5 and Lemma 3.2, we deduce that J_λ has at least two nontrivial critical points, and thus I_λ has at least two nontrivial critical points. This completes the proof. □

4 Proof of Theorem 1.2

In this section, we suppose that all the conditions of Theorem 1.2 are satisfied. By (g_1), there is $R_0>0$ such that $g(x)\le\frac{1}{2}(g_M+g_\infty)$ for all $|x|\ge R_0$. For any $d>0$, let $\rho=\rho(d)>R_0$ be such that $\Lambda_d\subset B_\rho(0)$. Define $\chi:\mathbb{R}^3\to\mathbb{R}^3$ as $\chi(x)=x$ for $|x|\le\rho$ and $\chi(x)=\rho x/|x|$ for $|x|>\rho$. We consider the barycenter map $\beta: E\setminus\{0\}\to\mathbb{R}^3$ given by

$$\beta(u)=\frac{\int_{\mathbb{R}^3}\chi(x)|u(x)|^6dx}{\int_{\mathbb{R}^3}|u(x)|^6dx}.$$

Since $\Lambda_d\subset B_\rho(0)$, by the definition of χ and Lebesgue's theorem, we have the following conclusion.

Lemma 4.1. *For any $d>0$, there exists $\lambda_d>0$ such that, if $\lambda\in(0,\lambda_d)$ and $u\in S_1$ with $J_\lambda(u)<c^*$, then $\beta(u)\in\Lambda_d$.*

Proof. Arguing indirectly, we assume that there exist $d_0>0$, $\lambda_n\downarrow 0$ and $(u_n)\subset S_1$ with $J_{\lambda_n}(u_n)<c^*$, but $\beta(u_n)\notin\Lambda_{d_0}$. From Lemma 2.1, there exists a unique $t_n>0$ such that $t_nu_n\in M_{\lambda_n}$. Take $v_n=t_nu_n$ and $w_n=v_n/|v_n|_6$. Following the steps contained in the proof of Lemma 3.1, we deduce

$$\int_{\mathbb{R}^3}|\nabla w_n|^2dx\to S,\qquad \int_{\mathbb{R}^3}|w_n|^6dx=1\quad\text{and}\quad\int_{\mathbb{R}^3}g(x)|w_n|^6dx\to g_M. \tag{4.1}$$

So, there exist $\{z_n\}\subset\mathbb{R}^3$, $\mu_n\in(0,+\infty)$ and $w\in D^{1,2}(\mathbb{R}^3)$ such that

$$\left\|w_n-\frac{1}{\mu_n^{\frac{1}{2}}}w\left(\frac{x-z_n}{\mu_n}\right)\right\|_{D^{1,2}}\to 0\qquad(n\to\infty). \tag{4.2}$$

Thus $\int_{\mathbb{R}^3}|\nabla w|^2dx=S$ and $\int_{\mathbb{R}^3}|w|^6dx=1$. Arguing as in Lemma 3.1 (Case 1), if $\mu_n\to\mu_0\in(0,+\infty]$ $(n\to\infty)$, one obtains a contradiction. Hence $\mu_n\to 0$ as $n\to\infty$, and we distinguish into two cases.

Case 1. $\mu_n\to 0$ as $n\to\infty$ and $|z_n|\le R_0$ for large n. Suppose that $z_n\to z_0$ as $n\to\infty$. Then $|z_0|\le R_0$ and $\chi(z_0)=z_0$. Applying (4.1), (4.2) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} g_M &= \lim_{n\to\infty}\int_{\mathbb{R}^3} g(x)\left|\frac{1}{\mu_n^{\frac{1}{2}}}w\left(\frac{x-z_n}{\mu_n}\right)\right|^6 dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^3} g(\mu_n x+z_n)|w|^6 dx \\ &= g(z_0)\int_{\mathbb{R}^3}|w|^6 dx \\ &= g(z_0), \end{aligned}$$

which implies that $z_0 \in \Lambda$. Moreover, by (4.2) and using the fact $\chi(z_0) = z_0$, we conclude that

$$\begin{aligned} \beta(w_n) &= \int_{\mathbb{R}^3}\chi(x)|w_n|^6 dx \\ &= \int_{\mathbb{R}^3}\chi(x)\left|\frac{1}{\mu_n^{\frac{1}{2}}}w\left(\frac{x-z_n}{\mu_n}\right)\right|^6 dx + o(1) \\ &= \int_{\mathbb{R}^3}\chi(\mu_n x+z_n)|w|^6 dx + o(1) \\ &= z_0\int_{\mathbb{R}^3}|w|^6 dx + o(1) \\ &= z_0 + o(1), \end{aligned}$$

which, together with $z_0 \in \Lambda$, yields that $\beta(w_n) \in \Lambda_{d_0}$ for large n. This contradicts the assumption that $\beta(w_n) = \beta(u_n) \notin \Lambda_{d_0}$ for all n.

Case 2. $\mu_n \to 0$ as $n \to \infty$ and there exits a subsequence of $\{z_n\}$ (still denoted by $\{z_n\}$) such that $|z_n| \geq R_0$ for all n. Applying (4.1) and (4.2), we deduce that

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3}(g_M - g(x))|w_n|^6 dx \\ &= \int_{\mathbb{R}^3}(g_M - g(x))\left|\frac{1}{\mu_n^{\frac{1}{2}}}w\left(\frac{x-z_n}{\mu_n}\right)\right|^6 dx + o(1) \\ &= \int_{\mathbb{R}^3}(g_M - g(\mu_n x+z_n))|w|^6 dx + o(1). \end{aligned}$$

Recall that $g(x) \leq \frac{1}{2}(g_\infty + g_M)$ for $|x| \geq R_0$. It follows from the Lebesgue dominated convergence theorem that

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^3}(g_M - g(\mu_n x+z_n))|w|^6 dx \geq \frac{1}{2}(g_M - g_\infty)\int_{\mathbb{R}^3}|w|^6 dx = \frac{1}{2}(g_M - g_\infty) > 0,$$

which is a contradiction. □

Now we are in a position to show that problem (1.1) admits at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ solutions. For this aim, we compare the topology of Λ and the topology of a suitable energy sublevel, and use the maps J_λ and β as they are introduced before. Moreover, we shall utilize a multiplicity result for critical points involving Lusternik–Schnirelmann category, e.g. see [15].

Lemma 4.2 (see [15]). *Let M be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space) and assume that $\Phi \in C^1(M,\mathbb{R})$ bounded from below and satisfies $-\infty < \inf_M \Phi < a < b < +\infty$. Suppose that Φ satisfies the Palais–Smale condition on the sublevel $\{u \in M : \Phi(u) \le b\}$ and that a is not a critical level for Φ. Then Φ has at least $\mathrm{cat}_{\Phi^a}(\Phi^a)$ critical points in Φ^a, where $\Phi^a := \{u \in M : \Phi(u) \le a\}$.*

Lemma 4.3 (see [3,4]). *Let A, B, M be closed sets with $A \subset B$. Let $F : A \to M$ and $G : M \to B$ be two continuous maps such that $G \circ F$ is homotopically equivalent to the embedding $J : A \to B$. Then $\mathrm{cat}_M(M) \ge \mathrm{cat}_B(A)$.*

Remark 4.4. Since $\mathcal{M}_\lambda$ is not a C^1 submanifold of E, we can not apply Lemma 4.2 directly. Fortunately, from Lemma 2.1, we know that the mapping m_λ is a homeomorphism between $\mathcal{M}_\lambda$ and S_1, but S_1 is a C^1 submanifold of E. Thus we can apply Lemma 4.2 to $J_\lambda(u) = I_\lambda(\hat{m}_\lambda(u))\,|_{S_1} = I_\lambda(m_\lambda(u))$, where J_λ is given in Lemma 2.2.

Proof of Theorem 1.2. For any $d > 0$, let $\lambda \in (0, \min\{\lambda_1, \lambda_d\})$ with λ_1 is given in Lemma 2.4 and λ_d is given in Lemma 4.1. It is easy to see that S_1 is a $C^{1,1}$ complete Riemann manifold and $J_\lambda \in C^1(S_1,\mathbb{R})$ is bounded from below. Set $l(t) = I_\lambda(tu_{\varepsilon,y})$, where $t > 0$, $y \in \Lambda$ and $\varepsilon \in (0,\varepsilon_0)$ with ε_0 is given in Lemma 2.6. In view of Lemma 2.1, $l(t)$ admits its maximum at a unique point t_y and $t_y u_{\varepsilon,y} \in \mathcal{M}_\lambda$. Hence, by Lemma 2.6,

$$J_\lambda\left(\frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}\right) = I_\lambda(t_y u_{\varepsilon,y}) = \sup_{t\ge 0} I_\lambda(tu_{\varepsilon,y}) \le c^* - \eta_0, \tag{4.3}$$

where $\eta_0 > 0$ is a constant independent of $y \in \Lambda$. From Corollary 2.5, we see that J_λ satisfies the (PS) condition on $\{u \in S_1 : J_\lambda(u) < c^*\}$. Therefore, by Lemma 4.2, J_λ has at least $\mathrm{cat}_{S_1(\eta_0)}(S_1(\eta_0))$ critical points, where $S_1(\eta_0) = \{u \in S_1 : J_\lambda(u) \le c^* - \eta_0\}$.

Define the mappings $F : \Lambda \to S_1$ and $G : S_1(\eta_0) \to \mathbb{R}^3$ by

$$F(y) = \frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}, \qquad G(u) = \beta(u).$$

Then F and G are continuous. It follows from Lemma 4.1 and (4.3) that $F(\Lambda) \subset S_1(\eta_0)$ and $G(S_1(\eta_0)) \subset \Lambda_d$. Define $\xi : [0,1] \times \Lambda \to \Lambda_d$ by

$$\xi(\theta,y) = \begin{cases} G\left(\frac{u_{(1-\theta)\varepsilon,y}}{\|u_{(1-\theta)\varepsilon,y}\|}\right), & \theta \in [0,1), \\ y, & \theta = 1. \end{cases}$$

Noting $y \in \Lambda \subset B_\rho(0)$, we obtain $\chi(y) = y$ and

$$\begin{aligned} \lim_{\theta\to 1^-} G\left(\frac{u_{(1-\theta)\varepsilon,y}}{\|u_{(1-\theta)\varepsilon,y}\|}\right) &= \lim_{\theta\to 1^-} \frac{\int_{\mathbb{R}^3} \chi(x)|u_{(1-\theta)\varepsilon,y}|^6 dx}{\int_{\mathbb{R}^3} |u_{(1-\theta)\varepsilon,y}|^6 dx} \\ &= \lim_{\theta\to 1^-} \frac{\int_{\mathbb{R}^3} \frac{\chi\left(\sqrt{(1-\theta)\varepsilon}z+y\right)\left|\psi\left(\sqrt{(1-\theta)\varepsilon}z+y\right)\right|^6}{(1+|z|^2)^3} dz}{\int_{\mathbb{R}^3} \frac{\left|\psi\left(\sqrt{(1-\theta)\varepsilon}z+y\right)\right|^6}{(1+|z|^2)^3} dz} = y. \end{aligned}$$

Thus $\xi \in C([0,1] \times \Lambda, \Lambda_d)$. Then we see that $\xi(\theta,y)$ with $(\theta,y) \in [0,1] \times \Lambda$ is a homotopy between $G \circ F$ and the inclusion map $j : \Lambda \to \Lambda_d$. This fact and Lemma 4.3 yield $\mathrm{cat}_{S_1(\eta_0)}(S_1(\eta_0)) \ge \mathrm{cat}_{\Lambda_d}(\Lambda)$. Hence, J_λ has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ critical points. Then, in view of Lemma 2.2 (iii), we conclude that I_λ has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ nontrivial critical points. Thus, problem (1.1) has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ nontrivial solutions. This completes the proof. □

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