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Multiplicity of weak solutions to degenerate weighted quasilinear elliptic equations with nonlocal terms and variable exponents

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Abstract. This paper studies the existence and multiplicity of weak solutions to degenerate weighted quasilinear elliptic equations with nonlocal nonlinearities and variable exponents. The equation involves a degenerate nonlinear operator with variable exponents, a nonlocal term, and growth conditions on the nonlinearity. Using critical point theory, we prove the existence of at least three weak solutions under general assumptions, extending the applicability of the results to a broad class of nonlinear problems in mathematical physics.

Keywords: variational methods, Hardy inequality, degenerate $p(x)$-Laplacian operators.

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1 Introduction

Significant challenges in the analysis and behavior of solutions arise from the presence of singularities and degeneracies in elliptic equations. Singularities, especially those arising at the origin or the boundary, can dramatically alter the properties of the operator, causing the solutions to become more sensitive to changes in the domain. For example, when $1 < p < N$, it is known that $u/|x| \in L^p(\mathbb{R}^N)$ if $u \in W^{1,p}(\mathbb{R}^N)$, or $u/|x| \in L^p(\Omega)$ when $u \in W^{1,p}(\Omega)$, where Ω is a bounded domain. This leads to the emergence of Hardy-type inequalities that control the singular behavior of solutions near critical points, particularly when the equation includes singular potential terms (see, e.g., [11,13,15,16]).

In addition to the challenges posed by singularities, the inclusion of nonlocal terms further complicates the situation. Nonlocal interactions, which typically arise from integral expressions or global coupling terms, create dependencies that affect the solution at each point in the domain, as well as its values across the entire domain. The nonlocal nature of these terms requires the use of advanced techniques that go beyond classical local methods to analyze the solution structure and its multiplicity.

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Furthermore, the operator's degeneracy, especially when coupled with a weighted function $\omega(x)$ in the p-Laplacian or $p(x)$-Laplacian operator, adds another layer of complexity. The degeneracy introduced by $\omega(x)$, whether singular or merely bounded, necessitates a shift in the choice of functional spaces. In such cases, traditional Sobolev spaces such as $W^{1,p}(\Omega)$ or $W^{1,p(x)}(\Omega)$ may no longer be suitable. Instead, we must consider alternative Sobolev spaces that accommodate the weight function, such as $W^{1,p}(\omega,\Omega)$, to handle the singularities or degeneracies (see [6] for further details).

This paper addresses these challenges by studying a class of weighted quasilinear elliptic equations with nonlocal nonlinearities and variable exponents. The goal is to establish the existence and multiplicity of weak solutions while taking into account the degeneracy of the operator, the Hardy-type singularities, and the nonlocal interactions that frequently arise in applied mathematical models.

In this paper, we investigate the existence of generalized solutions to a class of weighted quasilinear elliptic equations of the form:

$$\begin{cases} -\Delta_{p(x),a(x,u)}u + \frac{b(x)|u|^{q-2}u}{|x|^q} = \lambda f(x,u)\left(\int_\Omega F(x,u)\,dx\right)^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, $1 < q < N$, $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) is a bounded open subset, $\partial\Omega$ denotes its smooth boundary and $b(x)$ is a non-negative measurable function. The unknown function u satisfies a weighted quasilinear elliptic equation involving a variable exponent $p(x)$, a nonlinear term $f(x,u)$, and a non-local term involving an integral expression with a positive exponent r.

The operator $\Delta_{p(x),a(x,u)}u$ is a nonlinear generalization of the standard Laplacian, defined as:

$$\Delta_{p(x),a(x,u)}u = \operatorname{div}\left(a(x,u)|\nabla u|^{p(x)-2}\nabla u\right),$$

where $a(x,u)$ is a Carathéodory function satisfying the inequality

$$a_1\omega(x) \leq a(x,u) \leq a_2\omega(x) \quad \text{for some constants } a_1, a_2 > 0,$$

and a positive measurable function $\omega(x)$. The function $\omega(x)$ is assumed to belong to the local Lebesgue space $L^1_{\text{loc}}(\Omega)$ and satisfy additional growth conditions, such as $\omega^{-h(x)} \in L^1(\Omega)$, where $h(x)$ belongs to a range related to the variable exponent $p(x)$, more precisely, we assume that

(ω) $\omega^{-h(x)} \in L^1(\Omega)$, for $h(x) \in C(\overline{\Omega})$ and $h(x) \in (N/p(x), +\infty) \cap [1/(p(x)-1), +\infty)$,

The nonlinearity in the equation is characterized by the function $f(x,u)$, which satisfies growth conditions of the form:

$$(f_1) \qquad M_1|u|^{\alpha(x)-1} \leq |f(x,u)| \leq M_2|u|^{\beta(x)-1},$$

with $1 < \alpha(x) \leq \beta(x) < p_h^*(x)$, where $p_h^*(x) = \frac{Np_h(x)}{N-p_h(x)}$ and $p_h(x) = \frac{h(x)p(x)}{h(x)+1}$. These conditions allow for a broad class of nonlinearities, including those that are locally integrable and exhibit power-like behavior at infinity.

The parameter $\lambda > 0$ scales the non-local term, which depends on the integral

$$\int_\Omega F(x,u)\,dx$$

of another function $F(x, u)$, which itself is assumed to satisfy certain conditions. These non-local terms introduce further complexity into the problem, as they may give rise to both local and global interactions within the domain Ω.

The main objective of this paper is to establish the existence of at least three weak solutions to the problem under very general assumptions on the weighted function $\omega(x)$ and the non-linear non-local term. To achieve this, we apply critical point theory to the associated energy functional, which is constructed by integrating the relevant terms of the equation over Ω. The application of critical point theorems allows us to prove the existence of solutions without the need for strict assumptions on the regularity or structure of the nonlinearity, making the result highly general and applicable to a wide variety of problems in mathematical physics and differential equations.

2 Preliminaries and variational structure

Set

$$C_+(\overline{\Omega}) = \{p(x) | p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\},$$
$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

For $k > 0$, and $p(x) \in C_+(\overline{\Omega})$, we use the following notations

$$k^{\hat{p}} = \max\{k^{p^-},\ k^{p^+}\}, \quad k^{\check{p}} = \min\{k^{p^-},\ k^{p^+}\}.$$

It is easy to verify that (one can see [3], for further details)

(i) $k^{\frac{1}{\hat{p}}} = \max\left\{k^{\frac{1}{p^-}}, k^{\frac{1}{p^+}}\right\}$,

(ii) $k^{\frac{1}{\check{p}}} = \min\left\{k^{\frac{1}{p^-}}, k^{\frac{1}{p^+}}\right\}$,

(iii) $k^{\frac{1}{\hat{p}}} = a \Longleftrightarrow k = a^{\hat{p}}, k^{\frac{1}{\check{p}}} = a \Longleftrightarrow k = a^{\check{p}}$,

(iv) $(k\beta)^{\check{p}} \le k^{\check{p}}\beta^{\check{p}} \le (k\beta)^{\hat{p}} \le k^{\hat{p}}\beta^{\hat{p}}$.

Denote

$$L^{p(x)}(\omega, \Omega) = \left\{u:\ \text{measurable in } \Omega \text{ such that } \int_\Omega \omega(x)|u(x)|^{p(x)}\, dx < \infty\right\}$$

with a Luxemburg-type norm defined by

$$\|u\|_{L^{p(x)}(\omega,\Omega)} = \inf\left\{\eta > 0 : \int_\Omega \omega(x)\left|\frac{u(x)}{\eta}\right|^{p(x)} dx \le 1\right\}.$$

Now, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega):\ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)},$$

where $\|\nabla u\|_{p(x)} = \||\nabla u|\|_{p(x)}, |\nabla u| = \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}, \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$ is the gradient of u at $(x_1, x_2, \ldots, x_N)$.

Let

$$W^{1,p(x)}(\omega,\Omega)=\left\{u\in L^{p(x)}(\Omega):\omega^{\frac{1}{p(x)}}|\nabla u|\in L^{p(x)}(\Omega)\right\}$$

be the weighted Sobolev space, and denote by $W_0^{1,p(x)}(\omega,\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\omega,\Omega)$ with the norm

$$\|u\|=\inf\left\{\eta>0:\int_\Omega\omega(x)\left|\frac{\nabla u(x)}{\eta}\right|^{p(x)}dx\le 1\right\}.$$

Lemma 2.1 ([9])**.** *If $p_1(x),p_2(x)\in C_+(\overline{\Omega})$ such that $p_1(x)\le p_2(x)$ a.e. $x\in\Omega$, then there exists the continuous embedding $W^{1,p_2(x)}(\Omega)\hookrightarrow W^{1,p_1(x)}(\Omega)$.*

Proposition 2.2 ([10])**.** *For $p(x)\in C_+(\overline{\Omega}),u,u_n\in L^{p(x)}(\Omega)$, we have*

$$\min\left\{\|u\|_{p(x)}^{p^-},\|u\|_{p(x)}^{p^+}\right\}\le\int_\Omega|u(x)|^{p(x)}dx\le\max\left\{\|u\|_{p(x)}^{p^-},\|u\|_{p(x)}^{p^+}\right\}.$$

Let $0<d(x)\in S(\Omega)$ and $S(\Omega)$ be the set of all measurable real functions defined on Ω. Define

$$L_{d(x)}^{p(x)}(\Omega)=L^{p(x)}(d,\Omega)=\left\{u\in S(\Omega):\int_\Omega d(x)|u(x)|^{p(x)}\,dx<\infty\right\}$$

with a Luxemburg-type norm defined by

$$\|u\|_{L_{d(x)}^{p(x)}(\Omega)}=\|u\|_{(p(x),d(x))}:=\inf\left\{\eta>0:\int_\Omega d(x)\left|\frac{u(x)}{\eta}\right|^{p(x)}dx\le 1\right\}.$$

Proposition 2.3 ([7])**.** *If $p\in C_+(\overline{\Omega})$. Then*

$$\min\left\{\|u\|_{(p(x),d(x))}^{p^-},\|u\|_{(p(x),d(x))}^{p^+}\right\}\le\int_\Omega d(x)|u(x)|^{p(x)}dx\le\max\left\{\|u\|_{(p(x),d(x))}^{p^-},\|u\|_{(p(x),d(x))}^{p^+}\right\}$$

for any $u\in L_{d(x)}^{p(x)}(\Omega)$ and for a.e. $x\in\Omega$.

Combining Proposition 2.2 with Proposition 2.3, one has

Lemma 2.4. *Let*

$$\rho_\omega(u)=\int_\Omega\omega(x)\,|\nabla u(x)|^{p(x)}\,dx.$$

For $p\in C_+(\overline{\Omega}),u\in W^{1,p(x)}(\omega,\Omega)$, we have

$$\min\left\{\|u\|^{p^-},\|u\|^{p^+}\right\}\le\rho_\omega(u)\le\max\left\{\|u\|^{p^-},\|u\|^{p^+}\right\}.$$

From Proposition 2.4 of [13], if (ω) holds, $W^{1,p(x)}(\omega,\Omega)$ is a separable and reflexive Banach space.

From Theorem 2.11 of [14], if (ω) holds, the following embedding

$$W^{1,p(x)}(\omega,\Omega)\hookrightarrow W^{1,p_h(x)}(\Omega)\tag{2.1}$$

is continuous, where

$$p_h(x)=\frac{p(x)h(x)}{h(x)+1}<p(x).$$

Combining (2.1) with Proposition 2.7 and Proposition 2.8 in [8], we get the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

is continuous, where

$$1 \leq r(x) \leq p_h^*(x) = \frac{Np_h(x)}{N-p_h(x)} = \frac{Np(x)h(x)}{Nh(x)+N-p(x)h(x)}.$$

Furthermore, the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{t(x)}(\Omega)$$

is compact, when $1 \leq t(x) < p_h^*(x)$.

Define the functional $\mathcal{I}_\lambda \colon W_0^{1,p(x)}(\omega,\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) := \int_\Omega \frac{a(x,u)}{p(x)}|\nabla u|^{p(x)}dx + \frac{1}{q}\int_\Omega \frac{b(x)|u|^q}{|x|^q}dx,$$
$$\Psi(u) := \frac{1}{r+1}\Big(\int_\Omega F(x,u(x))dx\Big)^{r+1},$$

and $F(x,u) = \int_0^u f(x,\tau)d\tau, \forall(x,u) \in \Omega\times\mathbb{R}$.

It is clear that functionals Φ and Ψ are continuously Gâteaux differentiable with

$$\Phi'(u)(v) = \int_\Omega a(x,u)|\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_\Omega \frac{b(x)|u|^{q-2}uv}{|x|^q}dx,$$

and

$$\Psi'(u)(v) = \Big(\int_\Omega F(x,u)\,dx\Big)^r \int_\Omega f(x,u)vdx, \quad \forall u,v \in W_0^{1,p(x)}(\omega,\Omega).$$

We say that $u \in W_0^{1,p(x)}(\omega,\Omega)$ is a generalized solution of the problem (1.1) if

$$\mathcal{I}_\lambda'(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0, \quad \forall v \in W_0^{1,p(x)}(\omega,\Omega).$$

Lemma 2.5. *The functional Φ' is coercive and strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$.*

Proof. For any $u \in W_0^{1,p(x)}(\omega,\Omega)\setminus\{0\}$, by Lemma 2.4 one has

$$\begin{aligned}\Phi'(u)(u) &= \int_\Omega a(x,u)|\nabla u|^{p(x)-2}\nabla u\nabla u dx + \int_\Omega \frac{b(x)|u|^{q-2}u^2}{|x|^q}dx \\ &\geq a_1\rho_\omega(u) \\ &\geq a_1\cdot\min\{\|u\|^{p^+},\|u\|^{p^-}\},\end{aligned}$$

thus

$$\lim_{\|u\|\to\infty}\frac{\Phi'(u)(u)}{\|u\|} \geq a_1\cdot\lim_{\|u\|\to\infty}\frac{\min\{\|u\|^{p^+},\|u\|^{p^-}\}}{\|u\|} = +\infty,$$

then Φ' is coercive in view of $p(x) \in C_+(\overline{\Omega})$.

According to (2.2) of [17], for all $x, y \in \mathbb{R}^N$, there is a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq C_p|x-y|^p, \quad \text{if } p \geq 2,$$

and

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}}, \quad \text{if } 1<p<2, \quad \text{and} \quad (x,y) \neq (0,0),$$

where $\langle .,.\rangle$ is the usual inner product in $\mathbb{R}^N$. Thus, for any $u, v \in X$ satisfying $u \neq v$, by standard arguments we can obtain

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u-v\rangle &= \int_\Omega a(x,u)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(\nabla u - \nabla v)dx \\ &\quad + \int_\Omega \frac{b(x)}{|x|^q}(|u|^{q-2}u - |v|^{q-2}v)(u-v))dx \\ &> 0,\end{aligned}$$

hence we have Φ' is strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$. □

Lemma 2.6. *The functional Φ' is a mapping of (S_+)-type, i.e. if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u)\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\omega,\Omega)$.*

Proof. Let $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$.

Noting that Φ' is strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$, one has

$$\lim_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0,$$

while

$$\begin{aligned}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle &= \int_\Omega a(x,u)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \\ &\quad + \int_\Omega \Big(\frac{b(x)|u_n|^{q-2}}{|x|^q}u_n(u_n-u) - \frac{b(x)|u|^{q-2}}{|x|^q}u(u_n-u)\Big)dx,\end{aligned}$$

thus we get

$$\overline{\lim}_{n\to\infty}\int_\Omega a(x,u)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \leq 0.$$

Further, by (1.2) one has

$$\overline{\lim}_{n\to\infty}\int_\Omega \omega(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \leq 0,$$

then $u_n \to u$ in $W_0^{1,p(x)}(\omega,\Omega)$ via Lemma 3.2 in [12]. □

Lemma 2.7. *The functional Φ' is a homeomorphism.*

Proof. The strict monotonicity of Φ' implies that it is injective. Since Φ' is coercive, it is also surjective, and hence Φ' has an inverse mapping.

Next, we show that the inverse mapping $(\Phi')^{-1}$ is continuous.

Let $\tilde{f}_n, \tilde{f} \in (W_0^{1,p(x)}(\omega,\Omega))^*$ such that $\tilde{f}_n \to \tilde{f}$. We aim to prove that $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$.

Indeed, let $(\Phi')^{-1}(\tilde{f}_n) = u_n$ and $(\Phi')^{-1}(\tilde{f}) = u$, so that $\Phi'(u_n) = \tilde{f}_n$ and $\Phi'(u) = \tilde{f}$. By the coercivity of Φ', the sequence u_n is bounded. Without loss of generality, assume $u_n \rightharpoonup u_0$, which implies

$$\lim_{n\to\infty} \left(\Phi'(u_n) - \Phi'(u), u_n - u_0\right) = \lim_{n\to\infty} \left(\tilde{f}_n - \tilde{f}, u_n - u_0\right) = 0.$$

Thus, $u_n \to u_0$ because Φ' is of (S_+)-type, which ensures that $\Phi'(u_n) \to \Phi'(u_0)$. Combining this with $\Phi'(u_n) \to \Phi'(u)$, we deduce that $\Phi'(u) = \Phi'(u_0)$. Since Φ' is injective, it follows that $u = u_0$, and hence $u_n \to u$. Therefore, we have $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$, proving that $(\Phi')^{-1}$ is continuous. □

Lemma 2.8 (Hölder type inequality [1,8])**.** *Let $p, q, s \geq 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad \text{for a.e. } x \in \Omega.$$

If $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$, then $fg \in L^{s(x)}(\Omega)$ and

$$\|fg\|_{s(x)} \leq 2\|f\|_{p(x)}\|g\|_{q(x)}. \tag{2.2}$$

Lemma 2.9. *The functional $\Psi' : X := W_0^{1,p(x)}(\omega,\Omega) \to (W_0^{1,p(x)}(\omega,\Omega))^*$ is compact.*

Proof. The condition (f_1) and the compact embedding $W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\Omega)$, where $1 \leq \beta(x) < p_h^*(x)$, imply the compactness of $\Psi'(u)$. Specifically, let $(u_k)_k \subset X$ be a sequence such that $u_k \rightharpoonup u$. Since the embedding $W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\Omega)$ is compact, there exists a subsequence, still denoted by $(u_k)_k$, such that $u_k \to u$ strongly in $L^{\beta(x)}(\Omega)$ and $u_k(x) \to u(x)$ almost everywhere. The continuity of $F(x,u)$ with respect to u ensures that

$$F(x,u_k) \to F(x,u) \quad \text{for almost every } x.$$

Moreover, there exists $C > 0$ such that

$$|F(x,u_k)| \leq C|u_k|^{\beta(x)}.$$

Applying the dominated Convergence theorem we can conclude

$$\int_\Omega F(x,u_k)\,dx \to \int_\Omega F(x,u)\,dx \quad \text{as } k \to +\infty. \tag{2.3}$$

From condition (f_1), it follows that the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ is continuous, as $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies (f_1). Consequently, we conclude that $N_f(u_k) \to N_f(u)$ in $L^{\frac{\beta(x)}{\beta(x)-1}}(\Omega)$.

Using Hölder's inequality, for any $v \in W_0^{1,p(x)}(\omega,\Omega)$, we obtain

$$\begin{aligned}\left|\int_\Omega f(x,u_k)v dx - \int_\Omega f(x,u)v dx\right| &\leq \int_\Omega |(f(x,u_k) - f(x,u))v| dx \\ &\leq 2\|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}} \|v\|_{\beta(x)} \\ &\leq 2c_\beta \|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}} \|v\|,\end{aligned}$$

where c_β is the embedding constant of the embedding $W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{\beta(x)}(\Omega), 1 \le \beta(x) < p_h^*(x)$. Thus, from (2.3) and the above inequality, $\Psi'(u_k) \to \Psi'(u)$ in X^*, i.e. Ψ' is completely continuous, thus Ψ' is compact. □

The following critical point theorems constitute the principal tools used to obtain our result.

Theorem 2.10 ([5, Theorem 3.6]). *Let X be a reflexive real Banach space, and let $\Phi : X \to \mathbb{R}$ be a coercive functional that is continuously Gâteaux differentiable and weakly lower semicontinuous in the sequential sense. Assume that the Gâteaux derivative of Φ has a continuous inverse on the dual space X^*. Additionally, let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume the following conditions hold:*

(a_0)
$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

There exist constants $d > 0$ and a point $\overline{x} \in X$ such that $d < \Phi(\overline{x})$, and the following conditions are satisfied:

(a_1)
$$\frac{\sup_{\Phi(x)<d} \Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})},$$

(a_2) *For each $\lambda \in \Lambda_d := \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup_{\Phi(x)\le d}\Psi(x)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.*

Then, for every $\lambda \in \Lambda_d$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3 Main results

In this section, a theorem about the existence of at least three weak solutions to the problem (1.1) are obtained.

Recall the Hardy inequality (see Lemma 2.1 in [11] for more details), which states that for $1 < p < N$, the following inequality holds:

$$\int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx \le \frac{1}{H}\int_\Omega |\nabla u|^p\, dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $H = \left(\frac{N-p}{p}\right)^p$ is the optimal constant.

By combining this with Lemma 2.1 and using the fact that $1 < q < p_h(x) < N$, we get the continuous embeddings

$$W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow W_0^{1,p_h(x)}(\Omega) \hookrightarrow W_0^{1,q}(\Omega),$$

which leads to the inequality

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\, dx \le \frac{1}{H}\int_\Omega |\nabla u|^q\, dx, \quad \forall u \in W_0^{1,p(x)}(\omega,\Omega),$$

where $H = \left(\frac{N-q}{q}\right)^q$.

We are now ready to present our primary result. To this end, we define

$$D(x) := \sup\{D > 0 \mid B(x,D) \subseteq \Omega\}$$

for each $x \in \Omega$, where $B(x, D)$ denotes a ball centered at x with radius D. It is clear that there exists a point $x^0 \in \Omega$ such that $B(x^0, R) \subseteq \Omega$, where

$$R = \sup_{x \in \Omega} D(x).$$

In the remainder of the paper, the symbol m will represent the constant

$$m = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)},$$

with Γ denoting the Gamma function.

Theorem 3.1. *Assume that $p^- > \beta^+(r+1)$ and that there exist $d, \delta > 0$, such that*

$$\frac{a_1}{p^+}\left(\frac{2\delta}{R}\right)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} \geq d,$$

and

$$A_\delta := \frac{\frac{a_2}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}} \|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^{q} \frac{\|b\|_\infty}{qH} m \left(R^N - \left(\frac{R}{2}\right)^N\right)}{\frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left((\delta)^{\check{\alpha}} m \left(\frac{R}{2}\right)^N\right)^{r+1}} < B_d := \frac{d}{\frac{(M_2)^{r+1}\left(c_\beta^{\hat{\beta}}\right)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\left(\left[\left(\frac{p^+}{a_1}d\right)^{\frac{1}{\check{p}}}\right]^{\hat{\beta}}\right)^{r+1}},$$

then for any $\lambda \in]A_\delta, B_d[$, problem (1.1) admits at least three weak solutions.

Proof. It is worth noting that Φ and Ψ satisfy the regularity assumptions outlined in Theorem 2.10. We will now establish the fulfillment of conditions (a_1) and (a_2). To this end, let's consider

$$\frac{a_1}{p^+}\left(\frac{2\delta}{R}\right)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} \geq d$$

and consider $v_\delta \in X$ such that

$$v_\delta(x) := \begin{cases} 0 & x \in \Omega \backslash B\left(x^0, R\right) \\ \frac{2\delta}{R}\left(R - \left|x - x^0\right|\right) & x \in \mathfrak{B} := \overline{B}\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right), \\ \delta & x \in \overline{B}\left(x^0, \frac{R}{2}\right). \end{cases}$$

Then, by the definition of Φ , we have

$$\begin{aligned} \frac{a_1}{p^+}\left(\frac{2\delta}{R}\right)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} &< \Phi(v_\delta) \\ &\leq \frac{a_2}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}} \|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^{q} \frac{\|b\|_\infty}{qH} m \left(R^N - \left(\frac{R}{2}\right)^N\right) \end{aligned}$$

Therefore, $\Phi(v_\delta) > d$. However, it is important to consider the following

$$\begin{aligned} \Psi(v_\delta) \geq \frac{1}{r+1}\left(\int_\Omega F(x, v_\delta)dx\right)^{r+1} &\geq \frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left(\int_{B(x^0, \frac{R}{2})} |\delta|^{\alpha(x)} dx\right)^{r+1}, \\ &\geq \frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left((\delta)^{\check{\alpha}} m \left(\frac{R}{2}\right)^N\right)^{r+1}, \end{aligned} \tag{3.1}$$

which yields to $\frac{\Phi(v_\delta)}{\Psi(v_\delta)} \leq A_\delta < \lambda$. In addition, for every $u \in \Phi^{-1}(]-\infty, d])$, one has the following

$$\frac{a_1}{p^+}\|u\|^{\check{p}} \leq d. \tag{3.2}$$

therefore,

$$\|u\| \leq \left(\frac{p^+}{a_1}\Phi(u)\right)^{\frac{1}{\check{p}}} < \left(\frac{p^+}{a_1}d\right)^{\frac{1}{\check{p}}}.$$

Furthermore, under the assumption (f_1), we can conclude the following

$$\begin{aligned}\Psi(u) &\leq \frac{1}{r+1}\Big(\int_\Omega F(x,u(x))dx\Big)^{r+1}, \\ &\leq \frac{(M_2)^{r+1}}{(r+1)(\beta^-)^{r+1}}\Big(\int_\Omega |u|^{\beta(x)}dx\Big)^{r+1}, \\ &\leq \frac{(M_2)^{r+1}}{(r+1)(\beta^-)^{r+1}}(\|u\|_\beta^{\hat{\beta}})^{r+1}, \\ &\leq \frac{(M_2)^{r+1}(c_\beta^{\hat{\beta}})^{(r+1)}}{(r+1)(\beta^-)^{r+1}}(\|u\|^{\hat{\beta}})^{r+1}. \end{aligned} \tag{3.3}$$

This leads to the following result

$$\sup_{\Phi(u)<d}\Psi(u) \leq \frac{(M_2)^{r+1}(c_\beta^{\hat{\beta}})^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\left(\left[\Big(\frac{p^+}{a_1}d\Big)^{\frac{1}{\check{p}}}\right]^{\hat{\beta}}\right)^{r+1},$$

and

$$\frac{1}{d}\sup_{\Phi(u)<d}\Psi(u) < \frac{1}{\lambda}.$$

Furthermore, we can establish the coerciveness of $\mathcal{I}_\lambda$ for any positive value of λ by employing inequality (3.3) once more. This yields the following result

$$\Psi(u) \leq \frac{(M_2)^{r+1}(c_\beta^{\hat{\beta}})^{(r+1)}}{(r+1)(\beta^-)^{r+1}}(\|u\|^{\hat{\beta}})^{r+1}.$$

When $\|u\|$ is great enough, the following can be inferred

$$\Phi(u) - \lambda\Psi(u) \geq \frac{a_1}{p^+}\|u\|^{p^-} - \lambda\frac{(M_2)^{r+1}(c_\beta^{\hat{\beta}})^{(r+1)}}{(r+1)(\beta^-)^{r+1}}(\|u\|^{\hat{\beta}})^{r+1}.$$

By considering the fact that $p^- > \beta^+(r+1)$, we can reach the desired conclusion. In conclusion, considering the aforementioned fact that

$$\bar{\Lambda}_{d,\delta} := (A_\delta, B_d) \subseteq \left(\frac{\Phi(v_\delta)}{\Psi(v_\delta)}, \frac{d}{\sup_{\Phi(u)<d}\Psi(u)}\right),$$

Based on Theorem 2.10, it can be deduced that for any $\lambda \in \bar{\Lambda}_{d,\delta}$, the function $\Phi - \lambda\Psi$ possesses at least three critical points in $X := W_0^{1,p}(\omega, \Omega)$. These critical points correspond to weak solutions of problem (1.1). □

Remark 3.2. Setting $d = \frac{a_1}{p^+}$, then B_d becomes

$$B_d = \frac{\frac{a_1}{p^+}}{\frac{(M_2)^{r+1}\left(c_\beta^{\hat{\beta}}\right)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}},$$

moreover, one has

$$p^+ \geq p^- > \beta^+(r+1) \geq \alpha^+(r+1) \quad \text{and} \quad q < p^+,$$

thus $\lim_{\delta \to 0^+} A_\delta = 0$, consequently, $]A_\delta, B_d[\neq \emptyset$ and Theorem 3.1, can be rewritten as follows: Assume that $p^- > \beta^+(r+1)$ and that there exist $\delta > 0$, such that

$$\delta \geq \frac{R}{2}\left(\frac{1}{\|\omega\|_{L^1(\mathfrak{B})}}\right)^{\frac{1}{\hat{p}}}$$

and

$$A_\delta := \frac{\frac{a_2}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}}\|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^q \frac{\|b\|_\infty}{qH} m\left(R^N - \left(\frac{R}{2}\right)^N\right)}{\frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left((\delta)^{\check{\alpha}} m\left(\frac{R}{2}\right)^N\right)^{r+1}} < B_d := \frac{\frac{a_1}{p^+}}{\frac{(M_2)^{r+1}\left(c_\beta^{\hat{\beta}}\right)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}},$$

then for any $\lambda \in]A_\delta, B_d[$, problem (1.1) admits at least three weak solutions.

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References

[1] S. N. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions: existence uniqueness, localization, blow-up*, Atlantis Studies in Differential Equations, Vol. 4, Atlantis Press, Paris, 2015. https://doi.org/10.2991/978-94-6239-112-3

[2] G. Bonanno, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.* **1**(2012), 205–220. https://doi.org/10.1515/anona-2012-0003

[3] G. Bonanno, A. Chinnì, V. D. Rădulescu, Existence of two non-zero weak solutions for a $p(\cdot)$-biharmonic problem with Navier boundary conditions, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.* **34**(2023), 727–743. https://doi.org/10.4171/RLM/1025

[4] G. Bonanno, G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, *Z. Anal. Anwend.* **35**(2016), 449–464. https://doi.org/10.4171/ZAA/1573

[5] G. Bonanno, S. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.* **89**(2010), 1–10. https://doi.org/10.1080/00036810903397438

[6] P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 5, W. de Gruyter, Berlin–New York, 1997. https://doi.org/10.1515/9783110804775

[7] X. Fan, Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.* **312**(2005), 464–477. https://doi.org/10.1016/j.jmaa.2005.03.057

[8] X. Fan, X. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.* **59**(2004), 173–188. https://doi.org/10.1016/j.na.2004.07.009

[9] X. Fan, J. S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}$, *J. Math. Anal. Appl.* **262**(2001), 749–760. https://doi.org/10.1006/jmaa.2001.7618

[10] X. Fan, D. Zhao, On the spaces $L^{p(x)}$ and $W^{m,p(x)}$, *J. Math. Anal. Appl.* **263**(2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617

[11] P. J. García Azorero, I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, *J. Differential Equations* **144**(1998), 441–476. https://doi.org/10.1006/jdeq.1997.3375

[12] K. Ho, I. Sim, Existence and some properties of solutions for degenerate elliptic equations with exponent variable, *Nonlinear Anal.* **98**(2014), 146–164. https://doi.org/10.1016/j.na.2013.12.003

[13] K. Ho, I. Sim, Existence results for degenerate $p(x)$-Laplace equations with Leray–Lions type operators, *Sci. China Math.* **60**(2017), 133–146. https://doi.org/10.1007/s11425-015-0385-0

[14] Y. H. Kim, L. Wang, C. Zhang, Global bifurcation for a class of degenerate elliptic equations with variable exponents, *J. Math. Anal. Appl.* **371**(2010), 624–637. https://doi.org/10.1016/j.jmaa.2010.05.058

[15] J. Liu, Z. Zhao, Existence of triple solutions for elliptic equations driven by p-Laplacian-like operators with Hardy potential under Dirichlet–Neumann boundary conditions, *Bound. Value Probl.* **2023**, Article 3. https://doi.org/10.1186/s13661-023-01692-8

[16] J. Liu, Z. Zhao, Leray–Lions type $p(x)$-biharmonic equations involving Hardy potentials, *Appl. Math. Lett.* **149**(2024), 108907. https://doi.org/10.1016/j.aml.2023.108907

[17] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, in: P. Bénilan, J. Robert (Eds.), *Journées d'analyse non linéaire*, Lecture Notes in Mathematics, Vol. 665, Springer, Berlin, 1978, pp. 205–227. https://doi.org/10.1007/BFb0061807