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On the surjectivity conditions of a linear functional differential operator of the second order

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Abstract. Linear second-order functional differential equations without the Volterra condition are studied. Sufficient conditions for the everywhere solvability of the equation (surjectivity of the corresponding functional differential operator) are obtained in terms of norms of the positive and negative parts of the functional operators. These conditions are shown to be sharp in the sense that if they are not satisfied, then there exists an equation with no solution. The obtained solvability conditions are formulated directly for the equation itself, without considering specific boundary value problems.

Keywords: functional differential equations, solvability, boundary value problems, surjectivity.

2020 Mathematics Subject Classification: 34K05, 34K06, 34K10.

1 Introduction.

Functional differential equations are very important because they successfully model many phenomena in physics, engineering, biology, and economics [3, 7, 13, 20] (of course, they are also important as an independent mathematical object [3, 4, 6, 32]). Often, the most important aspect of the study is the study of solutions of boundary value problems for functional differential equations. Many works are devoted to the theory of boundary value problems for such equations [6, 16, 31]. However, given the widespread use of modeling in all fields of knowledge, a functional differential equation can be interesting even without being tied to a specific boundary value problem. In this case, first of all, the question of the existence of a solution to the equation for any right-hand side (the surjectivity of the operator) is of interest; only then can the equation be used for appropriate modeling.

So, we consider linear functional differential equations

$$(\mathcal{L}x)(t) = f(t), \quad t \in [0,1], \tag{1.1}$$

where the operator $\mathcal{L}$ acts from the space $\mathbf{AC}^1$ (real functions that are absolutely continuous together with their first derivative on the interval $[0,1]$) to the space $\mathbf{L}$ of integrable real functions and is defined by the equality

$$(\mathcal{L}x)(t) \equiv \ddot{x}(t) - (T^+x)(t) + (T^-x)(t), \quad t \in [0,1],$$

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where T^+, T^- are linear bounded positive operators acting from the space of continuous real functions $\mathbf{C}$ into the space $\mathbf{L}$. A linear operator $T : \mathbf{C} \to \mathbf{L}$ is called positive if it maps non-negative functions into almost everywhere non-negative ones. The norm of such a positive operator is defined by the equality

$$\| T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1})(s)\, ds,$$

where $\mathbf{1} : \mathbb{R} \to \mathbb{R}$ is the unit function. Here in the spaces $\mathbf{AC}^1$, $\mathbf{C}$, $\mathbf{L}$ the standard norms are used:

$$\| x\|_{\mathbf{AC}^1} = |x(0)| + |\dot{x}(0)| + \int_0^1 |\ddot{x}(t)|\, dt,\ \| x\|_{\mathbf{C}} = \max_{t\in[0,1]} |x(t)|,\ \| z\|_{\mathbf{L}} = \int_0^1 |z(t)|\, dt.$$

Next, we will also need the space $\mathbf{L}_\infty[0,1]$ of real functions essentially bounded on $[0,1]$ and the space $\mathbf{AC}_\infty$ of absolutely continuous real functions on $[0,1]$ with derivative $\dot{x} \in \mathbf{L}_\infty[0,1]$,

$$\| z\|_{\mathbf{L}_\infty} = \operatorname*{ess\,sup}_{s\in[0,1]} |z(t)|, \quad \| x\|_{\mathbf{AC}_\infty} = |x(0)| + \| \dot{x}\|_{\mathbf{L}_\infty}.$$

Definition 1.1. A solution of equation (1.1) is any function $x \in \mathbf{AC}^1$ that satisfies equation (1.1) for almost all $t \in [0,1]$.

Definition 1.2. We call equation (1.1) everywhere solvable if it has a solution for all $f \in \mathbf{L}$. In this case, the operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ is surjective.

Definition 1.3. A linear boundary value problem for equation (1.1) is a system of equations consisting of equation (1.1) and two equations of the form

$$\ell_1 x = \alpha_1, \quad \ell_2 x = \alpha_2, \tag{1.2}$$

where $\ell_1, \ell_2 : \mathbf{AC}^1 \to \mathbb{R}$ are linearly independent linear bounded functionals, α_1, $\alpha_2 \in \mathbb{R}$ are given numbers.

Definition 1.4. Boundary value problem (1.1)–(1.2) is called Fredholm if the operator of this problem $[\mathcal{L}, \ell_1, \ell_2] : \mathbf{AC}^1 \to \mathbf{L} \times \mathbb{R}^2$ is a Noetherian operator of zero index [5,21].

The space $\mathbf{AC}^1$ is compactly embedded in the space $\mathbf{C}$ (see, for example, [25]). Hence, any boundary value problem (1.1)–(1.2) is Fredholm. So, a necessary and sufficient condition for the unique solvability of this problem is that the corresponding homogeneous boundary value problem

$$\mathcal{L}x = 0, \quad \ell_1 x = 0, \quad \ell_2 = 0,$$

has only the trivial solution. in the space $\mathbf{AC}^1$. Note that in [5] it was shown that even under weaker natural assumptions on the operator $\mathcal{L}$ boundary value problem (1.1)–(1.2) is Fredholm.

We are interested not in the solvability condition of boundary value problems for equation (1.1), but rather in the solvability conditions of equation (1.1) itself. Since we do not require the Volterra property [5, p. 200] of the operators T^+, T^-, the properties of our equation may not directly correspond to those of linear ordinary differential equations [5, p. 100], for which the unique solvability of the Cauchy problem guarantees solvability everywhere. It is easy to give an example of an unsolvable equation (1.1). Consider the equation

$$\ddot{x}(t) - 4\,x(0) + 8\,x(1/2) - 4\,x(1) = 1, \quad t \in [0,1]. \tag{1.3}$$

This equation has form (1.1) for

$$(T^+x)(t) = 4\,x(0) + 4\,x(1), \quad (T^-x)(t) = 8x(1/2), \quad t \in [0,1],$$

thus, $\| T^+\|_{\mathbf{C}\to\mathbf{L}} = 8$, $\| T^-\|_{\mathbf{C}\to\mathbf{L}} = 8$. Since for the solution x of equation (1.3) the function $\ddot{x}$ must be constant, then $x(t) = a_0 + a_1\,t + a_2\,t^2$, $t \in [0,1]$, for some constants a_i. It is easy to check that $\mathcal{L}x = 0$ for this x. Then $\mathcal{L}x(t) \equiv 0 \neq 1$. The contradiction shows that equation (1.3) has no solution.

Definition 1.5. Let $\mathcal{P} \geqslant 0$, $\mathcal{Q} \geqslant 0$. Denote by $\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ the set of operators $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ defined by the equality

$$(\mathcal{L}x)(t) \equiv \ddot{x}(t) - (T^+x)(t) + (T^-x)(t), \quad t \in [0,1], \tag{1.4}$$

such that

$$\| T^+\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{P}, \quad \| T^-\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{Q}. \tag{1.5}$$

Denote by $\mathcal{S}$ the set of surjective operators (1.4).

Definition 1.6. The set $\Omega \equiv \{(\mathcal{P},\mathcal{Q}) \in [0,\infty)^2 \mid \mathcal{M}_{\mathcal{P},\mathcal{Q}} \subset \mathcal{S}\}$ is called the set of everywhere solvability (or the set of surjectivity).

Remark 1.7. From the proof of Lemmas 2.3, 2.4, 2.5, and Theorem 3.1 it follows that if $(\mathcal{P},\mathcal{Q}) \in \Omega$, then $(\mathcal{P}_1,\mathcal{Q}_1) \in \Omega$ for all $\mathcal{P}_1 \in [0,\mathcal{P}]$, $\mathcal{Q}_1 \in [0,\mathcal{Q}]$.

Therefore, the equalities (1.5) in Definition 1.5 can be replaced by the inequalities

$$\| T^+\|_{\mathbf{C}\to\mathbf{L}} \leqslant \mathcal{P}, \quad \| T^-\|_{\mathbf{C}\to\mathbf{L}} \leqslant \mathcal{Q}.$$

Our goal is to find the set Ω. The following Theorem 3.1 gives a complete description of Ω.

The problem of constructing a set of everywhere solvability, apparently, has not attracted attention due to the fact that there is a simple necessary and sufficient condition for the operator $\mathcal{L}$ to be everywhere solvable.

Lemma 1.8 ([5]). *Equation* (1.1) *is everywhere solvable if and only if there exist linearly independent linear bounded functionals* $\ell_1, \ell_2 : \mathbf{AC}^1 \to \mathbb{R}$ *such that boundary value problem* (1.1)–(1.2) *is uniquely solvable.*

By choosing different functionals ℓ_1, ℓ_2 and finding solvability conditions for boundary value problem (1.1)–(1.2) in terms of the norms of the operators T^+ and T^-, using Lemma 1.8 we can construct only subsets of the set of everywhere solvability Ω. Finding the set of everywhere solvability is, apparently, a new problem that has not yet been solved in this formulation. Its solution can be useful in modeling various processes, including in economics, when it is required that a solution exists for each external influence from the space under consideration.

In the work [18], surjectivity was studied for a partial differential equation with constant and distributed delay. In [1], the surjectivity of functional differential operators was studied using the surjectivity module [28, p. 26] $q(\mathcal{L}) = \inf_{y\neq 0} \| \mathcal{L}^*y\| / \| y\|$, which is related to the minimal norm of the Green operator of boundary value problems for the equation $\mathcal{L}x = f$. Here it is possible to estimate the norm of the adjoint operator from below only in very special cases. For example, one effective condition for the surjectivity of a neutral functional differential operator was obtained in [2]. In the papers [29,30], conditions for surjectivity of operators in a Banach space are obtained, and the results are applied to boundary value problems. In

[14,15], perturbations of linear surjective operators are studied. Thus, surjectivity conditions for linear operators can also be useful in studying boundary value problems. Coefficient surjectivity criteria (including those for families of equations) without special requirements for functional operators have not been obtained, as far as the author knows.

A convenient criterion for surjectivity is formulated using the adjoint operator.

Lemma 1.9 ([5]). *The following statements are equivalent:*

1) *the operator $\mathcal{L}$ is surjective;*

2) $\dim\ker\mathcal{L} = 2$;

3) $\ker\mathcal{L}^* = \{0\}$.

The construction of the solvability set Ω is apparently a rather difficult problem. To the author's knowledge, it has only been solved for the first-order equation [9]. The method used here is completely different from the method of [9]. It is based on the fact that if the family $\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ contains a non-surjective operator, then this family contains finite-dimensional operators T^+ and T^- for which the adjoint boundary value problem has a non-trivial solution.

There are relatively many works on the conditions for the solvability of boundary value problems for functional differential equations (see, for example, [8,10–12,26,27]).

If in Lemma 1.8 we take the periodic boundary value problem $x(0) = x(1)$, $\dot{x}(0) = \dot{x}(1)$, then the solvability condition [22] of this problem gives the following conditions for the surjectivity of $\mathcal{L}$:

$$\frac{4\mathcal{P}}{4-\mathcal{P}} \leqslant \mathcal{Q} \leqslant 8+4\sqrt{4-\mathcal{P}},\ \mathcal{P} < 4, \quad \text{or} \quad \frac{4\mathcal{Q}}{4-\mathcal{Q}} \leqslant \mathcal{P} \leqslant 8+4\sqrt{4-\mathcal{Q}},\ \mathcal{Q} < 4. \tag{1.6}$$

Our main result, formulated in Theorem 3.1 (section 3), will significantly improve inequalities (1.6) (see Fig. 3.1, Corollaries (3.2), (3.3), (3.4)).

The general scheme of this work is as following. First we show that if $(\mathcal{P},\mathcal{Q}) \notin \Omega$, there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the equation $\mathcal{L}^*g = 0$ has a non-trivial solution (Lemmas 2.1, 2.2). Then we prove that in this case for some $\mathcal{P}_1 \leqslant \mathcal{P}$, $\mathcal{Q}_1 \leqslant \mathcal{Q}$ there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P}_1,\mathcal{Q}_1}$ such that the equation $\mathcal{L}^*g = 0$ has a non-trivial piecewise linear solution (Lemma 2.3). In Lemmas 2.4, 2.5, is proved that it is sufficient to consider the case of a piecewise linear solution g whose graph consists of three line segments. This last case admits a simple explicit solution of the adjoint equation, which allows us to explicitly describe the set Ω (Theorem 3.1).

2 Auxiliary statements

If $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$, then

$$\mathcal{L}x = \ddot{x} - Tx, \tag{2.1}$$

where the operator $T : T^+ - T^-$ has the representation [19, p. 303–304] in the form of the Riemann–Stieltjes integral

$$(Tx)(t) = \int_0^1 x(s)\, d_s r(t,s), \quad t \in [0,1],$$

$$r(t,s) = p^+(t)k^+(t,s) - p^-(t)k^-(t,s), \quad t \in [0,1],$$

- the functions $p^+, p^- \in \mathbf{L}$ are nonnegative,
- $\| p^+\|_{\mathbf{L}} = \mathcal{P}$, $\| p^-\|_{\mathbf{L}} = \mathcal{Q}$,
- the functions $k^+(t,\cdot)$, $k^-(t,\cdot)$ are nondecreasing for all $t \in [0,1]$,
- $k^+(t,0) = 0$, $k^-(t,0) = 0$, $k^+(t,1) = 1$, $k^-(t,1) = 1$.

The operators T^+, T^- have the representations

$$(T^{+/-}x)(t) = \int_0^1 x(s)d_s r^{+/-}(t,s), \quad t \in [0,1],$$

where $r^+(t,s) = p^+(t)k^+(t,s)$, $r^-(t,s) = p^-(t)k^-(t,s)$, $s,t \in [0,1]$, $\int_0^1 r^+(t,1)\,dt = \mathcal{P}$, $\int_0^1 r^-(t,1)\,dt = \mathcal{Q}$.

2.1 Adjoint equation

Let $\mathcal{L}^* : (\mathbf{L})^* \to (\mathbf{AC}^1)^*$ be the adjoint operator. The $\mathbf{AC}^1$ is isomorphic to the space $\mathbf{L} \times \mathbb{R}^2$. Indeed, let $(\delta x)(t) = \ddot{x}(t)$, $t \in [0,1]$; $r_0 x = x(0)$, $r_1 x = \dot{x}(0)$. Then the operator $[\delta, r_0, r_1] : \mathbf{AC}^1 \to \mathbf{L}_1[0,1] \times \mathbb{R}^2$ defined by the equality

$$[\delta, r_0, r_1]x = \{\ddot{x}, x(0), \dot{x}(0)\},$$

is continuously invertible. Let $(\Lambda z)(t) = \int_0^t (t-s)z(s)\,ds$, $t \in [0,1]$, for every $z \in \mathbf{L}$. Then $\Lambda : \mathbf{L} \to \mathbf{AC}^1$ is a bounded operator and the Cauchy problem

$$\begin{cases} \delta x = z \\ r_0 x = c_0, \quad r_1 x = c_1, \end{cases}$$

has a unique solution

$$x(t) = c_0 + c_1 t + (\Lambda z)(t), \quad t \in [0,1]. \tag{2.2}$$

So, we have

$$\left([\delta, r_0, r_1]^{-1}\{z, c_0, c_1\}\right)(t) = c_0 + c_1\, t + (\Lambda z)(t), \quad t \in [0,1].$$

Thus, $(\mathbf{AC}^1)^* \simeq (\mathbf{L} \times \mathbb{R}^2)^* = (\mathbf{L})^* \times \mathbb{R}^2 = \mathbf{L}_\infty[0,1] \times \mathbb{R}^2$. Therefore, $\mathcal{L}^*$ acts from $(\mathbf{L})^* \simeq \mathbf{L}_\infty[0,1]$ into the space $\mathbf{L}_\infty[0,1] \times \mathbb{R}^2$.

Lemma 2.1. *Let an operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ be defined by (2.1). Then $\mathcal{L}$ is not surjective if and only if there exists a non-trivial $g \in \mathbf{L}_\infty[0,1]$ satisfying the homogeneous adjoint equation $\mathcal{L}^* g = 0$:*

$$g(t) - \int_0^1 \left(\int_t^1 (\tau - t)d_\tau r(s,t)\right) g(s)\,ds = 0, \quad t \in [0,1], \tag{2.3}$$

$$\int_0^1 \int_0^1 d_s r(t,s) g(t)\,dt = \int_0^1 r(s,1)g(s)\,ds = 0, \tag{2.4}$$

$$\int_0^1 \int_0^1 s\, d_s r(t,s) g(t)\,dt = 0. \tag{2.5}$$

Proof. Let $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ from (2.1), $x \in \mathbf{AC}^1$, $g \in \mathbf{L}_\infty[0,1]$. Using the representation

$$x(t) = x(0) + t\dot{x}(0) + \int_0^1 (t-s)\ddot{x}(s)\,ds, \quad t \in [0,1],$$

we obtain

$$
\begin{aligned}
&\int_0^1 g(t)(\mathcal{L}x)(t)\,dt \\
&= \int_0^1 g(t)\left[\ddot{x}(t) - \int_0^1 d_s r(t,s)\,x(0) - \int_0^1 s\,d_s r(t,s)\,\dot{x}(0) - \int_0^1 \left(\int_0^s (s-\tau)\ddot{x}(\tau)\,d\tau\right) d_s r(t,s)\right] dt \\
&= \int_0^1 \ddot{x}(t)\left[g(t) - \int_0^1 \left(\int_t^1 (s-t) d_s r(t,s)\right) g(\tau)\,d\tau\right] dt \\
&\quad - \int_0^1 \int_0^1 d_s r(t,s) g(t)\,dt\,x(0) - \int_0^1 \int_0^1 s\,d_s r(t,s) g(t)\,dt\,\dot{x}(0) = (\mathcal{L}^* g)(x),
\end{aligned} \tag{2.6}
$$

where $\mathcal{L}^* : \mathbf{L}_\infty[0,1] \to \mathbf{L}_\infty[0,1] \times \mathbb{R}^2$ is the adjoint operator of the operator $\mathcal{L}$.

If the operator $\mathcal{L}$ is not surjective, then there exists a non-trivial $g \in \mathbf{L}_\infty[0,1]$ such that $\int_0^1 g(t)(\mathcal{L}x)(t)\,dt = 0$ for all $x \in \mathbf{AC}^1$. From (2.6), it follows that $\int_0^1 g(t)(\mathcal{L}x)(t)\,dt = 0$ for all $\ddot{x} \in \mathbf{L}$, all $x(0) \in \mathbb{R}$, for all $\dot{x}(0) \in \mathbb{R}$. Therefore, in this case, there exists a non-zero solution $g \in \mathbf{L}_\infty[0,1]$ to the adjoint equation $\mathcal{L}^* g = 0$, that is g satisfies equalities (2.3), (2.4), (2.5).

If the system (2.3), (2.4), (2.5) has only the trivial solution, then there is no nontrivial g such that $\int_0^1 g(t)(\mathcal{L}x)(t)\,dt = 0$ for all $x \in \mathbf{AC}^1$. Therefore, the operator $\mathcal{L}$ is surjective. □

From equation (2.3) it follows that its solution g is differentiable and $\dot{g} \in \mathbf{L}_\infty$. Therefore, $g \in \mathbf{AC}_\infty$. Substituting $t = 1$ into equation (2.3), we obtain $g(1) = 0$. Substituting $t = 0$ into equation (2.3) and taking into account equality (2.5), we obtain $g(0) = 0$. Differentiating equality (2.3) and taking into account equality (2.4), we obtain

$$
\dot{g}(t) = -\int_0^1 \left(\int_t^1 d_\tau r(s,t)\right) g(s)\,ds = -\int_0^1 (r(s,1) - r(s,t)) g(s)\,ds = \int_0^1 r(s,t) g(s)\,ds.
$$

Therefore, system (2.3), (2.4), (2.5) is equivalent to the system

$$
\dot{g}(t) = \int_0^1 r(s,t) g(s)\,ds, \quad t \in [0,1], \tag{2.7}
$$

$$
g(0) = 0, \quad g(1) = 0 \tag{2.8}
$$

$$
\int_0^1 r(s,1) g(s)\,ds = 0 \tag{2.9}
$$

where $g \in \mathbf{AC}_\infty$.

From here we obtain the following lemma.

Lemma 2.2. *Let an operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ be defined by (2.1). Then $\mathcal{L}$ is not surjective if and only if there exists a non-trivial $g \in \mathbf{AC}_\infty$ satisfying to equalities (2.7), (2.8), (2.9) that is satisfying the homogeneous adjoint equation $\mathcal{L}^* g = 0$.*

In simplest case, when

$$
(\mathcal{L}x)(t) = \ddot{x}(t) - \sum_{j=1}^{n} p_j(t) x(t_j), \quad t \in [0,1].
$$

constructing the adjoint operator is not difficult. Let assume that $g \in (\mathbf{L})^* = \mathbf{L}_\infty[0,1]$. We use the representation (2.2) for $x \in \mathbf{AC}^1$. We have

$$\begin{aligned} g(\mathcal{L}x) &= \int_0^1 \ddot{x}(t)g(t)\,dt - \sum_{j=1}^n p_j(t)\left(x(0) + t_j\dot{x}(0) + \int_0^{t_j}(t_j - s)\ddot{x}(s)\,ds\right) \\ &= \int_0^1 \ddot{x}(s)\left(g(s) - \sum_{j=1}^n (t_j - s)^+ \int_0^1 g(\tau)p_j(\tau)\,d\tau\right)\,ds \\ &\quad - x(0)\int_0^1 g(t)\sum_{j=1}^n p_j(t)\,dt - \dot{x}(0)\int_0^1 g(t)\sum_{j=1}^n t_j p_j(t)\,dt = (\mathcal{L}^*g)(x). \end{aligned}$$

Thus, $g \in \ker \mathcal{L}^*$ if and only if $g(\mathcal{L}x) = 0$ for all $x(0) \in \mathbb{R}$, all $\dot{x}(0) \in \mathbb{R}$, and for all $\ddot{x} \in \mathbf{L}$, that is

$$g(s) - \sum_{j=1}^n (t_j - s)^+ \int_0^1 g(\tau)p_j(\tau)\,d\tau = 0, \quad s \in [0,1], \tag{2.10}$$

$$\int_0^1 g(t)\sum_{j=1}^n p_j(t)\,dt = 0, \tag{2.11}$$

$$\int_0^1 g(t)\sum_{j=1}^n t_j p_j(t)\,dt = 0, \tag{2.12}$$

where

$$\alpha^+ = \begin{cases} \alpha & \text{if } \alpha > 0; \\ 0 & \text{if } \alpha \leqslant 0. \end{cases}$$

From (2.10), it follows that any solution to $\mathcal{L}^*g = 0$ is piecewise linear. Moreover, $g(1) = 0$ from (2.10) and $g(0) = 0$ from (2.10) and (2.12).

2.2 Lemmas on the existence of piecewise linear solutions of a homogeneous adjoint equation

Lemma 2.3. *Let $(\mathcal{P}, \mathcal{Q}) \notin \Omega$. Then there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the equation $\mathcal{L}^*g = 0$ has a non-trivial solution g in the form*

$$g(t) = \sum_{j\in J} \varphi_j(t), \quad t \in [0,1], \tag{2.13}$$

where $J \subset \{1, 2, \dots\}$,

$$\varphi_j(t) = \begin{cases} \frac{t-a_j}{c_j-a_j}F_j, & t \in (a_j, c_j), \\ \frac{b_j-t}{b_j-a_j}F_j, & t \in (c_j, b_j), \\ 0, & t \in [0,1]\backslash(a_j, b_j), \end{cases} \tag{2.14}$$

$c_j \in (a_j, b_j)$, $F_j \in \mathbb{R}$, $j \in J$, the intervals (a_j, b_j) are disjoint.

Proof. Since $(P, Q) \notin \Omega$, there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the problem (2.7)–(2.9) has a nontrivial solution g. Denote by $\{e_j\}_{j\in J}$ the set of intervals within which g preserves its strict sign: $e_j = [a_j, b_j]$, $g(a_i) = g(b_j) = 0$, $g(t) \neq 0$ for $t \in (a_j, b_j)$, where a_j, b_j are pairs of adjacent zeros of g. There exist at most countably many such pairs. We will demonstrate how to modify the operator $\mathcal{L}$ such that the adjoint equation $\tilde{\mathcal{L}}^*\tilde{g} = 0$ of the modified operator $\tilde{\mathcal{L}}$

has a solution g satisfying the conditions of the lemma. It turns out that $|\tilde{g}(t)| \geqslant |g(t)|$ for each $t \in e_j$.

Let a, b be adjacent zeros of g, and $g(t) > 0$ for $t \in (a,b)$ (the case $g(t) < 0$, $t \in (a,b)$ is similar).

Denote $E^+ = \{\tau \in [0,1] : g(\tau) \geqslant 0\}$, $E^- = \{\tau \in [0,1] : g(\tau) < 0\}$.

We define a function $L : [a,b] \to \mathbb{R}$ by the following equalities: $L(a) = 0$; and for all $t \in (a,b]$

$$L(t) = \int_0^1 \max_{\substack{k^+(s,a)\leqslant \tilde{k}^+(s,t)\leqslant k^+(s,t)\\ k^-(s,a)\leqslant \tilde{k}^-(s,t)\leqslant k^-(s,t)}} (p^+(s)\tilde{k}^+(s,t) - p^-(s)\tilde{k}^-(s,t))g(s)\,ds$$
$$= \int\limits_{E^+} (p^+(s)k^+(s,t) - p^-(s)k^-(s,a))g(s)ds + \int\limits_{E^-} (p^+(s)k^+(s,a) - p^-(s)k^-(s,t))g(s)ds.$$

Since the functions $k^+(s,\cdot)$, $k^-(s,\cdot)$ are non-decreasing, the functions L and $\tilde{L}(t) \equiv \int_0^t L(s)\,ds$ are also non-decreasing, and the function $\tilde{L}$ is convex downwards. Therefore

$$\frac{\tilde{L}(c)(t-a)}{c-a} \leqslant \tilde{L}(t), \quad c \in (a,b], \quad t \in [a,c].$$

From the definition of the function L, it follows that

$$L(t) \geqslant \dot{g}(t), \quad \tilde{L}(t) \geqslant g(t), \quad t \in [a,b].$$

Similarly, we define the function R: $R(b) = 0$, when $t \in [a,b)$

$$R(t) = \int_0^1 \max_{\substack{k^+(s,t)\leqslant \tilde{k}^+(s,t)\leqslant k^+(s,b)\\ k^-(s,t)\leqslant \tilde{k}^-(s,t)\leqslant k^-(s,b)}} (p^+(s)\tilde{k}^+(s,t) - p^-(s)\tilde{k}^-(s,t))g(s)\,ds$$
$$= \int\limits_{E^+} (p^+(s)k^+(s,t) - p^-(s)k^-(s,b))g(s)ds + \int\limits_{E^-} (p^+(s)k^+(s,b) - p^-(s)k^-(s,t))g(s)ds.$$

It is obvious that $R(t) \leqslant \dot{g}(t)$, $t \in [a,b)$, R is non-decreasing, the function $\tilde{R}(t) \equiv -\int_t^b R(s)\,ds$ is non-increasing and convex downwards. Therefore

$$\frac{\tilde{R}(c)(b-t)}{b-c} \geqslant R(t) \geqslant g(t), \quad c \in [a,b), \quad t \in [c,b].$$

It follows from the definition of the function R that

$$R(t) \leqslant \dot{g}(t), \quad \tilde{R}(t) \geqslant g(t), \quad t \in [a,b].$$

Let $c \in (a,b)$ be a unique solution to the equation $\tilde{L}(c) = \tilde{R}(c)$.

Define a new function

$$\tilde{g}(t) \equiv \begin{cases} g(t), & t \in [0,1] \setminus [a,b], \\ \frac{\tilde{L}(c)(t-a)}{c-a}, & t \in [a,c], \\ \frac{\tilde{R}(c)(b-t)}{b-c}, & t \in [c,b]. \end{cases}$$

Define a new operator $\tilde{\mathcal{L}}$ by the equality $\tilde{\mathcal{L}}x \equiv \ddot{x} - \int_0^1 x(s)\,d_s\tilde{r}(\cdot,s)$, where

$$\tilde{r}(t,s) \equiv \tilde{p}^+(s)\tilde{k}^+(s,t) - \tilde{p}^-(s)\tilde{k}^-(s,t), \quad s,t \in [0,1],$$

and

$$\tilde{p}^+(t) \equiv \begin{cases} p^+(t), & t \in [0,1] \setminus (a,b), \\ \frac{p^+(t)g(t)}{\tilde{g}(t)}, & t \in (a,b), \end{cases} \qquad \tilde{p}^-(t) \equiv \begin{cases} p^-(t), & t \in [0,1] \setminus (a,b), \\ \frac{p^-(t)g(t)}{\tilde{g}(t)}, & t \in (a,b), \end{cases}$$

$$\tilde{k}^+(s,t) \equiv \begin{cases} k^+(s,a), \; s \in E^-, \; t \in [a,c], \\ k^+(s,b), \; s \in E^-, \; t \in [c,b], \\ k^+(s,t), \; \text{otherwise}, \end{cases} \qquad \tilde{k}^-(s,t) \equiv \begin{cases} k^-(s,a), \; s \in E^+, \; t \in [a,c], \\ k^-(s,b), \; s \in E^+, \; t \in [c,b], \\ k^-(s,t), \; \text{otherwise}. \end{cases}$$

Then $\tilde{\mathcal{L}} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$, since $\tilde{p}^+(s) \leqslant p^+(s)$, $\tilde{p}^-(s) \leqslant p^-(s)$, $s \in [0,1]$. Moreover, the function $\tilde{g}$ is a solution to the equation $\tilde{\mathcal{L}}^*\tilde{g} = 0$.

The procedure described above can be applied to all adjacent zeros (of which there are at most countably many). After this, we obtain an operator, which we denote by $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$, such that the equation $\mathcal{L}^*g = 0$ has a solution of the form (2.13), where ϕ_j are defined by (2.14). □

Lemma 2.4. *Let $(\mathcal{P},\mathcal{Q}) \notin \Omega$. Then there exist points $0 \leqslant a_1 < c_1 < b_1 \leqslant a_2 < c_2 < b_2 \leqslant 1$ and an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the adjoint equation $\mathcal{L}^*g = 0$ has a piecewise linear solution satisfying conditions $g(t) = 0$ for $t \in [0,1] \cup [b_1,a_2] \cup [b_2,1]$, $g(t) < 0$ for $t \in (a_1,b_1)$, $g(t) > 0$ for $t \in (a_2,b_2)$.*

Proof. By Lemma 2.3 there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the adjoint equation $\mathcal{L}^*g = 0$ has a piecewise linear solution g of the form (2.13). Let $\{e_j = [a_j,b_j]\}_{j\in J}$ be the set of all intervals between adjacent zeros of the function g, $\{e_j\}_{j\in J^+}$ be the set of all intervals within which $g(t) > 0$, $J^- = J \setminus J^+$.

Assume that the solution g takes its minimum and maximum at the points $c_1 < c_2$ respectively:

$$g_{\min} \equiv \min_{t\in[0,1]} g(t) = g(c_1), \quad g_{\max} \equiv \max_{t\in[0,1]} g(t) = g(c_2).$$

Let

$$\widetilde{e}_1 = [a_1,c_1], \; \widetilde{e}_2 = [c_1,b_1], \; \widetilde{e}_3 = [a_2,c_2], \; \widetilde{e}_4 = [c_2,b_2].$$

For $t \in \widetilde{e}_i$, $i = 1,2,3,4$, we have

$$r(s,t) = p^+(s)k_i^+(s) - p^-(s)k_i^-(s), \quad s \in [0,1],$$

where $k_i^+(s) \leqslant k_{i+1}^+(s)$, $k_i^-(s) \leqslant k_{i+1}^-(s)$, $i = 1,2,3$.

We will write the equation for g at $t \in \widetilde{e}_i$:

$$\begin{aligned} \dot{x}(t) &= \int_0^1 r(s,t)g(s)\,ds = \int_0^1 (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))\,ds \\ &= \sum_{j\in J^+} \int_{e_j} (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))g(s)\,ds + \sum_{j\in J^-} \int_{e_j} (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))g(s)\,ds. \end{aligned}$$

Let us stretch or compress (that is, make a suitable change of variable) all segments $[a_j,c_j]$, $[c_j,b_j]$ so that they coincide with the segments $[a_1,c_1]$, $[c_1,b_1]$, respectively, if $j \in J^-$, and coincide with the segments $[a_2,c_2]$, $[c_2,b_2]$ if $j \in J^+$. On each of these segments we multiply the functions p^+, p^- by the ratio of the corresponding segments so that the values

$$\int_{e_j} p^+(s)\,ds, \quad \int_{e_j} p^-(s)\,ds, \quad \int_{e_j} (p^+(s) - p^-(s))\,ds, \; \int_{e_j} (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))\,ds,$$

do not change after the change of variable.

Therefore, for $t \in \widetilde{e}_i$ we have

$$
\begin{aligned}
\dot{g}(t) &= \sum_{j\in J^-} \int_{e_1} (p_j^+(s)k_{i,j}^+(s) - p_j^-(s)k_{i,j}^-(s))\, g(s)\frac{g_j}{g_{\min}} ds \\
&\quad + \sum_{j\in J^+} \int_{e_2} (p_j^+(s)k_{i,j}^+(s) - p_j^-(s)k_{i,j}^-(s))\, g(s)\frac{g_j}{g_{\max}} ds \\
&= \int_{e_1} \left(\sum_{j\in J^-} \frac{p_j^+(s)g_j}{g_{\min}} k_{i,j}^+(s) - \sum_{j\in J^-} \frac{p_j^-(s)g_j}{g_{\min}} k_{i,j}^-(s) \right) g(s)ds \\
&\quad + \int_{e_2} \left(\sum_{j\in J^+} \frac{p_j^+(s)g_j}{g_{\max}} k_{i,j}^+(s) - \sum_{j\in J^+} \frac{p_j^-(s)g_j}{g_{\max}} k_{i,j}^-(s) \right) g(s)ds \\
&= \int_{e_1} \left(\left[\sum_{j\in J^-} \frac{p_j^+(s)g_j}{g_{\min}} \right] \widetilde{k}_i^+(s) - \left[\sum_{j\in J^-} \frac{p_j^-(s)g_j}{g_{\min}} \right] \widetilde{k}_i^-(s) \right) g(s)ds \\
&\quad + \int_{e_2} \left(\left[\sum_{j\in J^+} \frac{p_j^+(s)g_j}{g_{\max}} \right] \widetilde{k}_i^+(s) - \left[\sum_{j\in J^+} \frac{p_j^-(s)g_j}{g_{\max}} \right] \widetilde{k}_i^-(s) \right) g(s)ds.
\end{aligned}
$$

Here $p_j^{+/-}$, $k_{i,j}^{+/-}$ denote the functions $p^{+/-}$, $k_i^{+/-}$ restricted on the set e_j in the coordinates of one of the corresponding segments $\widetilde{e}_i$, $i = 1,2,3,4$; $g_j = g(c_j)$; the functions $\widetilde{k}_i^{+/-}$ are determined by the equalities

$$
\sum_{j\in J^{+/-}} \frac{p_j^{+/-}(s)g_j}{g_{\max/\min}} k_{i,j}^{+/-}(s) = \left(\sum_{j\in J^{+/-}} \frac{p_j^{+/-}(s)g_j}{g_{\max/\min}} \right) \widetilde{k}_i^{+/-}(s).
$$

Moreover,

$$
\widetilde{k}_i^{+/-}(s) \leqslant \widetilde{k}_{i+1}^{+/-}(s), \quad s \in [0,1], \quad i = 1,2,3,
$$

since

$$
k_{i,j}^{+/-}(s) \leqslant k_{i+1,j}^{+/-}(s), \quad s \in [0,1], \quad i = 1,2,3, \quad j \in J.
$$

We can now define

$$
\widetilde{p}^+(s) \equiv \begin{cases} \sum\limits_{j\in J^-} \frac{p_j^+(s)g_j}{g_{\min}}, & s \in e_1, \\ \sum\limits_{j\in J^+} \frac{p_j^+(s)g_j}{g_{\max}}, & s \in e_2, \\ 0, & s \in [0,1] \setminus (e_1 \cup e_2), \end{cases} \qquad \widetilde{p}^-(s) \equiv \begin{cases} \sum\limits_{j\in J^-} \frac{p_j^-(s)g_j}{g_{\min}}, & s \in e_1, \\ \sum\limits_{j\in J^+} \frac{p_j^-(s)g_j}{g_{\max}}, & s \in e_2, \\ 0, & s \in [0,1] \setminus (e_1 \cup e_2). \end{cases}
$$

Let $\widetilde{k}^{+/-}(s,t) = k_i(s)$ for $s \in [0,1]$, $t \in \widetilde{e}_j$. On the remaining part of the interval with respect to t, the function $\widetilde{k}^{+/-}(s,t)$ is extended in such a way that the functions $\widetilde{k}^{+/-}(s,\cdot)$ are non-decreasing for each $s \in [0,1]$.

In this case $\| \widetilde{p}^+ \|_{\mathbf{L}} \leqslant \| p^+ \|_{\mathbf{L}}$, $\| \widetilde{p}^- \|_{\mathbf{L}} \leqslant \| p^- \|_{\mathbf{L}}$, and

$$
\widetilde{g}(t) = \begin{cases} g(t), & t \in e_1 \cup e_2, \\ 0, & t \in [0,1] \setminus (e_1 \cup e_2), \end{cases}
$$

is a solution to the boundary value problem

$$\dot{\widetilde{g}}(t) = \int_0^1 \widetilde{r}(s,t)\widetilde{g}(s)\,ds, \quad t \in [0,1], \quad \widetilde{g}(0) = 0, \quad \widetilde{g}(1) = 0,$$
$$\int_0^1 (\widetilde{p}^+(s) - \widetilde{p}^-(s))g(s)\,ds = 0.$$

Thus, $(\mathcal{P}, \mathcal{Q}) \notin \Omega$, and there exists an operator $\widetilde{\mathcal{L}} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the equation $\widetilde{\mathcal{L}}^* \widetilde{g} = 0$ has a piecewise linear solution satisfying the conditions: $\widetilde{g}(t) = 0$ if $t \in [0, a_1] \cup [b_1, a_2] \cup [b_2, 1]$, $\widetilde{g}(t) < 0$ if $t \in (a_1, b_1)$, $\widetilde{g}(t) > 0$ if $t \in (a_2, b_2)$, where $0 \leqslant a_1 < c_1 < b_1 \leqslant a_2 < c_2 < b_2 \leqslant 1$. □

Lemma 2.5. *Let $(\mathcal{P}, \mathcal{Q}) \notin \Omega$. There exist points $0 = t_1 < t_2 < t_3 < t_4 = 1$, and functions $p_j \in \mathbf{L}$ such that the operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ defined by the equality*

$$\mathcal{L}x \equiv \ddot{x}(t) - \sum_{j=1}^{4} p_j(t) x(t_j), \quad t \in [0,1],$$

belongs to the set $\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ and is not surjective.

Proof. The operator $\mathcal{L}$, whose existence is asserted in the Lemma 2.4, has the form

$$(\mathcal{L}x)(t) = \ddot{x}(t) - \sum_{j \in J_1} p_j(t) x(t_j), \quad t \in [0,1],$$

where $p_j = p_j^+ - p_j^-$, p_j^+, $p_j^- \in \mathbf{L}$, $p_j^+(t) \geqslant 0$, $p_j^-(t) \geqslant 0$, $t \in [0,1]$, $\sum_{j \in J_1} \| p_j^+ \|_{\mathbf{L}} \leqslant \mathcal{P}$, $\sum_{j \in J_1} \| p_j^- \|_{\mathbf{L}} \leqslant \mathcal{Q}$. The set $\{t_j | j \in J_1\} = \{0, a_1, c_1, b_1, a_2, c_2, b_2, 1\}$ is finite.

The adjoint boundary value problem has the form (2.10), (2.11), (2.12).

By Lemma 2.4 the adjoint boundary value problem

$$g(s) - \sum_{j \in J_1} (t_j - s)^+ \int_0^1 g(\tau) p_j(\tau)\,d\tau = 0, \quad s \in [0,1], \tag{2.15}$$

$$\int_0^1 g(t) \sum_{j \in J_1} p_j(t)\,dt = 0, \tag{2.16}$$

$$\int_0^1 g(t) \sum_{j \in J_1} t_j p_j(t)\,dt = 0. \tag{2.17}$$

has a nontrivial piecewise linear solution g.

We see that problem (2.15)–(2.17) depends only on the values $A_j = \int_0^1 g(t) p_j(t)\,dt$. We know that $\min_{t \in [0,1]} g(t) = g(c_1)$, $\max_{t \in [0,1]} g(t) = g(c_2)$.

It is obvious that we can change the functions p_j so that the values A_j, $\| p_j^+ \|_L$, $\| p_j^- \|_L$ are preserved, and the support of each of the functions p_j is in any sufficiently small neighborhoods of four points, $t = 0$, $t = 1$, c_1, c_2. Let this condition be satisfied.

We will show that from the set $\{t_j | j \in J_1\}$ we can successively exclude the points a_1, b_1, a_2, b_2 by setting $p_j = 0$ for the corresponding j. Let $t_k \in (0,1)$ be one of such points, and let the supports of all functions p_k not intersect the interval (t_{k-1}, t_{k+1}). Then for some $\xi \in (0,1)$ the equality $t_k = \xi t_{k-1} + (1 - \xi) t_{k+1}$ holds. Let $\widetilde{p}_k = 0$, $\widetilde{p}_{k-1} = p_{k-1} + \xi p_k$, $\widetilde{p}_{k+1} = p_{k+1} + (1 - \xi) p_k$, $\widetilde{p}_i = p_i$ for $i \in J_1 \setminus \{t_{k-1}, t_k, t_{k+1}\}$. Let

$$\widetilde{g}(s) - \sum_{j \in J_1} (t_j - s)^+ \int_0^1 g(\tau) \widetilde{p}_j(\tau)\,d\tau = 0, \quad s \in [0,1].$$

It is easy to check that $\widetilde{g}$ satisfies the equation $\widetilde{\mathcal{L}}^*\widetilde{g}=0$ for

$$(\widetilde{\mathcal{L}}x)(t)=\ddot{x}(t)-\sum_{j\in J_1}\widetilde{p}_j(t)x(t_j),\quad t\in[0,1].$$

Obviously, $\widetilde{g}$ satisfies the boundary conditions (2.16), (2.17). □

3 Main result

Let us formulate the main assertion.

Theorem 3.1. *Let non-negative $\mathcal{P}$ and $\mathcal{Q}$ be given. $(\mathcal{P},\mathcal{Q})\in\Omega$ if and only if*

$$\mathcal{Q}\in[0,4],\quad \mathcal{P}\leqslant\min_{k\in(0,1]}\frac{(1+\sqrt{k}+\sqrt{k+1})^2(k+1)-\mathcal{Q}}{k},$$

or

$$\mathcal{Q}\in(4,12+8\sqrt{2}],\quad \mathcal{P}\leqslant\min_{k\in[0,1]}\Big((1+\sqrt{k}+\sqrt{k+1})^2(k+1)-\mathcal{Q}k\Big).$$

Proof. By Lemma 2.5, if $(\mathcal{P},\mathcal{Q})\notin\Omega$, then there exists a non-surjective operator $\mathcal{L}\in\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ defined by

$$(\mathcal{L}x)(t)\equiv\ddot{x}(t)-\sum_{j=1}^{4}p_j(t)x(t_j),\quad t\in[0,1],\tag{3.1}$$

where $0=t_1<t_2<t_3<t_4=1$,

$$p_j=p_j^+-p_j^-,\quad p_j^+,p_j^-\in\mathbf{L},\quad p_j^+(t)\geqslant0,\quad p_j^-(t)\geqslant0,\quad t\in[0,1].$$

We will assume that

$$\int_0^1 p_j^+(s)\,ds=\mathcal{P}_j,\quad j=1,\dots,4,\quad \sum_{j=1}^4\mathcal{P}_j=\mathcal{P},$$

$$\int_0^1 p_j^-(s)\,ds=\mathcal{Q}_j,\quad j=1,\dots,4,\quad \sum_{j=1}^4\mathcal{Q}_j=\mathcal{Q}.$$

For the equation (3.1) the adjoint equation is constructed in part 2.1. Its solution satisfies the equalities (2.10), (2.11), (2.12). For convenience, we present here the adjoint boundary value problem

$$g(t)=\sum_{j=2,3,4}(t_j-t)^+\int_0^1 g(s)p_j(s)\,ds,\quad t\in[0,1],\tag{3.2}$$

$$g(0)=0,\quad g(1)=0,\tag{3.3}$$

$$\sum_{j=1}^4\int_0^1 g(s)p_j(s)\,ds=0,\tag{3.4}$$

$$\sum_{j=2}^4 t_j\int_0^1 g(s)p_j(s)\,ds=0,\tag{3.5}$$

which has a non-trivial solution g. Let us denote the maximum of this solution by $M\geqslant0$ (taken at the point $t=t_3$) and the minimum by $-m\leqslant0$ (taken at the point $t=t_2$). The function g is linear on each of the intervals $[0, t_2]$, $[t_2, t_3]$, $[t_3, 1]$. At least one of the numbers M and m is positive. Let $m > 0$.

Let us find for what minimal $\mathcal{P}$, $\mathcal{Q}$ a non-zero solution to the problem (3.2)–(3.5) can exist.

Let $A_j = \int_0^1 g(s) p_j(s)\, ds$, $j = 1, 2, 3, 4$.

From (3.2) for $t \in [t_3, 1]$ it follows that

$$\dot{g}(t) = -A_4 = \frac{-M}{1 - t_3}, \tag{3.6}$$

$t \in (t_3, 1)$, therefore $A_4 \geqslant 0$.

From (3.2) for $t \in (t_2, t_3)$ it follows that $\dot{g}(t) = -A_4 - A_3 = \frac{M+m}{t_3 - t_2} > 0$. Since $A_4 \geqslant 0$, then $A_3 < 0$.

From (3.2) for $t \in (0, t_2)$ it follows that

$$\dot{g}(t) = -A_4 - A_3 - A_2 = \frac{-m}{t_2} < 0. \tag{3.7}$$

Therefore, from (3.4) it follows that $\dot{g}(t) = A_1 = \frac{-m}{t_2}$, therefore $A_1 < 0$. In addition, we have $A_3 + A_4 < 0$, $A_4 + A_3 + A_2 > 0$. Therefore, $A_2 > 0$.

So, $A_1 < 0$, $A_2 > 0$, $A_3 < 0$, $A_4 \geqslant 0$.

To satisfy these conditions with minimal $\mathcal{P}$ and $\mathcal{Q}$, the supports of the functions p_1^+, p_2^-, p_3^+, p_4^- should be concentrated in a sufficiently small neighborhood of the point $t = t_2$, which is the minimum point of the function g. Similarly, the supports of the functions p_1^-, p_2^+, p_3^-, p_4^+ should be concentrated in a sufficiently small neighborhood of the point $t = t_3$, which is the maximum point of the function g. We have

$$\inf_{\substack{\|p_i^+\|_{\mathbf{L}} = P_i,\\ \|p_i^-\|_{\mathbf{L}} = Q_i}} A_i = -M\,Q_i - m\,P_i, \quad i = 1, 3; \qquad \sup_{\substack{\|p_i^+\|_{\mathbf{L}} = P_i,\\ \|p_i^-\|_{\mathbf{L}} = Q_i}} A_i = M\,P_i + m\,Q_i, \quad i = 2, 4, \tag{3.8}$$

while inf and sup in (3.8) cannot be achieved for the integrable functions p_j.

Consider the limiting case

$$A_i = -M\,Q_i - m\,P_i, \quad i = 1, 3; \quad A_i = M\,P_i + m\,Q_i, \quad i = 2, 4. \tag{3.9}$$

From (3.4) and (3.9), it follows that

$$A_1 + A_2 + A_3 + A_4 = M\,(P_2 + P_4 - Q_1 - Q_3) - m\,(P_1 + P_3 - Q_2 - Q_4) = 0. \tag{3.10}$$

From (3.5) and (3.9), we obtain

$$t_2\,(M\,P_2 + m\,Q_2) - t_3\,(M\,Q_3 + m\,P_3) + M\,P_4 + m\,Q_4 = 0. \tag{3.11}$$

Now, for $k \equiv \frac{M}{-m} \geqslant 0$ from (3.6), (3.7), (3.10), and (3.11), it follows that

$$k = \frac{Q_4(1 - t_3)}{1 - P_4(1 - t_3)} = \frac{1 - P_1\,t_2}{Q_1 t_2} = \frac{P_1 + P_3 - Q_2 - Q_4}{P_2 + P_4 - Q_1 - Q_3} = \frac{P_3\,t_2 - Q_2\,t_2 - Q_4}{P_2\,t_2 + P_4 - Q_3\,t_3}.$$

These inequalities are equivalent to the system

$$\begin{cases} P_1 + k\,Q_1 = \gamma_1, \\ k\,P_2 + Q_2 = \gamma_1 + (1 + k)\gamma_2, \\ P_3 + k\,Q_3 = k\,\gamma_3 + (1 + k)\gamma_2, \\ k\,P_4 + Q_4 = k\,\gamma_3, \end{cases} \tag{3.12}$$

where $\gamma_i \equiv \frac{1}{\Delta_i}$, $\Delta_i \equiv t_{i+1} - t_i$, $i = 1,2,3$.

From here and from the non-negativity of P_i, we obtain the conditions for Q_i:

$$\begin{cases} Q_1 \leqslant \frac{\gamma_1}{k}, \\ Q_2 \leqslant \gamma_1 + (1+k)\gamma_2, \\ Q_3 \leqslant \gamma_3 + \frac{(1+k)}{k}\gamma_2, \\ Q_4 \leqslant k\,\gamma_3. \end{cases} \tag{3.13}$$

From (3.12) we get

$$\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 = \mathcal{R} - \frac{1}{k}(Q_2 + Q_4) - k(Q_1 + Q_3), \tag{3.14}$$

where

$$\mathcal{R} \equiv \gamma_1 \frac{1+k}{k} + \gamma_2 \frac{(1+k)^2}{k} + \gamma_3(1+k).$$

We need to find the minimum of $\mathcal{P}$ over $k \in (0,1]$, over all γ_i, $i = 1,2,3$, $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \leqslant 1$, and over all $Q_i \geqslant 0$ such that $Q_1 + Q_2 + Q_3 + Q_4 = \mathcal{Q}$ and conditions (3.13) are satisfied. Let us find the maximum value of Q for which conditions (3.13) are satisfied. We have

$$Q = Q_1 + Q_2 + Q_3 + Q_4 \leqslant \mathcal{R}.$$

Set $\Delta_2 = 1 - \Delta_1 - \Delta_3$ and find the minimum

$$\mathcal{R} = (1+k)\left(\frac{1}{\Delta_1 k} + \frac{1+k}{(1-\Delta_1-\Delta_3)k} + \frac{1}{\Delta_3}\right)$$

relative to the variables Δ_1, Δ_3, with the other arguments fixed. From the conditions $\frac{\partial\mathcal{P}}{\partial\Delta_1} = \frac{\partial\mathcal{P}}{\partial\Delta_3} = 0$, we have

$$\Delta_1 = \frac{1}{1+\sqrt{k}+\sqrt{k+1}}, \quad \Delta_3 = \Delta_1\sqrt{k}, \quad \Delta_2 = \Delta_1\sqrt{k+1}.$$

Introduce the notations

$$G(k) \equiv (1+\sqrt{k}+\sqrt{k+1})^2(k+1), \tag{3.15}$$

and for every Q

$$H_1(k) \equiv \frac{G(k) - Q}{k}, \tag{3.16}$$

$$H_2(k) \equiv G(k) - k\,Q. \tag{3.17}$$

Then

$$\mathcal{R} = (1+k)\frac{(1+\sqrt{k}-\sqrt{k+1})^2}{(\sqrt{k+1}-\sqrt{k})^2(\sqrt{k+1}-1)^2} = \frac{G(k)}{k}.$$

We can compute the derivative:

$$\frac{dR}{dk} = k^{-5/2}\frac{1+\sqrt{k}+\sqrt{k+1}}{\sqrt{k+1}}(k^2(\sqrt{k}+\sqrt{k+1}) - \sqrt{k}(\sqrt{k+1}+1) < 0.$$

It is easy to check that $\frac{dR}{dk} < 0$ for all $k \in (0,1)$. Therefore, $\min_{k\in(0,1]} \mathcal{R}(k) = \mathcal{R}(1) = 2(2+\sqrt{2})^2 = 12 + 8\sqrt{2} \equiv \mathcal{Q}^{\max}$.

Thus, if $\mathcal{Q} = \mathcal{Q}^{\max}$, then equalities (3.12) are possible only if $\mathcal{P} = 0$ and not for integrable functions p_j. If $\mathcal{Q} > \mathcal{Q}^{\max}$, then equalities (3.12) are impossible.

Let $\mathcal{Q} \in [0, \mathcal{Q}^{\max})$. Find the minimum of $\mathcal{P}$ in (3.14) for $\mathcal{Q}_i \geqslant 0$ such that $\sum_{i=1}^{4} \mathcal{Q}_i = \mathcal{Q}$ and with the other variables fixed. Increasing Q_2 and Q_4 until the constraints (3.13) are satisfied and decreasing Q_1 and Q_3 to zero, while keeping $\mathcal{Q}$ the same, we obtain one of the following two cases:

i) if $\mathcal{Q} \leqslant \gamma_1 + \gamma_2(1+k) + k\gamma_3$, then $\mathcal{Q}_1 = \mathcal{Q}_3 = 0$ and

$$\mathcal{P} = \mathcal{R} - \frac{\mathcal{Q}}{k},$$

ii) if $\mathcal{Q} \geqslant \gamma_1 + \gamma_2(1+k) + k\gamma_3$, then $\mathcal{Q}_2 = \gamma_1 + \gamma_2(1+k)$, $\mathcal{Q}_4 = k\gamma_3$ and

$$\mathcal{P} = \mathcal{R} - k(\mathcal{Q} - (\gamma_1 + \gamma_2(1+k) + k\gamma_3)) - \frac{\gamma_1 + \gamma_2(1+k) + k\gamma_3}{k} = k\mathcal{R} - k\mathcal{Q}.$$

For a fixed k, the minimum value of $\mathcal{R}$ is $\frac{G(k)}{k}$, and the following equality holds for the constraint on $\mathcal{Q}$

$$\gamma_1 + \gamma_2(1+k) + k\gamma_3 = (1 + \sqrt{k} + \sqrt{k+1})^2.$$

Thus, in case i) one should minimize $\mathcal{P} = H_1(k)$ subject to

$$\mathcal{Q} \leqslant (1 + \sqrt{k} + \sqrt{k+1})^2, \tag{3.18}$$

in the second case, one should minimize $\mathcal{P} = H_2(k)$ subject to

$$\mathcal{Q} \geqslant (1 + \sqrt{k} + \sqrt{k+1})^2. \tag{3.19}$$

It is obvious that for $\mathcal{Q} \leqslant 4$ at the minimum point of $H_1(k)$, $k \in (0,1]$, the inequality (3.18) is satisfied.

It is easy to check that at the minimum point of $H_2(k)$, $k \in (0,1]$, the inequality (3.19) is satisfied if $\mathcal{Q} \geqslant 4$.

Since the limiting case (3.9) is not achieved on integrable coefficients p_j, the obtained boundaries belong to the set of everywhere solvability Ω. $\square$

Now we obtain a representation for the boundaries of the set Ω, which in the notation (3.15), (3.16), (3.17) in accordance with Theorem 3.1 look like this: $(\mathcal{P}, \mathcal{Q}) \in \Omega$ if and only if

$$\mathcal{Q} \in [0,4], \quad \mathcal{P} \leqslant \widetilde{\mathcal{P}_1}(\mathcal{Q}) \equiv \min_{k \in (0,1]} H_1(k).$$

or

$$\mathcal{Q} \in (4, 12 + 8\sqrt{2}], \quad \mathcal{P} \leqslant \widetilde{\mathcal{P}_2}(\mathcal{Q}) \equiv \min_{k \in [0,1]} H_2(k).$$

For all $\mathcal{Q} \in [0,4]$, the function H_1 takes its minimum at the point $k \in [0,1]$, which satisfies the equation $H_1'(k) = 0$, that is if

$$G'(k)k - G(k) + \mathcal{Q} = 0 \tag{3.20}$$

(here and below $(\cdot)' \equiv \frac{d(\cdot)}{dk}$). Therefore, the dependence $\widetilde{\mathcal{P}_1}(\mathcal{Q})$ for $Q \in [0,4]$ can be defined parametrically using the equalities (3.16), (3.20) .

First, we define the number $k_0 = k_1^2 \approx 0.43$, where $k_1 \in [0,1]$ is the only root of the equation $k^4 + 6k^3 + 5k^2 - k = 0$ on the interval $[0,1]$. The number k_0 satisfies the equation (3.20) for $\mathcal{Q} = 4$. The function $\mathcal{Q}(k) = G(k) - G'(k)k$ maps $[k_0, 1]$ onto $[0,4]$.

Let us define the function $\widetilde{\mathcal{P}_1}(\mathcal{Q})$ parametrically:

$$\mathcal{Q} = G(k) - G'(k)k, \quad \widetilde{\mathcal{P}_1} \equiv G'(k), \quad k \in [k_0, 1]. \tag{3.21}$$

Denote $\mathcal{P}_4 \equiv \widetilde{\mathcal{P}_1}(4) = G'(k_0) \approx 17.7$.

We have, in particular,

$$\widetilde{\mathcal{P}_1}(0) = 12 + 8\sqrt{2} \quad (k = 1), \quad \widetilde{\mathcal{P}_1}(27/8) = 19 \quad (k = 9/16), \quad \widetilde{\mathcal{P}_1}(4) = \mathcal{P}_4.$$

For $\mathcal{Q} \in (4, \mathcal{P}_4]$, the function $H_2(k)$ takes a minimum value for $k \in [0, 1]$ at the point $k = 0$, therefore $\widetilde{\mathcal{P}_2}(\mathcal{Q}) = H_2(0) = G(0) = 4$. If $\mathcal{Q} \in (\mathcal{P}_4, 12 + 8\sqrt{2}]$, then the function $H_2(k)$ takes a minimum value at the maximum point satisfying the equality

$$H_2'(k) = G'(k) - \mathcal{Q} = 0. \tag{3.22}$$

Therefore, the dependence $\widetilde{\mathcal{P}_2}(\mathcal{Q})$ for $\mathcal{Q} \in (\mathcal{P}_4, 12 + 8\sqrt{2}]$ can also be specified parametrically using the equalities (3.17),(3.22):

$$\mathcal{Q} = G'(k), \quad \widetilde{\mathcal{P}_2} = G(k) - G'(k)k, \quad k \in [k_0, 1]. \tag{3.23}$$

From the parameterizations (3.21) and (3.23) it is clear that the dependencies $\widetilde{\mathcal{P}_1}(Q)$ and $\widetilde{\mathcal{P}_2}(\mathcal{Q})$ are mutually inverse.

If $\mathcal{Q} > 12 + 8\sqrt{2}$, the minimum value of $H_2(k)$ is negative (and is accepted for $k = 1$). Thus, operator (1.4) cannot be surjective for all linear positive operators satisfying equalities (1.5) for any P.

Fig. 3.1 shows the boundaries of the set Ω.

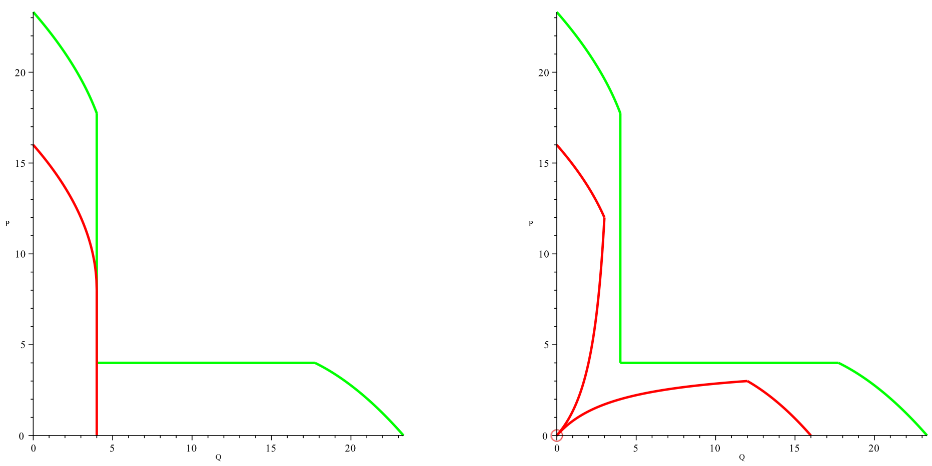

Figure 3.1: The boundaries of the set Ω are marked in green. On the left, the boundaries of the set of solvability of the two-point problem $x(0) = 0$, $x(1) = 0$ [23, 24] are marked in red. On the right, the boundaries of the set of solvability of the periodic problem (1.6) [17] are marked in red.

Let us formulate the statements just obtained as consequences of Theorem 3.1.

Corollary 3.2. *If* $(\mathcal{P},\mathcal{Q}) \in \Omega$, *then* $(\mathcal{Q},\mathcal{P}) \in \Omega$.

Corollary 3.3.

1. *If* $\mathcal{Q} = 0$, *then* $(\mathcal{P},\mathcal{Q}) \in \Omega$ *if and only if* $0 \leqslant \mathcal{P} \leqslant 12 + 8\sqrt{2}$.
2. *If* $\mathcal{Q} \in [0,4]$, *then* $(\mathcal{P},\mathcal{Q}) \in \Omega$ *if and only if* $\mathcal{P} \in [0, G'(k)]$, *where* $k \in (0,1]$ *is a unique solution to the equation* $G(k) - G'(k)k = \mathcal{Q}$.
3. *If* $\mathcal{Q} \in [4, \mathcal{P}_4]$, *then* $(\mathcal{P},\mathcal{Q}) \in \Omega$ *if and only if* $\mathcal{P} \in [0,4]$.
4. *If* $\mathcal{Q} \in [\mathcal{P}_4, 12 + 8\sqrt{2}]$, *then* $(\mathcal{P},\mathcal{Q}) \in \Omega$ *if and only if* $\mathcal{P} \in [0, G(k) - G'(k)k]$, *where* $k \in (0,1]$ *is the maximum root of the equation* $G'(k) = \mathcal{Q}$.

There are also rational points on the boundary of Ω.

Corollary 3.4. *If* $\mathcal{Q} = 27/8$, *then* $(\mathcal{P},\mathcal{Q}) \in \Omega$ *if and only if* $\mathcal{P} \in [0,19]$.
If $\mathcal{Q} = 19$, *then* $(\mathcal{P},\mathcal{Q}) \in \Omega$ *if and only if* $\mathcal{P} \in [0,27/8]$.

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References

[1] A. R. Abdullaev, N. A. Bragina, Green operators with minimal norm, *Russian Math.* **4**(2003), 3–7. MR1992157; Zbl 1088.47060

[2] A. R. Abdullaev, O. A. Nevolina, On a quasilinear differential equation of neutral type in the critical case, *Yaroslavl Pedagogical Bulletin* **2**(2013), 39–42.

[3] R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, *Nonoscillation theory of functional differential equations with applications*, Springer Science & Business Media, 2012. https://doi.org/10.1007/978-1-4614-3455-9; MR2908263; Zbl 1253.34002

[4] N. V. Azbelev, Impact of certain traditions on development of the theory of differential equations, *Comput. Math. Appl.* **37**(1999), No. 4–5, 1–8. https://doi.org/10.1016/S0898-1221(99)00054-1; MR1677170; Zbl 0936.34049

[5] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to the theory of functional differential equations* (in Russian), Nauka, Moscow, 1991. MR1144998; Zbl 0725.34071

[6] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to the theory of functional differential equations: methods and applications*, Contemp. Math. Appl., Vol. 3, Hindawi Publishing Corporation, Cairo, 2007. https://doi.org/10.1155/9789775945495; MR2319815; Zbl 1202.34002

[7] N. V. Azbelev, V. P. Maksimov, P. M. Simonov, Theory of functional differential equations and applications, *Int. J. Pure Appl. Math.* **69**(2011), No. 2, 203–235. MR2841706; Zbl 1228.34001

[8] A. Boichuk, J. Diblík, D. Khusainov, M. Růžičková, Fredholm's boundary-value problems for differential systems with a single delay, *Nonlinear Anal.* **22**(2010), No. 5, 2251–2258. https://doi.org/10.1016/j.na.2009.10.025; MR2577792; Zbl 1190.34073

[9] E. I. Bravyi, Conditions of everywhere solvability for a linear functional differential equation of the first order (in Russian), *Applied Mathematics and Control Sciences* **3**(2024), 38–52. https://doi.org/10.15593/2499-9873/2024.3.03

[10] A. Cabada, D. O'Regan, R. L. Pouso, Second order problems with functional conditions including Sturm–Liouville and multipoint conditions, *Math. Nachr.* **281**(2008), No. 9, 1254–1263. https://doi.org/10.1002/mana.200510675; MR2442703; Zbl 1160.34013

[11] A. Cabada, J. J. Nieto, A generalization of the monotone iterative technique for nonlinear second order periodic boundary value problems, *J. Math. Anal. Appl.* **151**(1990), No. 1, 181–189. https://doi.org/10.1016/0022-247X(90)90249-F; MR1069454; Zbl 0719.34039

[12] A. Domoshnitsky, S. Padhi, S. N. Srivastava, Vallee–Poussin theorem for fractional functional differential equations, *Fract. Calc. Appl. Anal.* **25**(2022), No. 4, 1630–1650. https://doi.org/10.1007/s13540-022-00061-z; MR4468528; Zbl 1503.34143

[13] C. Corduneanu, Y. Li, M. Mahdavi, *Functional differential equations. Advances and applications*, John Wiley & Sons, Hoboken, NJ, 2016. https://doi.org/10.1002/9781119189503; MR3525768; Zbl 1353.34001

[14] B. D. Gel'man, On a class of operator equations, *Math. Notes* **70**(2001), No. 4, 494–501. https://doi.org/10.1023/A:1012376602648; MR1884191; Zbl 1035.47045

[15] B. D. Gel'man, S. N. Kalabukhova, On condensing perturbations of closed linear surjective operators, *Glob. Stoch. Anal.* **2**(2012), No. 1, 1–16. Zbl 1296.47057

[16] R. Hakl, A. Lomtatidze, J. Šremr, *Some boundary value problems for first order scalar functional differential equations*, Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica, Vol. 10, Masaryk University, Brno, 2002. Zbl 1048.34004

[17] R. Hakl, S. Mukhigulashvili, A periodic boundary value problem for functional differential equations of higher order, *Georgian Math. J.* **16**(2009), No. 4, 651–665. MR2640795; Zbl 1187.34085

[18] A. G. Jeong, S. Nakagiri, H. Tanabe, Structural operators and semigroups associated with functional-differential equations in Hilbert spaces, *Osaka J. Math.* **30**(1993), No. 3, 365–395. MR1240003; Zbl 0818.47039

[19] L. V. Kantorovich, *Descriptive theory of sets and functions. Functional analysis in semi-ordered spaces*, Gordon and Breach Publishers, Amsterdam, 1996. Zbl 1061.01505

[20] V. Kolmanovskii, A. Myshkis, *Introduction to the theory and applications of functional differential equations*, Kluwer Academic Publishers, Dordrecht, 1999. https://doi.org/10.1007/978-94-017-1965-0; MR1680144; Zbl 0917.34001

[21] S. G. Krein, *Linear equations in Banach space* (in Russian), Nauka, Moscow, 1971. MR374949; Zbl 0233.47001

[22] A. Lomtatidze, S. Mukhigulashvili, On periodic solutions of second order functional differential equations, *Mem. Differ. Equ. Math. Phys.* **5**(1995), 125–126. Zbl 0866.34054

[23] S. V. Mukhigulashvili, On the unique solvability of the Dirichlet problem for a second-order linear functional-differential equation, *Differ. Equ.* **40**(2004), No. 4, 515–523. https://doi.org/10.1023/B:DIEQ.0000035789.25629.86; MR2153646; Zbl 1080.34050

[24] S. V. Mukhigulashvili, J. Šremr, On the solvability of the Dirichlet problem for nonlinear second-order functional-differential equations, *Differ. Equ.* **41**(2005), No. 10, 1425–1435. https://doi.org/10.1007/s10625-005-0294-z; MR2241459; Zbl 1141.34038

[25] K. T. Mynbaev, M. O. Otelbaev, *Weighted functional spaces and the spectrum of differential operators* (in Russian), Nauka, Moscow, 1988. Zbl 0651.46037

[26] J. J. Nieto, R. Rodríguez-López, Green's function for second-order periodic boundary value problems with piecewise constant arguments, *J. Math. Anal. Appl.* **304**(2005), 33–57. https://doi.org/10.1016/j.jmaa.2004.09.023; MR2124647; Zbl 1078.34046

[27] S. K. Ntouyas, Y. G. Sficas, P. Ch. Tsamatos, Boundary value problems for functional differential equations, *J. Math. Anal. Appl.* **199**(1996), No. 1, 213–230. https://doi.org/10.1006/jmaa.1996.0137; MR1381388; Zbl 0854.34066

[28] A. Pietsch, *Operator ideals*, Elsevier (North-Holland), Amsterdam, 1980. MR582655; Zbl 0434.47030

[29] V. Šeda, Surjectivity and boundary value problems, *Equadiff 6, Proc. 6th Int. Conf.*, 1985, Lect. Notes Math., Vol. 1192, pp. 161–170. Zbl 0636.34061

[30] V. Šeda, Surjectivity of an operator, *Czechoslovak Math. J.* **40**(1990), No. 1, 46–63. https://doi.org/10.21136/CMJ.1990.102358; MR1037350; Zbl 0739.47030

[31] A. L. Skubachevsky, Boundary value problems for elliptic functional-differential equations and their applications, *Russian Math. Surveys* **71**(2016), No. 5, 801–906. https://doi.org/10.1070/rm9739; MR3588929; Zbl 1364.35413

[32] V. V. Vlasov, D. A. Medvedev, Functional-differential equations in Sobolev spaces and related questions of spectral theory, *J. Math. Sci. (N.Y.)* **164**(2010), No. 5, 659–841. https://doi.org/10.1007/s10958-010-9768-5; MR2472270; Zbl 1288.35004