Electronic Journal of Qualitative Theory of Differential Equations
2025, No. **10**, 1–18; https://doi.org/10.14232/ejqtde.2025.1.10 www.math.u-szeged.hu/ejqtde/

Periodic solutions in a linear delay difference system

Dedicated to the memory of Professor István Győri (1943–2022)

Jan Čermák✉, **Lucie Fedorková** and **Luděk Nechvátal**

Institute of Mathematics, Brno University of Technology,
Technická 2896/2, 616 69 Brno, Czech Republic

Received 17 February 2025, appeared 5 April 2025
Communicated by Mihály Pituk

Abstract. The paper investigates periodicity properties of a linear autonomous difference system with two delayed terms. Assuming that the system matrices are simultaneously triangularizable, we formulate necessary and sufficient conditions guaranteeing the existence of a nonzero periodic solution (with an *a priori* given period) of the studied system. The analytical form of such conditions is shown to generalize the existing results on this topic. Moreover, it is supported by a geometric reformulation, offering a better understanding of the derived periodicity conditions. Information on the form of the searched periodic solution (including its prime period) is also provided.

Keywords: difference equation, delay, periodic solution.

2020 Mathematics Subject Classification: 39A23, 39A06.

1 Introduction and preliminaries

The study of periodic solutions in differential and difference systems belongs among frequent research topics. It is well known that the appearance of periodic solutions forms an important part of the bifurcation analysis of dynamical systems that is often accompanied by other interesting phenomena (such as the loss of asymptotic stability of the corresponding equilibria). There are various methods for detecting a periodic behavior in the studied systems, often based on fixed-point theorems. Since the procedures utilizing this approach do not usually give analytical expressions of the searched periodic solutions, other methods were found and applied to obtain the desired expressions (see, e.g., the Carvalho method, initially developed for delay differential equations but applicable also to discrete equations [6,7,14]). We refer to [2,8–12,17–19,21,22] for a survey of some results and techniques used in the existence theory of periodic solutions.

In the autonomous linear case, the problem of the existence of periodic solutions is related to the existence of a specific characteristic root. In particular, the delay differential system

$$y'(t) + \sum_{s=1}^{k} A_s y(t-s) = 0, \qquad t \in \mathbb{R}^+,$$

✉Corresponding author. Email: cermak.j@fme.vutbr.cz

where $A_s \in \mathbb{R}^{d\times d}$, has a q-periodic solution $y : \mathbb{R}^+ \to \mathbb{R}^d$ (i.e., a solution satisfying $y(t+q) = y(t)$ for all $t \in \mathbb{R}^+$ and some $q \in \mathbb{R}^+$) if and only if the corresponding characteristic quasi-polynomial

$$F(\lambda) = \det\left(\lambda I + \sum_{s=1}^{k} \exp(-\lambda s) A_s\right)$$

(I being the identity matrix) has a pure imaginary root. Similarly, the difference system

$$x(n) + \sum_{s=1}^{k} A_s x(n-s) = 0, \qquad n \in \mathbb{Z}^+, \tag{1.1}$$

$A_s \in \mathbb{R}^{d\times d}$, admits a q-periodic solution $x : \mathbb{Z}^+ \to \mathbb{R}^d$ (i.e., a solution satisfying $x(n+q) = x(n)$ for all $n \in \mathbb{Z}^+$ and some $q \in \mathbb{Z}^+$) if and only if the family of roots of

$$P(\lambda) = \det\left(\lambda^k I + \sum_{s=1}^{k} \lambda^{k-s} A_s\right)$$

involves a q-th root of unity. That is, there exists a root $\lambda_{j,q}$ of P in the form

$$\lambda_{j,q} = \exp\left(\mathrm{i}\frac{2\pi j}{q}\right) \quad \text{for some } j = \left\lfloor -\frac{q}{2} \right\rfloor + 1, \left\lfloor -\frac{q}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{q}{2} \right\rfloor. \tag{1.2}$$

However, converting these theoretical conditions into efficient forms is usually regarded as questionable, therefore other approaches are preferred.

The aim of this paper is to pose arguments supporting a deeper analysis of the above stated polynomial properties when detecting a periodic behavior of the studied systems. To do this, we consider a particular case of the difference system (1.1) and apply a simple method converting the above polynomial condition into a more efficient form. In the sequel, we precise what is meant by this particular case.

If $k = 1$ in (1.1), then the roots of P are eigenvalues of the only system matrix, namely $-A_1$. Thus, (1.1) with $k = 1$ admits a periodic solution just when $-A_1$ (hence also A_1) has an eigenvalue with a unitary modulus and a rational argument. We can take this classical result as a pattern for our next investigations, and search for periodicity conditions in terms of eigenvalues of the system matrices.

Further, we consider (1.1) with two nonzero system matrices, i.e., the system

$$x(n) + Ax(n-m) + Bx(n-k) = 0 \tag{1.3}$$

with $A, B \in \mathbb{R}^{d\times d}$ and coprime $k, m \in \mathbb{Z}^+$, $k > m$ (note that if $\gcd(k,m) > 1$, then (1.3) can be easily converted to the case with coprime delays). Now, the characteristic polynomial P becomes

$$P^*(\lambda) = \det(\lambda^k I + \lambda^{k-m} A + B).$$

Contrary to the case $k = 1$, the formulation of efficient conditions on eigenvalues of A, B, ensuring that P^* has a root (1.2), seems to be extremely complicated (and perhaps lying beyond theoretical possibilities). Therefore, other methods have been developed to detect a periodic behavior in (1.3). In particular, the paper [10] successfully employed the method of circulant matrices and, for specific choices $A = -I$ and $m = 1$, extended some earlier results (see, e.g., [15,16]) to obtain complete periodicity conditions for the studied system. We recall Theorem 4.1 of [10] as the most relevant result to the topic of our paper.

Theorem 1.1. *The system* (1.3) *with* $A = -I$ *and* $m = 1$ *has a nontrivial periodic solution* x *if and only if at least one of the following conditions holds:*

(a) *B has a zero eigenvalue or has a real eigenvalue $\beta = b$ such that*

$$b \in \left\{ 2(-1)^{\ell} \sin\left(\frac{(2\ell+1)\pi}{2(2k-1)}\right),\ \ell = 0, 1, \dots, k-1 \right\}; \tag{1.4}$$

(b) *B has a pure imaginary eigenvalue $\beta = \mathrm{i}b$ ($b \in \mathbb{R}$) such that*

$$b \in \left\{ 2(-1)^{\ell} \sin\left(\frac{\ell\pi}{2k-1}\right),\ \ell = 1, \dots, 2k-2 \right\}; \tag{1.5}$$

(c) *B has a complex eigenvalue $\beta = b\exp(\mathrm{i}\phi)$ ($b, \phi \in \mathbb{R}$, $|\phi| < \pi/2$) such that, for some positive integers $\xi < \eta$,*

$$|\phi| = \frac{\xi\pi}{2\eta} \quad \textit{and} \quad b \in \left\{ 2(-1)^{\ell} \sin\left(\frac{(\eta-\xi)\pi}{2\eta(2k-1)} + \frac{\ell\pi}{2k-1}\right),\ \ell = 0, 1, \dots, 2k-2 \right\}. \tag{1.6}$$

Following the conclusions of Theorem 1.1, we return to the characteristic polynomial approach. In particular, we show that Theorem 1.1 can be obtained from the root analysis of P^* as well, and moreover, this analysis appears to be applicable also in some more general situations. In fact, it utilizes a straightforward procedure how to simplify the form of P^* under a specific restriction imposed on A, B, namely their simultaneous triangularizability. This property assumes the existence of a regular matrix T such that both $T^{-1}AT$ and $T^{-1}BT$ are upper triangular. The existence of such a matrix T is guaranteed whenever A, B are commuting (for some less restrictive conditions on A, B, see, e.g., [3]). Notice that the matrices $A = -I$ and B considered in Theorem 1.1 obviously share this property. Further, if A, B are simultaneously triangularizable, then, after a few straightforward computational steps, one can decompose P^* into a significantly simpler form, namely

$$P^{**}(\lambda) = \prod_{i=1}^{d} (\lambda^k + \alpha_i \lambda^{k-m} + \beta_i), \tag{1.7}$$

where $(\alpha_i, \beta_i) \in \mathbb{C}^2$, $i = 1, \dots, d$, are simultaneously ordered couples of eigenvalues of A, B (i.e., couples involving mutually corresponding diagonal elements of the matrices $T^{-1}AT$ and $T^{-1}BT$, respectively). Thus, (1.3) with simultaneously triangularizable matrices A, B has a q-periodic solution x if and only if at least one of the trinomials involved in the product (1.7) has a root of the form (1.2).

To summarize the previous considerations, the mathematical core of discussions on the existence of periodic solutions of (1.3) with simultaneously triangularizable A, B consists in analyzing the following problem: *Find conditions under which a complex trinomial*

$$Q(\lambda) = \lambda^k + \alpha\lambda^{k-m} + \beta \tag{1.8}$$

admits a root (1.2).

Based on the previous notes, the paper is organized as follows. In Section 2, we consider (1.3) with simultaneously triangularizable A, B, and formulate a general result on the existence of a q-periodic solution of (1.3), including its form and prime period. Section 3 reveals a geometric background of this result, namely a relationship between the derived periodicity conditions and analytical description of some roulette curves. Two particular cases of (1.3) are investigated in Section 4. Here, we provide a generalization of Theorem 1.1 to (1.3) with $A = -I$ and a general m, and derive fully explicit conditions for detecting its periodic solution. An application of the presented results in control theory is stated as well. The final section, involving survey remarks and possible perspectives of the research, concludes the paper.

2 Basic analytical conditions for the existence of a periodic solution of (1.3)

First, we consider the characteristic polynomial P that is associated with the general linear system (1.1). A simple but efficient method for analyzing the distribution of characteristic roots with respect to the unit circle is the D-partition method (in the discrete case, it is also referred to as the boundary locus technique). This method is standardly employed in stability analysis of (1.1) but can also be applied to reveal periodic behavior.

Indeed, assume that P has a unimodular root $\lambda = \exp(\mathrm{i}\phi)$, $-\pi < \phi \le \pi$. A direct substitution into $P(\lambda) = 0$ yields

$$\det\left(\exp(\mathrm{i}k\phi)I + \sum_{s=1}^{k}\exp(\mathrm{i}(k-s)\phi)A_s\right) = 0. \tag{2.1}$$

In particular, if $\lambda = \lambda_{j,q}$ is given by (1.2), i.e., if we put $\phi = 2\pi j/q$, then (2.1) becomes

$$\det\left(\exp\left(\mathrm{i}\frac{2\pi jk}{q}\right)I + \sum_{s=1}^{k}\exp\left(\mathrm{i}\frac{2\pi j(k-s)}{q}\right)A_s\right) = 0,$$

hence,

$$\det\left(I + \sum_{s=1}^{k}\exp\left(-\mathrm{i}\frac{2\pi js}{q}\right)A_s\right) = 0. \tag{2.2}$$

Thus, (1.1) admits a q-periodic solution if and only if there exists an integer j (specified in (1.2)) such that (2.2) holds. Notice that the same conclusion was formulated in Theorem 3.1 of [10] by the use of tools of circulant matrices (contrary to Theorem 3.1 of [10], now we do not need to assume pairwise commutativity of the system matrices).

Of course, (2.2) is still rather a theoretical condition. The following assertion converts (2.2) into an efficient form if we consider (1.3) instead of (1.1), and assume that A, B are simultaneously triangularizable. In what follows, Arg denotes the principal argument of a complex number (i.e., $-\pi < \mathrm{Arg}(z) \le \pi$ for $z \in \mathbb{C}$), and we introduce a function $(\cdot)_{2\pi} : \mathbb{R} \to (-\pi, \pi]$ as the 2π-periodic extension of the identity function defined on $(-\pi, \pi]$ (i.e., $(\cdot)_{2\pi}$ provides modulo 2π operation).

Lemma 2.1. *Let $A, B \in \mathbb{R}^{d\times d}$ be simultaneously triangularizable matrices, $k, m \in \mathbb{Z}^+$, $k > m$, be coprime and let $q \in \mathbb{Z}^+$. Then (1.3) has a nontrivial q-periodic solution if and only if there exists a couple of simultaneously ordered eigenvalues (α, β) of A, B satisfying, for some integer j such that $\lfloor -q/2\rfloor + 1 \le j \le \lfloor q/2 \rfloor$, any of the following conditions:*

(i) $\alpha = 0, \quad |\beta| = 1, \quad \mathrm{Arg}(\beta) = \left(\pi + \dfrac{2\pi jk}{q}\right)_{2\pi};$

(ii) $\beta = 0, \quad |\alpha| = 1, \quad \mathrm{Arg}(\alpha) = \left(\pi + \dfrac{2\pi jm}{q}\right)_{2\pi};$

(iii) $\alpha\beta \neq 0, \quad \big||\alpha| - |\beta|\big| \leq 1 \leq |\alpha| + |\beta|,$ *and*

$$\begin{aligned} \cos\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) &= \frac{|\beta|^2 - |\alpha|^2 - 1}{2|\alpha|}, \\ \cos\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right) &= \frac{|\alpha|^2 - |\beta|^2 - 1}{2|\beta|}, \end{aligned} \tag{2.3}$$

where $(\operatorname{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$*,* $(\operatorname{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ *have opposite signs if both are nonzero.*

Proof. Because of the reduction of P into P^{**}, we need to describe the situation when the complex trinomial Q (with appropriate eigenvalues α and β of A and B, respectively) has a unimodular root (1.2). First note that Q has a unimodular root $\lambda = \exp(\mathrm{i}\phi)$, $-\pi < \phi \leq \pi$, just when

$$\exp(\mathrm{i}k\phi) + \alpha \exp(\mathrm{i}(k-m)\phi) + \beta = 0 \tag{2.4}$$

(see also (2.1)). Since λ should be a q-th root of unity, we put again $\phi = 2\pi j/q$, and (2.4) becomes

$$\exp\left(\mathrm{i}\frac{2\pi jk}{q}\right) + \alpha \exp\left(\mathrm{i}\frac{2\pi j(k-m)}{q}\right) + \beta = 0. \tag{2.5}$$

This immediately implies the lemma's conditions (i) and (ii).

Now assume $\alpha\beta \neq 0$. Then (2.5) can be rewritten as

$$1 + |\alpha| \exp\left(\mathrm{i}\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right)\right) = -|\beta| \exp\left(\mathrm{i}\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right)\right)$$

which is (by comparing the real and imaginary parts) equivalent to the system

$$\begin{aligned} 1 + |\alpha| \cos\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) &= -|\beta| \cos\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right), \\ |\alpha| \sin\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) &= -|\beta| \sin\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right). \end{aligned} \tag{2.6}$$

Summing the squares of both the lines of (2.6) yields

$$1 + 2|\alpha| \cos\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) + |\alpha|^2 = |\beta|^2,$$

which implies $(2.3)_1$.

Now, substituting $(2.3)_1$ to the left-hand side of $(2.6)_1$, we obtain

$$1 + |\alpha| \frac{|\beta|^2 - |\alpha|^2 - 1}{2|\alpha|} = -|\beta| \cos\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right),$$

which, after a straightforward rearrangement, becomes $(2.3)_2$. Moreover, the couple (α, β) has to also satisfy the stated sign correspondence of the cosine arguments because $(2.6)_2$ implies that $(\operatorname{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ is in $(0, \pi)$ (that is positive) just when $(\operatorname{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ is in $(-\pi, 0)$ (that is negative). $\square$

Remark 2.2. If we fix $\alpha \neq 0$, then (2.3) allows to express β explicitly (in terms of its modulus and argument). Indeed, taking into account the sign relation between $(\mathrm{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ and $(\mathrm{Arg}(\beta) - 2\pi jk/q)_{2\pi}$, we get

$$\begin{aligned} |\beta| &= \sqrt{1 + |\alpha|^2 + 2|\alpha| \cos\left(\mathrm{Arg}(\alpha) - \frac{2\pi jm}{q}\right)}, \\ \mathrm{Arg}(\beta) &= \left(\kappa \arccos\left(\frac{|\alpha|^2 - |\beta|^2 - 1}{2|\beta|}\right) + \frac{2\pi jk}{q}\right)_{2\pi}, \end{aligned} \tag{2.7}$$

where $\kappa = -1$ if $(\mathrm{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ is in $(0, \pi]$, and $\kappa = 1$ if $(\mathrm{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ is in $(-\pi, 0]$. It means that, depending on the integer j, we have (for a given α) at most q possible positions of β in the complex plane such that a q-th root of unity belongs among the roots of Q.

Analogous explicit expressions can be obtained for α when $\beta \neq 0$ is fixed. More precisely, $(2.3)_2$, $(2.3)_1$ imply

$$\begin{aligned} |\alpha| &= \sqrt{1 + |\beta|^2 + 2|\beta| \cos\left(\mathrm{Arg}(\beta) - \frac{2\pi jk}{q}\right)}, \\ \mathrm{Arg}(\alpha) &= \left(\kappa \arccos\left(\frac{|\beta|^2 - |\alpha|^2 - 1}{2|\alpha|}\right) + \frac{2\pi jm}{q}\right)_{2\pi}, \end{aligned} \tag{2.8}$$

where $\kappa = -1$ if $(\mathrm{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ belongs to $(0, \pi]$, and $\kappa = 1$ if $(\mathrm{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ belongs to $(-\pi, 0]$. We add that a nice geometric interpretation of the conditions (2.7) and (2.8) will be discussed in the next section.

To complete the above basic periodic investigations of (1.3), we reveal the structure of the discussed q-periodic solution. As a first step, we state the following root property of P^{**}.

Proposition 2.3. *If $\lambda_{j,q} = \exp(\mathrm{i}2\pi j/q)$ (see (1.2)) is a root of P^{**}, then the complex conjugate number $\overline{\lambda}_{j,q} = \exp(-\mathrm{i}2\pi j/q)$ is also a root of P^{**}. Moreover, if v is a characteristic vector corresponding to $\lambda_{j,q}$, i.e., v solves the system $\big((\lambda_{j,q})^k I + (\lambda_{j,q})^{k-m} A + B\big)v = 0$, then its complex conjugate $\overline{v}$ is a characteristic vector corresponding to $\overline{\lambda}_{j,q}$, i.e., $\overline{v}$ satisfies $\big((\overline{\lambda}_{j,q})^k I + (\overline{\lambda}_{j,q})^{k-m} A + B\big)\overline{v} = 0$.*

Proof. Let $\lambda_{j,q}$ be a root of P^{**}. Since P^{**} and P^* are identically equal, and P^* is a real polynomial, we immediately get that $\overline{\lambda}_{j,q}$ is a root of P^{**} as well.

The complex conjugation of the corresponding characteristic vectors can be shown analogously as in the case of standard eigenvectors. Indeed, let v be a characteristic vector corresponding to $\lambda_{j,q}$, i.e., v satisfies

$$\big((\lambda_{j,q})^k I + (\lambda_{j,q})^{k-m} A + B\big)v = 0.$$

Then, taking the conjugates of both sides, we have

$$\overline{\big((\lambda_{j,q})^k I + (\lambda_{j,q})^{k-m} A + B\big)v} = \overline{0} \quad \Longrightarrow \quad \big((\overline{\lambda}_{j,q})^k I + (\overline{\lambda}_{j,q})^{k-m} A + B\big)\overline{v} = 0.$$

Therefore, $\overline{v}$ is a characteristic vector corresponding to $\overline{\lambda}_{j,q}$. □

Now, using Proposition 2.3, we can easily describe the family of all q-periodic solutions of (1.3) generated by $\lambda_{j,q}$, v, and their conjugates. Obviously, (1.3) has two such (linearly independent) solutions in the form

$$x_1^*(n) = v \exp\left(\mathrm{i}\frac{2\pi jn}{q}\right) \quad \text{and} \quad x_2^*(n) = \overline{v} \exp\left(-\mathrm{i}\frac{2\pi jn}{q}\right).$$

To avoid solutions with complex values, we put

$$x_1(n) = \frac{1}{2}\big(x_1^*(n) + x_2^*(n)\big) \quad \text{and} \quad x_2(n) = \frac{1}{\mathrm{i}2}\big(x_1^*(n) - x_2^*(n)\big)$$

to get a new pair of (linearly independent) real solutions

$$x_1(n) = v_1 \cos\left(\frac{2\pi jn}{q}\right) - v_2 \sin\left(\frac{2\pi jn}{q}\right), \qquad x_2(n) = v_2 \cos\left(\frac{2\pi jn}{q}\right) + v_1 \sin\left(\frac{2\pi jn}{q}\right), \tag{2.9}$$

where $v_1 = \Re(v)$, $v_2 = \Im(v)$. Then, any q-periodic solution x of (1.3) generated by $\lambda_{j,q}$, v, and their conjugates is given by a linear combination $x(n) = C_1x_1(n) + C_2x_2(n)$, $C_1, C_2 \in \mathbb{R}$. It is clear that the corresponding initial vectors

$$x^0(-k+1), x^0(-k+2), \dots, x^0(0)$$

have to match the form (2.9), i.e., we put

$$x_1^0(n) = v_1 \cos\left(\frac{2\pi jn}{q}\right) - v_2 \sin\left(\frac{2\pi jn}{q}\right), \qquad x_2^0(n) = v_2 \cos\left(\frac{2\pi jn}{q}\right) + v_1 \sin\left(\frac{2\pi jn}{q}\right) \tag{2.10}$$

for $n = -k+1, -k+2, \dots, 0$.

Moreover, we can easily decide whether the prescribed period q is also a prime period of the solutions (2.9). Obviously, x_1, x_2 are p-periodic ($p \in \mathbb{Z}^+$) whenever pj/q is an integer. Therefore, such a number p is a prime period if $p = q/\gcd(j,q)$. It means that if j and q are coprime, then q is a prime period.

3 Some geometric observations

In this section, we clarify a geometric background of the conditions (2.7) and (2.8). Doing this, we come back to the condition (2.4) characterizing the set of all complex α, β such that Q has a unimodular root $\lambda = \exp(\mathrm{i}\phi)$, $-\pi < \phi \le \pi$. As observed in [20], the geometric nature of such characterizations can be described in terms of some roulette curves. In the sequel, we elaborate on these issues in detail.

If we fix $\alpha \neq 0$ (as in Remark 2.2), the set of all β satisfying (2.4) is alternatively given by

$$\beta = |\alpha| \exp\big(\mathrm{i}(\pi + \operatorname{Arg}(\alpha) + (k-m)\phi)\big) - \exp(\mathrm{i}k\phi). \tag{3.1}$$

Set $s = \pi + \operatorname{Arg}(\alpha) + (k-m)\phi$ to obtain

$$\beta = |\alpha| \exp(\mathrm{i}s) - \exp\left(\mathrm{i}\frac{k}{k-m}\big(s - (\pi + \operatorname{Arg}(\alpha))\big)\right). \tag{3.2}$$

Further, the substitution $t = s - (\pi + \operatorname{Arg}(\alpha))k/m$ converts (3.2) to the form

$$\begin{aligned}\beta &= |\alpha| \exp\left(\mathrm{i}t + \mathrm{i}\frac{k}{m}\big(\pi + \operatorname{Arg}(\alpha)\big)\right) - \exp\left(\mathrm{i}\frac{k}{k-m}t + \mathrm{i}\frac{k}{m}\big(\pi + \operatorname{Arg}(\alpha)\big)\right) \\ &= \exp\left(\mathrm{i}\frac{k}{m}\big(\pi + \operatorname{Arg}(\alpha)\big)\right)\left(|\alpha| \exp(\mathrm{i}t) - \exp\left(\mathrm{i}\frac{k}{k-m}t\right)\right).\end{aligned} \tag{3.3}$$

Finally, we introduce the parameters $R = |\alpha|m/k$, $r = |\alpha|(k-m)/k$, $\delta = 1$ to rewrite (3.3) as

$$\beta = \exp\left(\mathrm{i}\frac{k}{m}\big(\pi + \operatorname{Arg}(\alpha)\big)\right)\left((R+r)\exp(\mathrm{i}t) - \delta \exp\left(\mathrm{i}\frac{R+r}{r}t\right)\right). \tag{3.4}$$

The second term in (3.4) represents the complex canonical parametrization of an epitrochoid, which is a roulette traced by a point attached to a circle of radius r, having distance δ from its center and rolling around the outside of a fixed circle of radius R. In addition, the appearance of the first term in (3.4) implies that the epitrochoid is rotated by the angle of rotation $\omega = \big(\frac{k}{m}(\pi + \operatorname{Arg}(\alpha))\big)_{2\pi}$. Thus, (3.1) is converted to the parametric equation of a rotated epitrochoid

$$\beta = \exp(\mathrm{i}\omega)\left((R+r)\exp(\mathrm{i}t) - \delta\exp\left(\mathrm{i}\frac{R+r}{r}t\right)\right), \tag{3.5}$$

where $t = s - (\pi + \operatorname{Arg}(\alpha))k/m = (k-m)\phi - \big(\pi + \operatorname{Arg}(\alpha)\big)(k-m)/m$. Note also that an epitrochoid is closed whenever r/R is a rational number. In our case, $r/R = (k-m)/m \in \mathbb{Q}$, and it is not difficult to check that the parametrization (3.5) has the prime period $2\pi(k-m)$.

To summarize, the complex number β satisfies (3.1) just when β is located on the (rotated) epitrochoid (3.5) with

$$R = |\alpha|\frac{m}{k}, \quad r = |\alpha|\frac{k-m}{k}, \quad \delta = 1, \quad \omega = \left(\frac{k}{m}(\pi + \operatorname{Arg}(\alpha))\right)_{2\pi}. \tag{3.6}$$

In particular, Q has a unimodular root (1.2) just when the position of β on this epitrochoid is specified via

$$t = (k-m)\frac{2\pi j}{q} - \big(\pi + \operatorname{Arg}(\alpha)\big)\frac{k-m}{m} \quad \text{for some } j = \left\lfloor -\frac{q}{2}\right\rfloor + 1, \left\lfloor -\frac{q}{2}\right\rfloor + 2, \dots, \left\lfloor \frac{q}{2}\right\rfloor. \tag{3.7}$$

Similarly, if we fix $\beta \neq 0$, then the set of all α satisfying (2.4) is given by

$$\alpha = |\beta|\exp\big(\mathrm{i}(\pi + \operatorname{Arg}(\beta) + (m-k)\phi)\big) - \exp(\mathrm{i}m\phi), \qquad -\pi < \phi \le \pi.$$

Putting $s = \pi + \operatorname{Arg}(\beta) + (m-k)\phi$, we obtain

$$\alpha = |\beta|\exp(\mathrm{i}s) - \exp\left(\mathrm{i}\frac{m}{m-k}\big(s - (\pi + \operatorname{Arg}(\beta))\big)\right). \tag{3.8}$$

Now, let

$$t = \frac{m}{m-k}\Big(s - \big(\pi + \operatorname{Arg}(\beta)\big)\Big) - \frac{m}{k}\operatorname{Arg}(\beta).$$

Then, (3.8) becomes

$$\begin{aligned}\alpha &= |\beta|\exp\left(\mathrm{i}\frac{m-k}{k}t + \mathrm{i}\Big(\pi + \frac{m}{k}\operatorname{Arg}(\beta)\Big)\right) - \exp\left(\mathrm{i}t + \mathrm{i}\frac{m}{k}\operatorname{Arg}(\beta)\right)\\ &= \exp\left(\mathrm{i}\Big(\pi + \frac{m}{k}\operatorname{Arg}(\beta)\Big)\right)\left(\exp(\mathrm{i}t) + |\beta|\exp\left(\mathrm{i}\frac{m-k}{m}t\right)\right).\end{aligned}$$

If we introduce $R = k/(k-m)$, $r = m/(k-m)$, $\delta = |\beta|$, and $\omega = (\pi + \operatorname{Arg}(\beta)m/k)_{2\pi}$, then

$$\alpha = \exp(\mathrm{i}\omega)\left((R-r)\exp(\mathrm{i}t) + \delta\exp\left(\mathrm{i}\frac{r-R}{r}t\right)\right), \tag{3.9}$$

where $t = \big(s - (\pi + \operatorname{Arg}(\beta))\big)m/(m-k) - \operatorname{Arg}(\beta)m/k = m\phi - \operatorname{Arg}(\beta)m/k$. Notice that the second term in (3.9) represents another roulette curve – hypotrochoid, which is a roulette traced by a point attached to a circle of radius r, having distance δ from its center, and rolling around the inside of a fixed circle of radius R ($R > r$). The first term in (3.9) again provides a rotation of the curve by the angle ω. Since the ratio $r/R = m/k$ is a rational number, the parametrization (3.9) is periodic and its prime period is $2\pi m$ (i.e., (3.9) represents a closed curve).

Overall, analogously as above, one can conclude that Q has a unimodular root (1.2) just when α is located on the hypotrochoid (3.9) with

$$R = \frac{k}{k-m}, \quad r = \frac{m}{k-m}, \quad \delta = |\beta|, \quad \omega = \left(\pi + \frac{m}{k}\operatorname{Arg}(\beta)\right)_{2\pi}, \tag{3.10}$$

where

$$t = m\frac{2\pi j}{q} - \frac{m}{k}\operatorname{Arg}(\beta) \quad \text{for some } j = \left\lfloor -\frac{q}{2} \right\rfloor + 1, \left\lfloor -\frac{q}{2} \right\rfloor + 2, \dots, \left\lfloor \frac{q}{2} \right\rfloor. \tag{3.11}$$

Thus, Lemma 2.1 and Remark 2.2, supported by the previous considerations, yield the following analytical and geometric characterization of the existence of a periodic solution of (1.3).

Theorem 3.1. *Let $A, B \in \mathbb{R}^{d\times d}$ be simultaneously triangularizable matrices, $k, m \in \mathbb{Z}^+$, $k > m$, be coprime and let $q \in \mathbb{Z}^+$. Then the following statements are equivalent.*

(i) *The system (1.3) has a nontrivial q-periodic solution.*

(ii) *Either* (iia) *or* (iib) *holds, where:*

  (iia) *There exists a zero eigenvalue α of A such that the corresponding eigenvalue β of B (in the sense of simultaneous ordering) is lying on a unit circle centered at the origin and the admissible values of arguments of β are specified via the conditions (i) of Lemma 2.1.*

  (iib) *There exists a nonzero eigenvalue α of A such that the corresponding eigenvalue β of B (in the sense of simultaneous ordering) is lying on the rotated epitrochoid (3.5) with R, r, δ, ω given by (3.6); moreover, all the admissible values of the parameter t are specified via (3.7). Alternatively, all such positions of β on this epitrochoid are described by (2.7).*

(iii) *Either* (iiia) *or* (iiib) *holds, where:*

  (iiia) *There exists a zero eigenvalue β of B such that the corresponding eigenvalue α of A (in the sense of simultaneous ordering) is lying on a unit circle centered at the origin and the admissible values of arguments of α are specified via the conditions (ii) of Lemma 2.1.*

  (iiib) *There exists a nonzero eigenvalue β of B such that the corresponding eigenvalue α of A (in the sense of simultaneous ordering) is lying on the rotated hypotrochoid (3.9) with R, r, δ, ω given by (3.10); moreover, all the admissible values of the parameter t are specified via (3.11). Alternatively, all such positions of α on this hypotrochoid are described by (2.8).*

Example 3.2. Consider the (planar) system (1.3) with $k = 5, m = 3$ and A, B given by

$$A = \frac{\sqrt{3}}{3}\begin{pmatrix} 8 & 9 \\ -6 & -7 \end{pmatrix}, \qquad B = \frac{\sqrt{3}}{3}\begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}.$$

It is easy to check that A and B are commuting (therefore simultaneously triangularizable), and the pairs of their simultaneously ordered eigenvalues are

$$(\alpha_1, \beta_1) = \left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right), \qquad (\alpha_2, \beta_2) = \left(\frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right).$$

Considering the first pair (α_1, β_1), we can check that the corresponding trinomial

$$\lambda^5 - \frac{\sqrt{3}}{3}\lambda^2 + \frac{2\sqrt{3}}{3},$$

appearing in the decomposition (1.7), has a unique unimodular root. More precisely, $\beta_1 = 2\sqrt{3}/3$ is located on the epitrochoid (3.5) with $R = \sqrt{3}/5$, $r = 2\sqrt{3}/15$, $\delta = 1$ and $\omega = -2\pi/3$. Moreover, its position corresponds to $t = 4\pi j/q - 4\pi/3$, where $j = 1$ and $q = 12$, see Figure 3.1 (left). Equivalently, $\alpha_1 = -\sqrt{3}/3$ is located on the hypotrochoid (3.9) with $R = 5/2$, $r = 3/2$, $\delta = 2\sqrt{3}/3$, $\omega = \pi$, and its position corresponds to $t = 6\pi j/q$, where $j = 1$ and $q = 12$, see Figure 3.1 (right). Thus, the eigenvalues β_1, α_1 obey (2.7), (2.8) with $j = 1$ and $q = 12$.

Considering the second pair (α_2, β_2), one can see that the corresponding trinomial

$$\lambda^5 + \frac{2\sqrt{3}}{3}\lambda^2 + \frac{\sqrt{3}}{3},$$

appearing in (1.7), has no unimodular root, hence β_2, α_2 do not meet (2.7), (2.8) for any integers j and q.

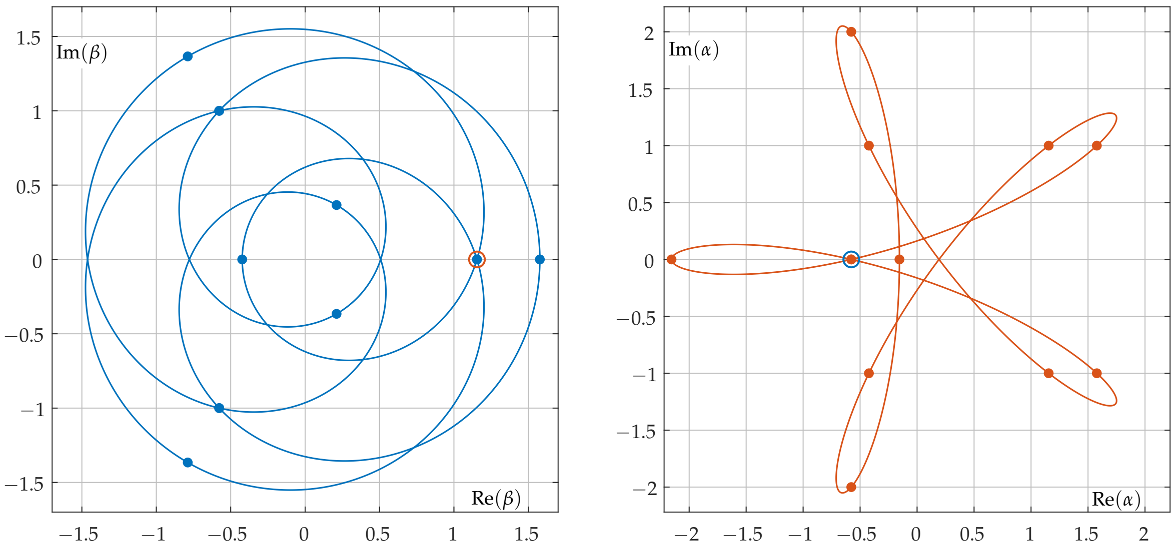

Figure 3.1: An epitrochoid (left) and a hypotrochoid (right) corresponding to the data of Example 3.2. The blue points in the left figure depict all the 12 possible positions of numbers β which can be an eigenvalue of B in a simultaneously ordered pair with α_1 of A (in fact, we have 9 positions only as one triple coincides with another one). Emphasized is the eigenvalue β_1 of B. The orange points in the right figure depict all the 12 positions of numbers α which can be an eigenvalue of A in a simultaneously ordered pair with β_1 of B (in fact, we have 11 positions only as one couple coincides). Emphasized is the eigenvalue α_1 of A.

By Theorem 3.1, the system has just a 12-periodic solution (and 12 is a prime period as $j = 1$ and $q = 12$ are coprime). Moreover, since it holds

$$(\lambda_{1,12})^k I + (\lambda_{1,12})^{k-m} A + B = \begin{pmatrix} \frac{\sqrt{3}}{2} + \mathrm{i}\frac{9}{2} & \frac{\sqrt{3}}{2} + \mathrm{i}\frac{9}{2} \\ -\frac{\sqrt{3}}{3} - \mathrm{i}3 & -\frac{\sqrt{3}}{3} - \mathrm{i}3 \end{pmatrix},$$

where $\lambda_{1,12} = \exp(\mathrm{i}2\pi/12)$, the characteristic vector corresponding to $\lambda_{1,12}$ can be taken simply as $v = (1, -1)^T$ (i.e., $v_1 = \Re(v) = (1, -1)^T$, $v_2 = \Im(v) = (0, 0)^T$). Thus, using the considerations performed at the end of Section 2, all (12-periodic) solutions of the studied system are of the form

$$x(n) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\frac{\pi n}{6}\right) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\left(\frac{\pi n}{6}\right), \qquad C_1, C_2 \in \mathbb{R}$$

(five initial vectors $x^0(-4), x^0(-3), x^0(-2), x^0(-1), x^0(0)$ have to be prescribed accordingly). No other periodic solution can be detected in this system. Figure 3.2 depicts both components of the periodic solution x with $C_1 = 1$ and $C_2 = 2$.

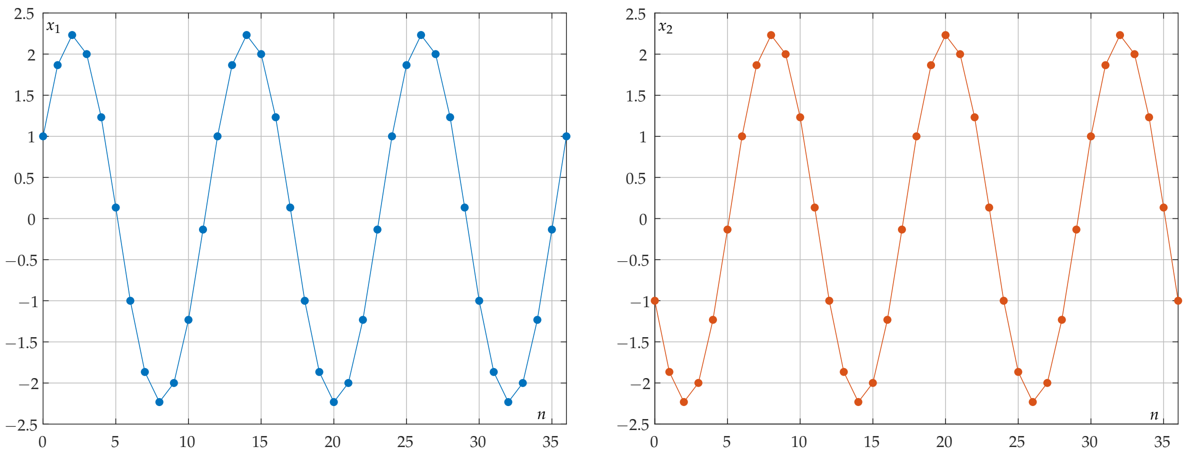

Figure 3.2: The first (left) and the second (right) component of the 12-periodic solution with respect to the data of Example 3.2.

4 Periodicity problem in two particular cases of (1.3)

In this section, we elaborate on the conclusions of Theorem 3.1 under specific choices of A (while B remains free). First, we consider the difference system

$$x(n) - x(n-m) + Bx(n-k) = 0, \tag{4.1}$$

and generalize the conditions (1.4)–(1.6) of Theorem 1.1 for arbitrary coprime integers k, m, $k > m$. Moreover, we give a new insight into a geometric structure of these conditions.

Theorem 4.1. *Let $B \in \mathbb{R}^{d\times d}$, and let k, $m \in \mathbb{Z}^+$, $k > m$, be coprime. Then the following statements are equivalent.*

(i) *The system (4.1) has a nontrivial periodic solution.*

(ii) *There exists an eigenvalue β of B lying on the epitrochoid (3.5), where $R = m/k$, $r = (k-m)/k$, $\delta = 1$, $\omega = (2\pi k/m)_{2\pi}$, and β corresponds to $t = 2\pi(k-m)(jm-q)/(qm)$ for some $q \in \mathbb{Z}^+$ and some $j = \lfloor -q/2 \rfloor + 1, \lfloor -q/2 \rfloor + 2, \dots, \lfloor q/2 \rfloor$.*

(iii) *B has a zero eigenvalue or has an eigenvalue $\beta = b\exp(\mathrm{i}\phi)$ such that, for some $\xi \in \mathbb{Z}_0^+, \eta \in \mathbb{Z}^+$, $\xi \le \eta$, we have*

$$|\phi| = \frac{\pi\xi}{2\eta} \qquad \text{and} \qquad b = 2(-1)^{\ell} \sin\left(\frac{\pi(\eta-\xi)m}{2\eta(2k-m)} + \frac{\pi\ell m}{2k-m}\right),$$

where ℓ is any integer satisfying

$$-k+\frac{m}{2}-\frac{1}{2}\left(1-\frac{\xi}{\eta}\right)\leq \ell \leq k-\frac{m}{2}-\frac{1}{2}\left(1-\frac{\xi}{\eta}\right),$$
$$\ell \neq \left(\frac{2k}{m}-1\right)s-\frac{1}{2}\left(1-\frac{\xi}{\eta}\right) \qquad \textit{for all integers } s \textit{ such that } \; -\left\lfloor\frac{m}{2}\right\rfloor \leq s \leq \left\lfloor\frac{m}{2}\right\rfloor. \tag{4.2}$$

Proof. If we put $A=-I$, then all the eigenvalues α_i of A are equal to -1 (i.e., $|\alpha_i|=1$, $\operatorname{Arg}(\alpha_i)=\pi$), and the equivalence between (i) and (ii) follows immediately from Theorem 3.1, the branches (i) and (iib). We show the equivalence between (i) and (iii).

If $\beta=0$ is an eigenvalue of B, then the corresponding trinomial (1.8) reduces to $Q(\lambda)=\lambda^k-\lambda^{k-m}$. Such a trinomial has a root $\lambda=1$ which is a q-th root of unity (for any $q\in\mathbb{Z}^+$), and the existence of a nontrivial periodic (constant) solution is straightforward.

Now, let $\beta=b\exp(\mathrm{i}\phi)$, $b\neq 0$, $|\phi|\leq\pi/2$, be an eigenvalue of B. The existence of a periodic solution is guaranteed by Theorem 3.1 (branches (i) and (iib) – its alternative supplement). Hence, let us find all the possible values of b and ϕ such that the conditions (2.7) are satisfied for some $q\in\mathbb{Z}^+$ and some $j\in\mathbb{Z}$ such that $\lfloor -q/2\rfloor+1\leq j\leq\lfloor q/2\rfloor$. Since $\alpha_i=-1$, then, taking into account the identity $2\cos^2(t)=1+\cos(2t)$, (2.7) simplifies to

$$\begin{aligned}|\beta|=|b|&=2\left|\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right)\right|,\\ \operatorname{Arg}(\beta)&=\left(\kappa\arccos\left(-\frac{|b|}{2}\right)+\frac{2\pi jk}{q}\right)_{2\pi}.\end{aligned} \tag{4.3}$$

If j is such that $\cos(\pi/2-\pi jm/q)>0$, then we can write $(4.3)_1$ as

$$\frac{|b|}{2}=\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right) \tag{4.4}$$

and $(4.3)_2$, with help of the identity $\arccos(-t)=\pi-\arccos(t)$, becomes

$$\operatorname{Arg}(\beta)=-\pi+\arccos\left(\frac{|b|}{2}\right)+\frac{2\pi jk}{q}+2\pi p_j=-\frac{\pi}{2}+(2k-m)\frac{\pi j}{q}+2\pi p_j \tag{4.5}$$

for a suitable $p_j\in\mathbb{Z}$. On the other hand, if j is such that $\cos(\pi/2-\pi jm/q)<0$, then

$$-\frac{|b|}{2}=\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right) \tag{4.6}$$

and $(4.3)_2$ becomes

$$\operatorname{Arg}(\beta)=\frac{\pi}{2}+(2k-m)\frac{\pi j}{q}+2\pi p_j \tag{4.7}$$

for a suitable $p_j\in\mathbb{Z}$. Clearly, in both the cases, $\operatorname{Arg}(\beta)$ has to be a rational multiple of π, therefore, let us write $|\phi|=\frac{\pi\xi}{2\eta}$ where ξ, η are integers such that $\eta>0$ and $0\leq\xi\leq\eta$. We distinguish four cases according to the sign of b and ϕ.

Let $b>0$, $\phi>0$ and $\cos(\pi/2-\pi jm/q)>0$. Then $\operatorname{Arg}(\beta)=\phi=\frac{\pi\xi}{2\eta}$ and (4.5), (4.4) yield

$$\frac{j}{q}=\frac{\xi+\eta-4\eta p_j}{2\eta(2k-m)}, \tag{4.8}$$

$$b=2\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right)=-2\sin\left(-\frac{\pi jm}{q}\right). \tag{4.9}$$

Substituting (4.8) into (4.9), we have

$$b = 2(-1)^{\ell_j} \sin\left(\frac{\pi(\eta-\xi)m}{2\eta(2k-m)} + \frac{\pi \ell_j m}{2k-m}\right), \tag{4.10}$$

where $\ell_j = 2p_j - 1$ is an odd integer.

Similarly, if $\cos(\pi/2 - \pi jm/q) < 0$, with help of (4.7) and (4.6) we get (4.10), where $\ell_j = 2p_j$ is now an even integer.

A discussion of the remaining cases ($b > 0$ and $\phi \le 0$, $b < 0$ and $\phi > 0$, $b < 0$ and $\phi \le 0$) is quite analogous, leading to the identical description (4.10) of the values b independently of the $\cos(\pi/2 - \pi jm/q)$ sign (both odd and even integers ℓ_j are always involved).

It remains to determine the range of the integers ℓ_j, which follows easily from the fact that $-\lfloor q/2 \rfloor < j \le \lfloor q/2 \rfloor$. Since q can be chosen as $q = 2\eta(2k-m)$, see (4.8), we actually have $\lfloor q/2 \rfloor = q/2$. Thus,

$$\left|\frac{\pi jm}{q}\right| \le \frac{\pi m}{2},$$

which implies

$$-k + \frac{m}{2} - \frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) \le \ell_j \le k - \frac{m}{2} - \frac{1}{2}\left(1 - \frac{\xi}{\eta}\right).$$

Finally, the condition (4.2) excludes those integers ℓ for which the argument of sine function in (4.10) is an integer multiple of π. Therefore, the case of $b = 0$ does not occur. □

Now, we put $m = 1$ and formulate a consequence of Theorem 4.1 for the case studied in [10].

Corollary 4.2. *Let $B \in \mathbb{R}^{d\times d}$, and let $k \in \mathbb{Z}^+$. Then the following statements are equivalent.*

(i) *The system (4.1) with $m = 1$ has a nontrivial periodic solution.*

(ii) *There exists an eigenvalue β of B lying on the epitrochoid (3.5), where $R = 1/k$, $r = (k-1)/k$, $\delta = 1$, $\omega = 0$, and $t = 2\pi(k-1)(j-q)/q$ for some $q \in \mathbb{Z}^+$ and some $j = \lfloor -q/2 \rfloor + 1, \lfloor -q/2 \rfloor + 2, \dots, \lfloor q/2 \rfloor$.*

(iii) *B has a zero eigenvalue or has an eigenvalue $\beta = b\exp(\mathrm{i}\phi)$ such that, for some $\xi \in \mathbb{Z}_0^+, \eta \in \mathbb{Z}^+$, $\xi \le \eta$, and $\ell \in \mathbb{Z}_0^+$, we have*

$$|\phi| = \frac{\pi\xi}{2\eta}, \quad b = 2(-1)^{\ell} \sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell}{2k-1}\right), \quad -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) < \ell \le 2k-2. \tag{4.11}$$

Proof. The stated equivalencies follow directly from Theorem 4.1 considering $m = 1$. Indeed, while the first two statements are straightforward, the conditions (iii) of Theorem 4.1 simplify to

$$|\phi| = \frac{\pi\xi}{2\eta} \quad \text{and} \quad b = 2(-1)^{\ell} \sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell}{2k-1}\right),$$

where ℓ is an integer such that

$$-k + \frac{\xi}{2\eta} \le \ell < -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) \quad \text{or} \quad -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) < \ell \le k - 1 + \frac{\xi}{2\eta}. \tag{4.12}$$

Now, to adjust the bounds from $(4.12)_1$, we can write

$$b = 2(-1)^{\ell}\sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell}{2k-1}\right) = 2(-1)^{\ell+1}\sin\left(\pi + \frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell}{2k-1}\right)$$
$$= 2(-1)^{\ell'}\sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell'}{2k-1}\right), \quad \text{where} \quad \ell' = \ell + 2k - 1.$$

Thus,

$$-k + \frac{\xi}{2\eta} \le \ell < -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) \quad \text{if and only if} \quad k - 1 + \frac{\xi}{2\eta} \le \ell' < 2k - \frac{1}{2}\left(3 - \frac{\xi}{\eta}\right).$$

Since $2 \le 3 - \frac{\xi}{\eta} \le 3$, the last inequality can be written as $\ell' \le 2k - 2$. □

Remark 4.3. To compare the conclusions of Corollary 4.2 and Theorem 1.1 (Theorem 4.1 of [10]), one can observe that the equivalency between (i) and (iii) of Corollary 4.2 actually agrees with the statement of Theorem 1.1. More precisely, when $\xi = 0$, we obtain $|\phi| = 0$ (i.e., $\beta = b$ is a real eigenvalue) and (4.11) reduces to

$$b = 2(-1)^{\ell}\sin\left(\frac{\pi(2\ell+1)}{2(2k-1)}\right), \qquad \ell = 0, 1, \dots, 2k-2.$$

It is not difficult to check that the values of b corresponding to $\ell = k, k+1, \dots, 2k-2$ are already contained in the set of all b corresponding to $\ell = 0, 1, \dots, k-1$, hence we get (1.4). Further, if $\xi = \eta$, then the conditions (4.11) directly imply (1.5), and if $0 < \xi < \eta$, then the conditions (4.11) agree with (1.6).

In addition to the assertion of Theorem 1.1, the part (ii) of Corollary 4.2 reveals a geometric nature of the condition (iii).

As the second illustration of Theorem 3.1, we consider a specific control problem for the linear autonomous difference system

$$x(n) + Ax(n-1) = 0. \tag{4.13}$$

We again restrict to the planar case and assume that A has the Jordan form

$$A = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}, \qquad a_1, a_2 \in \mathbb{R} \tag{4.14}$$

with the spectral norm $\|A\| = \sqrt{a_1^2 + a_2^2} \neq 1$. Thus, the trivial equilibrium of this system is a focus (attractive if $\|A\| < 1$ and repelling if $\|A\| > 1$). Obviously, such a system does not admit any (nontrivial) periodic solution.

Now we extend this system by a control term $u(n) = Bx(n)$, acting with an integer delay k, to obtain a delay feedback controlled system

$$x(n) + Ax(n-1) + Bx(n-k) = 0.$$

We put $B = \rho\Phi$ with a real scalar $\rho > 0$ and a rotation matrix

$$\Phi = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, \qquad -\pi < \vartheta \le \pi.$$

Our task is to describe the set of all control parameters ρ, ϑ and k such that the controlled system

$$x(n) + \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} x(n-1) + \rho \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} x(n-k) = 0 \tag{4.15}$$

admits a (nontrivial) q-periodic solution, a positive integer q being prescribed.

From the mathematical side, we analyze the existence of a q-periodic solution of (1.3) with the above specified A, B and $m = 1$. Obviously, the matrices A, B are commuting, and the simultaneous ordering of their eigenvalues is $(a_1 + \mathrm{i}a_2, \rho\exp(\mathrm{i}\vartheta)), (a_1 - \mathrm{i}a_2, \rho\exp(-\mathrm{i}\vartheta))$. Thus, if (ρ, ϑ) is a control couple, then $(\rho, -\vartheta)$ is also a control couple, and we can restrict to one sign variant only, say $(a_1 + \mathrm{i}a_2, \rho\exp(\mathrm{i}\vartheta))$.

Now, Theorem 3.1 (see also Remark 2.2) yields that the controlled system (4.15) has a q-periodic solution if and only if the control parameters ρ, ϑ, k meet, for an integer j such that $\lfloor -q/2 \rfloor + 1 \le j \le \lfloor q/2 \rfloor$, the conditions

$$\begin{aligned} \rho^2 &= 1 + \|A\|^2 + 2\|A\| \cos\left(\operatorname{Arg}(a_1 + \mathrm{i}a_2) - \frac{2\pi j}{q}\right), \\ \vartheta &= \left(\kappa \arccos\left(\frac{\|A\|^2 - \rho^2 - 1}{2\rho}\right) + \frac{2\pi jk}{q}\right)_{2\pi} \end{aligned} \tag{4.16}$$

where

$$\kappa = -1 \quad \text{if} \quad 0 < \left(\operatorname{Arg}(a_1 + \mathrm{i}a_2) - \frac{2\pi j}{q}\right)_{2\pi} \le \pi$$

and

$$\kappa = 1 \quad \text{if} \quad -\pi < \left(\operatorname{Arg}(a_1 + \mathrm{i}a_2) - \frac{2\pi j}{q}\right)_{2\pi} \le 0.$$

In particular, if $k = q$, then we directly have

$$\vartheta = \kappa \arccos\left(\frac{\|A\|^2 - \rho^2 - 1}{2\rho}\right).$$

Alternatively, Theorem 3.1 implies that such a control couple is lying on the epitrochoid (3.5) with $R = \|A\|/k$, $r = \|A\|(k-1)/k$, $\delta = 1$, $\omega = \left(k\pi + k\operatorname{Arg}(a_1 + \mathrm{i}a_2)\right)_{2\pi}$ and $t = (k-1)(2\pi j/q - \pi - \operatorname{Arg}(a_1 + \mathrm{i}a_2))$ for some $j = \lfloor -q/2 \rfloor + 1, \lfloor -q/2 \rfloor + 2, \dots, \lfloor q/2 \rfloor$.

Example 4.4. Consider the system (4.13) with A given by (4.14) and put $a_1 = -0.4$, $a_2 = -1$. The zero equilibrium of such a system is an unstable focus, therefore $\limsup_{n\to\infty} \|x(n)\| = \infty$ for any nontrivial solution x of (4.13), see Figure 4.1 (left). We wish to find control parameters ρ, ϑ, k generating a solution x of (4.15) with the period $q = 22$.

If we put $k = q = 22$ and $j = 3$, then (4.16) yields the control parameters $\rho \approx 0.353005$ and $\vartheta \approx 1.520652$. Under this choice of ρ, ϑ, k and j, the 22-nd root of unity $\lambda_{3,22} = \exp(\mathrm{i}6\pi/22)$ is also a root of the characteristic polynomial associated with the controlled system (4.15), hence, this system admits a 22-periodic solution ($q = 22$ is even a prime period since $\gcd(3, 22) = 1$). In view of the notes at the end of Section 2, any such periodic solution is a linear combination

$$x(n) = C_1 x_1(n) + C_2 x_2(n), \qquad C_1, C_2 \in \mathbb{R},$$

where x_1, x_2 are given by (2.9) with $q = 22$, $j = 3$, $v_1 = (1, 0)^T$, $v_2 = (0, 1)^T$, and the initial vectors are prescribed according to (2.10). Figure 4.1 (right) depicts a particular trajectory for $C_1 = -1$, $C_2 = 1$ (such a choice of C_1, C_2 implies $x_1(0) = -1$, $x_2(0) = 1$).

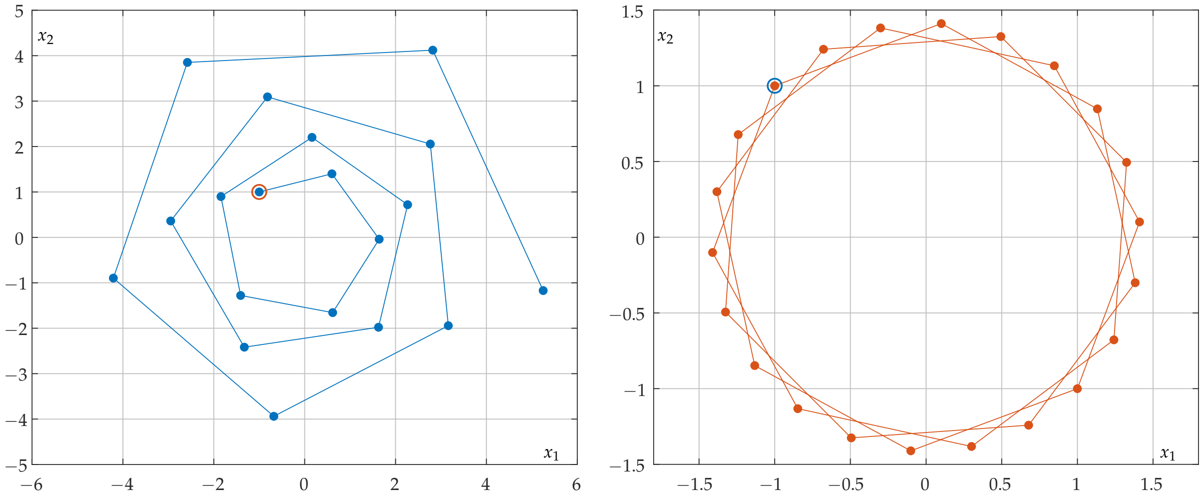

Figure 4.1: The left figure depicts a trajectory of the (uncontrolled) system (4.13) in the (x_1, x_2)-plane for the data of Example 4.4 and the initial vector $(-1, 1)^T$. The right figure then depicts a periodic trajectory (oriented clockwise) of the controlled system (4.15) (note that the prime period is $22/\gcd(3, 22) = 22$).

5 Final remarks

In this paper, we focused on detecting a periodic behavior in the linear difference system (1.3) with simultaneously triangularizable matrices A, B. Based on the analysis of the characteristic polynomial, we formulated efficient conditions for the existence of a periodic solution of (1.3). These conditions are given in terms of eigenvalues of A, B, and supported by its geometric characterization, calculating the revealed periodic solutions and their prime period.

Extending the periodicity problem from this paper to the general case of (1.3) (i.e., without the assumption on simultaneous triangularizability of A, B), or even to the system (1.1), is probably very complicated. Nice periodicity results were obtained for a scalar version of (1.3) in [11]. Similarly as in [10], the authors utilized the theory of circulant matrices in their investigations (note that those results seem to be achievable also by the D-partition technique applied in this paper). However, in a vector case, the formulation of such periodicity results in terms of eigenvalues of the system matrices is a considerably more challenging task.

The difficulty of the problem is underlined by the fact that similar problems for (1.3) and (1.1) remain open also in the related stability area. In this connection, we refer to the interesting paper [13] suggesting a method for detecting periodic solutions in some linear difference equations without knowledge of their characteristic roots. Of course, efficiency of the method for more general cases is questionable.

Taking into account recent developments on locating the trinomial roots (see, e.g., [1, 4, 5, 20]), we believe that the approach based on a suitable decomposition of the characteristic polynomial and the following analysis of the corresponding sparse polynomials has a promising potential in periodicity (as well as stability) theory of autonomous difference equations.

Acknowledgements

The research was supported by the project FSI-S-23-8161 of the Faculty of Mechanical Engineering, Brno University of Technology.

References

[1] W. Barrera, J. C. Magaña, J. P. Navarrete, Trinomials, torus knots and chains, *Trans. Amer. Math. Soc.* **376**(2023), No. 4, 2963–3004. https://doi.org/10.1090/tran/8834; MR4557887; Zbl 1520.30008

[2] L. Berg, Nonlinear difference equations with periodic solutions, *Rostock. Math. Kolloq.* **61**(2006), 13–20. MR2320473; Zbl 1145.39302

[3] G. Bourgeois, Pairs of matrices, one of which commutes with their commutator, *Electron. J. Linear Algebra* **22**(2011), 593–597. https://doi.org/10.13001/1081-3810.1460; MR2817799; Zbl 1223.15022

[4] J. Čermák, L. Fedorková, J. Jánský, On moduli and arguments of roots of complex trinomials, *Pacific J. Math.* **332**(2024), No. 1, 39–67. https://doi.org/10.2140/pjm.2024.332.39; MR4831741; Zbl 07949965

[5] J. Čermák, L. Fedorková, M. Kureš, Complete classification scheme for the distribution of trinomial zeros with respect to their moduli, *Publ. Math. Debrecen* **101**(2022), No. 1–2, 119–146. https://doi.org/10.5486/PMD.2022.9209; MR4475510; Zbl 1513.26019

[6] L. A. V. Carvalho, L. A. C. Ladeira, Difference equations: Detecting oscillations via higher order characteristic equations, *J. Difference Equ. Appl.* **8**(2002), No. 3, 277–291. https://doi.org/10.1080/10236190211953; MR1887907; Zbl 0996.39009

[7] K. L. Cooke, L. A. C. Ladeira, Applying Carvalho's method to find periodic solutions of difference equations, *J. Differ. Equations Appl.* **2**(1996), No. 2, 105–115. https://doi.org/10.1080/10236199608808047; MR1384562; Zbl 0856.39010

[8] T. Furumochi, M. Muraoka, Periodic solutions of periodic difference equations, in: *Advances in Discrete Dynamical Systems (Proceedings of the 11th International Conference on Difference Equations and Applications (ICDEA 06), Kyoto, Japan, July 24–28, 2006), Adv. Stud. Pure Math.* **53**(2009), 51–57. MR2582404; Zbl 1182.39013

[9] E. A. Groove, G. Ladas, *Periodicities in nonlinear difference equations*, Advances in Discrete Mathematics and Applications, Vol. 4, Chapman & Hall/CRC, Boca Raton, FL, 2005. MR2193366; Zbl 1078.39009

[10] I. Győri, L. Horváth, Existence of periodic solutions in a linear higher order system of difference equations, *Comput. Math. Appl.* **66**(2013), No. 11, 2239–2250. https://doi.org/10.1016/j.camwa.2013.06.018; MR3125370; Zbl 1350.39007

[11] I. Győri, L. Horváth, Utilization of circulant matrix theory in periodic autonomous difference equations, *Int. J. Difference Equ.* **9**(2014), No. 2, 163–185. MR3352981

[12] B. B. Kennedy, Symmetric periodic solutions for a class of differential delay equations with distributed delay, *Electron. J. Qual. Theory Differ. Equ.* **2014**(2014), No. 4, 1–18. https://doi.org/10.14232/ejqtde.2014.1.4; MR3183602; Zbl 1324.34135

[13] B. B. Kennedy, Finding periodic solutions without finding eigenvalues, *Elem. Math.* **73**(2018), No. 1, 1–14. https://doi.org/10.4171/EM/344; MR3746487; Zbl 1390.39004

[14] L. A. C. Ladeira, S. M. Tanaka, A method to calculate periodic solutions of functional differential equations, *J. Math. Anal. Appl.* **209**(1997), No. 1, 1–19. https://doi.org/10.1006/jmaa.1997.5176; MR1444507; Zbl 0877.34050

[15] H. Matsunaga, Exact stability criteria for delay differential and difference equations, *Appl. Math. Lett.* **20**(2007), No. 2, 183–188. https://doi.org/10.1016/j.aml.2006.03.012; MR2283908; Zbl 1114.34057

[16] H. Matsunaga, T. Hara, The asymptotic stability of a two-dimensional linear delay difference equation, *Dynam. Contin. Discrete Impuls. Systems* **6**(1999), No. 4, 465–473. MR1723884; Zbl 0952.39004

[17] Y. Nakata, Existence of a period two solution of a delay differential equation, *Discrete Contin. Dyn. Syst. Ser. S* **14**(2021), No. 3, 1103–1110. https://doi.org/10.3934/dcdss.2020392; MR4219100; Zbl 1460.34082

[18] R. D. Nussbaum, Periodic solutions of special differential delay equations: an example in non-linear functional analysis, *Proc. Roy. Soc. Edinburgh Sect. A* **81**(1978), No. 1–2, 131–151. https://doi.org/10.1017/S0308210500010490; MR0529382; Zbl 0402.34061

[19] S. Stević, A note on periodic character of a higher order difference equation, *Rostock. Math. Kolloq.* **61**(2006), 21–30. MR2320474; Zbl 1151.39012

[20] T. Theobald, T. de Wolff, Norms of roots of trinomials, *Math. Ann.* **366**(2016), No. 1–2, 219–247. https://doi.org/10.1007/s00208-015-1323-8; MR3552238; Zbl 1379.12002

[21] T. Yi, Z. Zhou, Periodic solutions of difference equation, *J. Math. Anal. Appl.* **286**(2003), No. 1, 220–229. https://doi.org/10.1016/S0022-247X(03)00473-6; MR2009631; Zbl 1033.39018

[22] J. Yu, A note on periodic solutions of the delay differential equation $x'(t) = -f(x(t-1))$, *Proc. Amer. Math. Soc.* **141**(2013), No. 4, 1281–1288. https://doi.org/10.1090/S0002-9939-2012-11386-3; MR3008875; Zbl 1271.34070