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Razumikhin-Type Stability Criteria for Differential Equations with Delayed Impulses

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ABSTRACT

This paper studies stability problems of general impulsive differential equations where time delays occur in both differential and difference equations. Based on the method of Lyapunov functions, Razumikhin technique and mathematical induction, several stability criteria are obtained for differential equations with delayed impulses. Our results show that some systems with delayed impulses may be exponentially stabilized by impulses even if the system matrices are unstable. Some less restrictive sufficient conditions are also given to keep the good stability property of systems subject to certain type of impulsive perturbations. Examples with numerical simulations are discussed to illustrate the theorems. Our results may be applied to complex problems where impulses depend on both current and past states.

Keywords: Systems with delayed impulses; Lyapunov - Razumikhin method; Global exponential stability; Impulsive stabilization

1 Introduction

During the last decades, the stability theory of impulsive delay differential systems has been undergoing fast development due to its important applications in various areas such as population management, disease control, image processing, and secure communication ([5], [9], [13], [14], [19], [20]). For general impulsive delay differential equations, existence and uniqueness results of solutions were obtained in [2] and [8]; uniform stability and uniform asymptotic stability criteria were established in [4] and [12]; sufficient conditions on exponential stability were discussed in [1], [3], [10], [11], [15]-[18], and [20]. However most of the current research on stability analysis has

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been focused on the impulsive delay differential equations with time delay occurred only in the differential equations. Recently, an impulsive delay differential model with delayed impulses has been investigated in impulsive synchronization of chaotic systems in secure communication where time delays appeared in both differential and difference equations of the error dynamics due to the presence of transmission delays in the process [6, 7]. This type of equations also have potential applications in other fields. For instance, time delays may be considered in the difference equation of the population growth model since the amount of harvesting or stocking may depend on the past and current population when the time needed for reproduction is not negligible; while in disease control models, time delays maybe be introduced into the impulses if the time needed for drugs to take effect is taken into consideration.

In this paper, we establish some global exponential stability criteria for impulsive delay differential equations with delayed impulses based on the the results and methods developed in [15]-[18]. The two impulsive stabilization results released the lower bounds for the length of impulsive intervals and the result dedicated to keeping the good stability property of systems subject to certain type of impulsive perturbations is less restrictive compared to some known results in that the magnitude of impulsive disturbances may be larger than the magnitude of the states before perturbation [12]. Generally speaking, the stability analysis of impulsive delay differential systems with delay in both differential and difference equations is more challenging than that of impulsive delay differential systems whose time delays only appear in the differential equations.

The rest of this paper is organized as follows. In Section 2, we introduce some notation and definitions, and then present several global exponential stability criteria for the general differential systems with delayed impulses in Section 3. Finally, some examples with numerical simulations are given to illustrate the effectiveness of our results in Section 4.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional real space and $\mathbb{R}_+ = [0, +\infty)$, and let $\mathbb{N}$ denote the set of positive integers, i.e., $\mathbb{N} = \{1, 2, \cdots\}$. Define $\psi(t^+) = \lim_{s\to t^+} \psi(s)$ and $\psi(t^-) = \lim_{s\to t^-} \psi(s)$. For $a, b \in \mathbb{R}$ with $a < b$ and for $S \subset \mathbb{R}^n$, we define the following classes of functions.

$$PC([a,b],S) = \Big\{ \psi : [a,b] \to S \,\Big|\, \psi(t) = \psi(t^+), \forall t \in [a,b); \psi(t^-) \text{ exists in } S, \forall t \in (a,b], \\ \text{and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b] \Big\},$$

$$PC([a,b),S)=\left\{\psi:[a,b)\to S\middle|\psi(t)=\psi(t^+),\forall t\in[a,b);\psi(t^-)\text{ exists in } S,\forall t\in(a,b),\right.$$
$$\left.\text{and}\,\psi(t^-)=\psi(t)\,\text{for all but at most a finite number of points } t\in(a,b)\right\},$$

and

$$PC([a,\infty),S)=\left\{\psi:[a,\infty)\to S\middle|\forall c>a,\psi|_{[a,c]}\in PC([a,c],S)\right\}.$$

Given a constant $\tau>0$, we equip the linear space $PC([-\tau,0],\mathbb{R}^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\psi\|_\tau=\sup_{-\tau\le s\le 0}\|\psi(s)\|$.

Consider the following impulsive system

$$\begin{cases} x'(t) &= F(t,x_t), \quad t\neq t_k,\\ \Delta x(t_k) &= I_k(x_{t_k^-}), \quad k\in\mathbb{N},\\ x_{t_0} &= \phi, \end{cases} \tag{2.1}$$

where $F,I_k:\mathbb{R}_+\times PC([-\tau,0],\mathbb{R}^n)\to\mathbb{R}^n;\phi\in PC([-\tau,0],\mathbb{R}^n);\quad 0\le t_0<t_1<t_2<\cdots<t_k<\cdots$, with $t_k\to\infty$ as $k\to\infty$; $\triangle x(t)=x(t)-x(t^-)$; and $x_t,x_{t^-}\in PC([-\tau,0],\mathbb{R}^n)$ are defined by $x_t(s)=x(t+s),x_{t^-}(s)=x(t^-+s)$ for $-\tau\le s\le 0$, respectively.

In this paper, we assume that functions $F,I_k,k\in\mathbb{N}$ satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t\ge t_0$ ([2]). Denote by $x(t)=x(t,t_0,\phi)$ the solution of (2.1) such that $x_{t_0}=\phi$. We further assume without loss of generality that all the solutions $x(t)$ of (2.1) are continuous except at $t_k,k\in\mathbb{N}$, at which $x(t)$ is right continuous (i.e., $x(t_k^+)=x(t_k),k\in\mathbb{N}$) and the left limit $x(t_k^-)$ exists.

Definition 2.1 *Function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$ is said to belong to the class ν_0 if*

i) V is continuous in each of the sets $[t_{k-1},t_k)\times\mathbb{R}^n$ and for each $x\in\mathbb{R}^n,t\in[t_{k-1},t_k),k\in N$, $\lim_{(t,y)\to(t_k^-,x)}V(t,y)=V(t_k^-,x)$ exists; and

ii) $V(t,x)$ is locally Lipschitzian in all $x\in\mathbb{R}^n$, and for all $t\ge t_0$, $V(t,0)\equiv 0$.

Definition 2.2 *Given a function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, the upper right-hand derivative of V with respect to system (2.1) is defined by*

$$D^+V(t,\psi)=\limsup_{h\to 0^+}\frac{1}{h}[V(t+h,\psi+hF(t,\psi))-V(t,\psi)],$$

for $(t,\psi)\in\mathbb{R}_+\times PC([-\tau,0],\mathbb{R}^n)$.

Definition 2.3 *The trivial solution of system (2.1) is said to be globally exponentially stable, if there exist some constants* $\alpha > 0$ *and* $M \geq 1$ *such that for any initial data* $x_{t_0} = \phi$

$$\|x(t, t_0, \phi)\| \leq M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \geq t_0,$$

where $(t_0, \phi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$.

3 Lyapunov-Razumkhin method

In this section, we shall present some Razumikhin-type theorems on global exponential stability for system (2.1) based on the Lyapunov-Razumikhin method and mathematical induction. Our results show that impulses play an important role in stabilizing some differential systems with delayed impulses.

Theorem 3.1 *Assume that there exist a function* $V \in \nu_0$ *and constants* $p, c, c_1, c_2 > 0$, d_k, $e_k \geq 0$, $k \in \mathbb{N}$, *such that the following conditions hold:*

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$, *for any* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$*;*

(ii) the upper right-hand derivative of V *with respect to system (2.1) satisfies*

$$D^+V(t, \varphi(0)) \leq -cV(t, \varphi(0)), \text{ for all } t \neq t_k \text{ in } \mathbb{R}_+,$$

whenever $V(t + s, \varphi(s)) \leq qV(t, \varphi(0))$ *for* $s \in [-\tau, 0]$, *where* $q \geq e^{c\tau}$*;*

(iii) for all $\varphi \in PC([-\tau, 0], \mathbb{R}^n)$ *and* $s \in [-\tau, 0]$,

$$V(t_k, \varphi(0) + I_k(\varphi(s))) \leq (1 + d_k)V(t_k^-, \varphi(0)) + e_kV(t_k^- + s, \varphi(s)).$$

Moreover $\sum_{k=1}^{\infty}(d_k + e_ke^{c\tau})$ *is finite.*

Then the trivial solution of system (2.1) is globally exponentially stable with convergence rate $\frac{c}{p}$.

Proof. Let $x(t) = x(t, t_0, \phi)$ be a solution of system (2.1) and $v(t) = V(t, x(t))$. We shall show

$$v(t) \leq c_2 \prod_{i=0}^{k-1}(1 + d_i + e_ie^{c\tau})\|\phi\|_\tau^p e^{-c(t-t_0)}, \quad t \in [t_{k-1}, t_k), \ k \in \mathbb{N},$$

where $d_0 = e_0 = 0$. Let

$$Q(t) = \begin{cases} v(t) - c_2 \prod_{i=0}^{k-1}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t-t_0)}, & t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}, \\ v(t) - c_2\|\phi\|_\tau^p e^{-c(t-t_0)}, & t \in [t_0 - \tau, t_0]. \end{cases}$$

We need to show $Q(t) \leq 0$ for all $t \geq t_0$. It is clear that $Q(t) \leq 0$ for $t \in [t_0 - \tau, t_0]$, since $v(t) \leq c_2\|x\|^p \leq c_2\|\phi\|_\tau^p$ by condition (i).

Next we show that $Q(t) \leq 0$ for $t \in [t_0, t_1)$. To this end, we let $\alpha > 0$ be any arbitrary constant and prove that $Q(t) \leq \alpha$ for $t \in [t_0, t_1)$. Suppose not, then there exists some $t \in [t_0, t_1)$ so that $Q(t) > \alpha$. Let $t^* = \inf\{t \in [t_0, t_1) : Q(t) > \alpha\}$. Since $Q(t) \leq 0 < \alpha$ for $t \in [t_0 - \tau, t_0]$, we know $t^* \in (t_0, t_1)$. Note that $Q(t)$ is continuous on $[t_0, t_1)$, we have $Q(t^*) = \alpha$ and $Q(t) \leq \alpha$ for $t \in [t_0 - \tau, t^*]$. Then $v(t^*) = Q(t^*) + c_2\|\phi\|_\tau^p e^{-c(t^*-t_0)} = \alpha + c_2\|\phi\|_\tau^p e^{-c(t^*-t_0)}$.

For any $s \in [-\tau, 0]$, we have

$$\begin{aligned} v(t^* + s) &= Q(t^* + s) + c_2\|\phi\|_\tau^p e^{-c(t^*+s-t_0)} \\ &\leq \alpha + c_2 e^{c\tau}\|\phi\|_\tau^p e^{-c(t^*-t_0)} \\ &\leq (\alpha + c_2\|\phi\|_\tau^p e^{-c(t^*-t_0)})e^{c\tau} \\ &= v(t^*)e^{c\tau} \leq qv(t^*). \end{aligned}$$

Thus we have $D^+v(t^*) \leq -cv(t^*)$ by condition (ii). And then we obtain

$$\begin{aligned} D^+Q(t^*) &= D^+v(t^*) + cc_2\|\phi\|_\tau^p e^{-c(t^*-t_0)} \\ &\leq -c(v(t^*) - c_2\|\phi\|_\tau^p e^{-c(t^*-t_0)}) \\ &\leq -cQ(t^*) = -c\alpha \\ &< 0, \end{aligned}$$

which contradicts the definition of t^*, and hence we have $Q(t) \leq \alpha$ for all $t \in [t_0, t_1)$. Let $\alpha \to 0^+$. We have $Q(t) \leq 0$ for $t \in [t_0, t_1)$.

Now we assume that $Q(t) \leq 0$ for $t \in [t_0, t_m)$, for $m \geq 1$ and $m \in \mathbb{N}$. We then show that $Q(t) \leq 0$ for $t \in [t_0, t_{m+1})$.

By condition (iii) with $\varphi(s) = x(t_m^- + s)$ and $s \in [-\tau, 0]$, we have,

$$\begin{aligned} &Q(t_m) = v(t_m) - c_2 \textstyle\prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t_m-t_0)} \\ &\leq (1 + d_m)v(t_m^-) + e_m v(t_m^- + s) - c_2 \textstyle\prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t_m-t_0)} \\ &\leq (1 + d_m)c_2 \textstyle\prod_{i=0}^{m-1}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t_m-t_0)} + e_m c_2 \textstyle\prod_{i=0}^{\hat{m}-1}(1 + d_i + e_i e^{c\tau}) \\ &\times\|\phi\|_\tau^p e^{-c(t_m-\tau-t_0)} - c_2 \textstyle\prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t_m-t_0)} \\ &\leq (1 + d_m + e_m e^{c\tau})c_2 \textstyle\prod_{i=0}^{m-1}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t_m-t_0)} \\ &-c_2 \textstyle\prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t_m-t_0)} = 0, \end{aligned}$$

where $\hat{m} \in \mathbb{N}$ and $\hat{m} \leq m$.

For any given $\alpha > 0$, we show that $Q(t) \le \alpha$ for $t \in (t_m, t_{m+1})$. Suppose not. Let $t^* = \inf\{t \in [t_m, t_{m+1}) : Q(t) > \alpha\}$. Since $Q(t_m) \le 0 < \alpha$, we have, by the continuity of $Q(t)$, that $Q(t^*) = \alpha$ and $Q(t) \le \alpha$ for $t \in [t_0, t^*]$. Then

$$\begin{array}{rcl} v(t^*) &=& Q(t^*) + c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*-t_0)} \\ &=& \alpha + c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*-t_0)}. \end{array}$$

Hence for $s \in [-\tau, 0]$, we have

$$\begin{array}{rcl} v(t^* + s) &\le& Q(t^* + s) + c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*+s-t_0)} \\ &\le& \alpha + c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*-t_0)} e^{c\tau} \\ &\le& (\alpha + c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*-t_0)}) e^{c\tau} \\ &\le& v(t^*) e^{c\tau} \ \le \ q v(t^*). \end{array}$$

Therefore, by condition (ii), we have $D^+ v(t^*) \le -c v(t^*)$ and

$$\begin{array}{rcl} D^+ Q(t^*) &=& D^+ v(t^*) + c c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*-t_0)} \\ &\le& -c[v(t^*) - c_2 \prod_{i=0}^{m}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t^*-t_0)}] \\ &\le& -c Q(t^*) \ = \ -c\alpha \\ &<& 0. \end{array}$$

Again this contradicts the definition of t^*, which implies $Q(t) \le \alpha$ for all $t \in [t_m, t_{m+1})$. Let $\alpha \to 0^+$. We have $Q(t) \le 0$ for all $t \in [t_m, t_{m+1})$. So $Q(t) \le 0$ for all $t \in [t_0, t_{m+1})$. Thus by the method of mathematical induction, we have

$$v(t) \le c_2 \prod_{i=0}^{k-1}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t-t_0)}, \quad t \in [t_{k-1}, t_k), \ \ k \in \mathbb{N}.$$

By conditions (i) and (iii), we have

$$\begin{array}{rcl} c_1 \|x\|^p &\le& v(t) \le c_2 \prod_{i=0}^{k-1}(1 + d_i + e_i e^{c\tau})\|\phi\|_\tau^p e^{-c(t-t_0)} \\ &\le& c_2 M \|\phi\|_\tau^p e^{-c(t-t_0)}, \quad t \ge t_0, \end{array}$$

which yields

$$\|x\| \le \left(\frac{c_2 M}{c_1}\right)^{\frac{1}{p}} \|\phi\|_\tau e^{-\frac{c}{p}(t-t_0)}, \quad t \ge t_0,$$

where $M = \prod_{i=1}^{\infty}(1 + d_i + e_i e^{c\tau}) < \infty$ since $\sum_{k=1}^{\infty}(d_k + e_i e^{c\tau}) < \infty$. Thus the proof is complete. ∎

Remark 3.1. Condition (iii) of Theorem 3.1 is less restrictive compared to some known results (see [12] for example) in that it allows the solution to jump up at the impulsive moments since $1 + d_k > 1$ and $e_k > 0$. This obviously cannot guarantee the stability of a delay differential system. Our result gives sufficient conditions on keeping the good stability property of the system under impulsive perturbations.

Theorem 3.2 *Assume that there exist a function $V \in \nu_0$, constants $p, c_1, c_2, \lambda > 0$ and $\alpha > 1$ such that*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$, *for any* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$*;*

(ii) the upper right-hand derivative of V with respect to system (2.1) satisfies

$$D^+V(t,\varphi(0)) \leq 0, \textit{for all}\, t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

whenever $qV(t,\varphi(0)) \geq V(t+s,\varphi(s))$ *for all* $s \in [-\tau, 0]$*, where* $q \geq \alpha e^{\lambda\tau}$ *is a constant;*

(iii) for all $\varphi \in PC([-\tau,0],\mathbb{R}^n)$ *and* $s \in [-\tau, 0]$*,*

$$V(t_k, \varphi(0) + I_k(\varphi(s))) \leq d_k V(t_k^-, \varphi(0)) + e_k V(t_k^- + s, \varphi(s)),$$

where d_k*,* e_k $(\forall k \in \mathbb{N})$ *are positive constants;*

(iv) $t_{k+1} - t_k < \frac{1}{\lambda} \min\left\{ \ln(\alpha),\ \ln(\frac{1}{d_k + e_k e^{\lambda\tau}})\right\}$.

Then the trivial solution of the impulsive system (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$.

Proof. Choose $M \geq 1$ such that

$$c_2 < Me^{-\lambda(t_1 - t_0)} \leq qc_2. \tag{3.1}$$

Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.1) with $x_{t_0} = \phi$, and $v(t) = V(t,x)$. We shall show

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)},\ \ t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{3.2}$$

We first show that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)},\ \ t \in [t_0, t_1). \tag{3.3}$$

From condition (i) and (3.1), we have, for $t \in [t_0 - \tau, t_0]$,

$$\begin{aligned} v(t) &\leq c_2\|x\|^p \leq c_2\|\phi\|_\tau^p \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}. \end{aligned}$$

If (3.3) is not true, then there must exist some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} v(\bar{t}) &> M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \\ &> c_2\|\phi\|_\tau^p \geq v(t_0 + s),\ \ s \in [-\tau, 0], \end{aligned}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad \text{and} \;\; v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \\ \text{for } t_0 - \tau \leq t \leq t^*, \tag{3.4}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p, \;\; \text{and} \;\; v(t) \geq c_2\|\phi\|_\tau^p, \; \text{for } t^{**} \leq t \leq t^*. \tag{3.5}$$

Then we obtain from (3.1), (3.4) and (3.5) that, for any $t \in [t^{**}, t^*]$,

$$v(t+s) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \leq qc_2\|\phi\|_\tau^p \leq qv(t), \;\; s \in [-\tau, 0].$$

Thus by condition (ii), we have $D^+v(t) \leq 0$ for $t \in [t^{**}, t^*]$, and then we obtain $v(t^{**}) \geq v(t^*)$, i.e., $c_2\|\phi\|_\tau^p \geq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}$, which contradicts (3.1). Hence (3.3) holds, and then (3.2) is true for $k = 1$.

Now we assume that (3.2) holds for $k = 1, 2, \cdots, m$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k), \; k = 1, 2, \cdots, m. \tag{3.6}$$

We show that (3.2) holds for $k = m+1$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{3.7}$$

For the sake of contradiction, suppose (3.7) is not true. Then we define

$$\bar{t} = \inf\{t \in [t_m, t_{m+1}) | v(t) > M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}\}.$$

By the continuity of $v(t)$ in the interval $[t_m, t_{m+1})$, we have

$$\begin{aligned} v(\bar{t}) &= M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \quad \text{and} \\ v(t) &\leq M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \; \text{for } t \in [t_m, \bar{t}). \end{aligned}$$

Since

$$\begin{aligned} v(t_m) &\leq d_m v(t_m^-) + e_m v(t_m^- + s) \\ &\leq d_m M\|\phi\|_\tau^p e^{-\lambda(t_m^- - t_0)} + e_m M\|\phi\|_\tau^p e^{-\lambda(t_m^- + s - t_0)} \\ &\leq (d_m + e_m e^{\lambda\tau}) M\|\phi\|_\tau^p e^{-\lambda(t_m - t_0)} \\ &< e^{-\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &\leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)}, \end{aligned}$$

i.e.

$$v(t_m) < M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} < v(\bar{t}).$$

Therefore, $\bar{t} \neq t_m$ and there exists some $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(t^*) \leq v(t) \leq v(\bar{t}) \ \text{ for } t \in [t^*, \bar{t}].$$

Then for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$, we either have $t+s \in [t_0-\tau, t_m)$ or $t+s \in [t_m, \bar{t})$. If $t+s \in [t_0-\tau, t_m)$, we have, by (3.6) that $v(t+s) \leq M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)}$; if $t+s \in [t_m, \bar{t})$, we again obtain that $v(t+s) \leq M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)}$ from the definition of $\bar{t}$.

Therefore we have in both cases, by conditions (ii) and (iv), that

$$\begin{aligned}
v(t+s) &\leq M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)} \\
&\leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)} e^{\lambda\tau} \\
&\leq M\|\phi\|_\tau^p e^{-\lambda(t-t_{m+1})} e^{\lambda(t_{m+1}-t_0)} e^{\lambda\tau} \\
&\leq e^{\lambda\tau} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{-\lambda(t_m-t_{m+1})} \\
&\leq qv(t^*) \leq qv(t), \quad s \in [-\tau, 0].
\end{aligned}$$

Thus we have $v(t+s) \leq qv(t)$ for all $s \in [-\tau, 0]$ and $t \in [t^*, \bar{t}]$. It follows from condition (ii) that $D^+v(t) \leq 0$, for $t \in [t^*, \bar{t}]$, which implies that $v(t^*) \geq v(\bar{t})$, i.e., $M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \geq M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}$, which contradicts the fact that $\bar{t} < t_{m+1}$. This implies that the assumption is not true, and hence (3.2) holds for $k = m+1$. Thus by mathematical induction, we obtain that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k).$$

Hence by condition (i), we have

$$\|x\| \leq M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

where $M^* \geq \max\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\}$. This implies that the trivial solution of system (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. ∎

Remark 3.2. It is well-known that, in the stability theory of functional differential equations, the condition $D^+V(t,x) \leq 0$ can not even guarantee the asymptotic stability of a functional differential system (see [9, 11]). However, as we can see from Theorem 3.2, impulses can contribute to the exponential stabilization a functional differential system.

Theorem 3.3 *Assume that there exist a function $V \in \nu_0$ and constants $\alpha > \tau$, $p, c_1, c_2 > 0$, and $\lambda \geq c > 0$ such that*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$, for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;

(ii) $D^+V(t,\varphi(0)) \leq cV(t,\varphi(0))$, for all $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, whenever $qV(t,\varphi(0)) \geq V(t+s,\varphi(s))$ for $s \in [-\tau, 0]$, where $q \geq e^{\lambda(2\alpha+\tau)}$ is a constant;

(iii) for all $\varphi \in PC([-\tau,0],\mathbb{R}^n)$ and $s \in [-\tau,0]$,

$$V(t_k, \varphi(0) + I_k(\varphi(s))) \le d_k V(t_k^-, \varphi(0)) + e_k V(t_k^- + s, \varphi(s)),$$

where $d_k,\ e_k$ $(\forall k \in \mathbb{N})$ are positive constants;

(iv) $t_{k+1} - t_k < \min\left\{\alpha,\ \frac{1}{\lambda}\ln(\frac{1}{d_k + e_k e^{\lambda\tau}}) - \alpha\right\}$.

Then the trivial solution of the impulsive system (2.1) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$.

Proof. Choose $M \ge 1$ such that

$$c_2 < Me^{-\lambda(t_1 - t_0)}e^{-\alpha c} < Me^{-\lambda(t_1 - t_0)} \le qc_2. \tag{3.8}$$

Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.1) with $x_{t_0} = \phi$, and $v(t) = V(t,x)$. We shall show that

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t - t_0)},\ \ t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{3.9}$$

We first prove that (3.9) holds for $k = 1$ by showing

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)},\ \ t \in [t_0, t_1). \tag{3.10}$$

From condition (i) and (3.8), we have, for $t \in [t_0 - \tau, t_0]$

$$v(t) \le c_2\|x\|^p \le c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}e^{-\alpha c}.$$

If (3.10) is not true, then there must exist some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} v(\bar{t}) &> M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} > M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}e^{-\alpha c} \\ &> c_2\|\phi\|_\tau^p \ge v(t_0 + s), \quad s \in [-\tau, 0], \end{aligned}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)},\ \text{ and }\ v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \quad t_0 - \tau \le t \le t^*; \tag{3.11}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p,\ \text{ and }\ v(t) \ge c_2\|\phi\|_\tau^p, \quad t^{**} \le t \le t^*. \tag{3.12}$$

Then by (3.8), (3.11) and (3.12), we have for any $t \in [t^{**}, t^*]$

$$v(t+s) \le M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} \le qc_2\|\phi\|_\tau^p \le qv(t),\ \ s \in [-\tau, 0], \tag{3.13}$$

thus by condition (ii), we obtain that $D^+v(t) \le cv(t)$ for $t \in [t^{**}, t^*]$. Therefore, $v(t^{**}) \ge v(t^*)e^{-\alpha c}$, i.e., $c_2\|\phi\|_\tau^p \ge M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c}$, which contradicts (3.8). Hence (3.10) holds, and then (3.9) is true for $k=1$.

Now we assume that (3.9) holds for $k = 1, 2, \cdots, m$ ($m \in \mathbb{N},\ m \ge 1$)

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \cdots, m. \tag{3.14}$$

From conditions (iii), (iv) and (3.14), we have

$$\begin{aligned} v(t_m) &\le d_m v(t_m^-) + e_m v(t_m^- + s) \\ &\le d_m M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} + e_m M\|\phi\|_\tau^p e^{-\lambda(t_m+s-t_0)} \\ &\le (d_m + e_m e^{\lambda\tau}) M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &< e^{-\lambda\alpha} e^{-\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)}, \end{aligned} \tag{3.15}$$

Next, we shall show that (3.9) holds for $k = m+1$, i.e.

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{3.16}$$

For the sake of contradiction, suppose (3.16) is not true. Then we define

$$\bar{t} = \inf\{t \in [t_m, t_{m+1}) | v(t) > M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}\}.$$

From (3.15), we know $\bar{t} \ne t_m$. By the continuity of $v(t)$ in the interval $[t_m, t_{m+1})$, we have

$$v(\bar{t}) = M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \quad \text{and} \quad v(t) \le M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \qquad t \in [t_m, \bar{t}]. \tag{3.17}$$

From (3.15) and (3.17), we have

$$v(t_m) < e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} < v(\bar{t}),$$

which implies that there exists some $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(t^*) \le v(t) \le v(\bar{t}), \quad t \in [t^*, \bar{t}].$$

For $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$, we have either $t+s \in [t_0-\tau, t_m)$ or $t+s \in [t_m, \bar{t}]$. If $t+s \in [t_0-\tau, t_m)$, we have from (3.14) that $v(t+s) \le M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)}$; if $t+s \in [t_m, \bar{t}]$, we also have $v(t+s) \le M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)}$ by the definition of $\bar{t}$.

By conditions (ii) and (iv), we have

$$\begin{aligned} v(t+s) &\le M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)} \\ &\le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)} e^{\lambda\tau} \\ &= M\|\phi\|_\tau^p e^{-\lambda(t-t_{m+1})} e^{\lambda(t_{m+1}-t_0)} e^{\lambda\tau} \\ &\le e^{\lambda\tau} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{-\lambda(t_m-t_{m+1})} \\ &\le e^{\lambda(\tau+\alpha)} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &\le qv(t^*) \le qv(t), \quad s \in [-\tau, 0]. \end{aligned}$$

Therefore in both cases we have $v(t+s) \leq qv(t)$. From condition (ii), we have that $D^+v(t) \leq cv(t)$. Since $\lambda \geq c$, we have

$$\begin{aligned} v(\bar{t}) &\leq v(t^*)e^{\alpha c} = e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{\alpha c} \\ &\leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &< v(\bar{t}). \end{aligned}$$

This contradiction implies the assumption is not true. Thus (3.9) holds for $k = m+1$ and by mathematical induction, we have

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k).$$

Then by condition (i), we get

$$\|x\| \leq M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

where $M^* \geq \max\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\}$, this implies that the trivial solution of system (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. ■

Remark 3.3. It is well-known that, in the stability theory of delay differential equations, the condition $D^+V(t,x) \leq cV(t,x)$ allows the derivative of Lyapunov function to be positive which may not even guarantee the stability of a delay differential system (see [9, 10, 11] and Example 4.2). However, as we can see from Theorem 3.3, impulses have played an important role in exponentially stabilizing a delay differential system.

Remark 3.4. The above stabilization theorems released the lower bounds for the length of impulsive intervals as required in the stability theorems in [10], [11], [15]-[17] and therefore the conditions are less restrictive. Our results are more applicable in that they deal with systems with time delays in both states and impulses.

4 Examples and Simulations

In this section, we give two examples and their numerical simulations to illustrate our results.

Example 4.1 *Consider the impulsive nonlinear delay differential equation with time delays in both differential and difference equations*

$$\begin{cases} x'(t) &= -e^2(t+1)x(t) + \frac{t}{1+x^2(t)}x(t-1), \quad t \geq t_0 = 0,\ t \neq 0.25k, \\ x(t_k) &= (1+\frac{1}{2^k})x(t_k^-) + (\frac{2}{3})^k x(t_k^- - 0.5), \quad t_k = 0.25k,\ k \in \mathbb{N}, \\ x_s &= 5\cos(s), \qquad s \in [-1, 0]. \end{cases} \tag{4.1}$$

Choose $V(t,x) = |x|$. Then condition (i) of Theorem 3.1 holds with $c_1 = c_2 = p = 1$. And we have

$$\begin{aligned} D^+V(t,\varphi(0)) &\leq \operatorname{sgn}(\varphi(0))[-e^2(t+1)\varphi(0) + \tfrac{t\varphi(-1)}{1+\varphi^2(0)}] \\ &\leq -e^2(t+1)|\varphi(0)| + \tfrac{t|\varphi(-1)|}{1+\varphi^2(0)} \\ &\leq -e^2(t+1)V(\varphi(0)) + tV(\varphi(-1)). \end{aligned} \tag{4.2}$$

For any solution $x(t)$ of equation (4.1) such that $V(t+s,\varphi(s)) \leq e^2V(t,\psi(0))$, for $s \in [-1,0]$ (i.e., $\tau = 1$ and $c = 2$ in notation of Theorem 3.1), we have $|\varphi(-1)| \leq e^2|\varphi(0)|$. Therefore,

$$D^+V(t,\varphi(0)) \leq [-e^2(t+1) + te^2]V(\varphi(0)) \leq -e^2V(\varphi(0)) \leq -2V(\varphi(0)).$$

This shows that condition (ii) of Theorem 3.1 holds.

Moreover,

$$V(t_k,\varphi(0)+I_k(\varphi)) \leq (1+\frac{1}{2^k})V(t_k^-,\varphi(0)) + (\frac{2}{3})^kV(t_k^- - 0.5,\varphi(-0.5)).$$

We see condition (iii) of Theorem 3.1 holds with $d_k = \frac{1}{2^k}$ and $e_k = (\frac{2}{3})^k$. Thus by Theorem 3.1, the trivial solution of system (4.1) is globally exponentially stable with convergence rate 2. The numerical simulations of this example are given in Figure 4 (impulse-perturbed system) and Figure 2 (nonimpulsive system). As we can see from the simulation, the system keeps the global exponential stability property under relatively small impulsive perturbations.

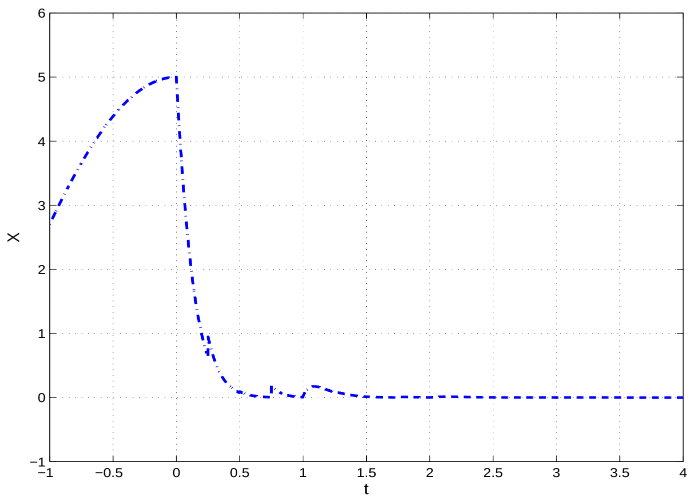

Figure 1: Numerical simulation of Example 4.1, impulse-perturbed system.

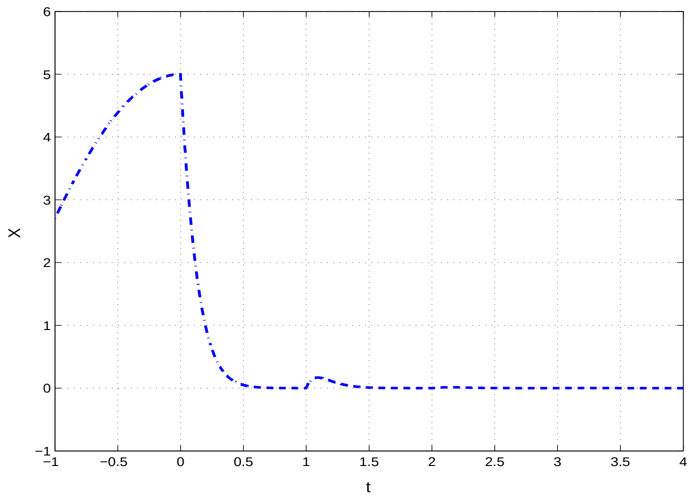

Figure 2: Numerical simulation of Example 4.1, nonimpulsive system.

Example 4.2 *Consider the nonlinear impulsive delay system with delayed impulses*

$$\begin{cases} X'(t) &= AX(t) + f(t, X(t), X_t),\ t \neq 0.2k,\ s \in [-0.2, 0], \\ \Delta X(t_k) &= B_k X(t_k^-) + C_k X(t_k^- - 0.1),\ t_k = 0.2k,\ k \in \mathbb{N}^*, \\ X_{t_0} &= \phi, \end{cases} \tag{4.3}$$

where $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, $\phi \in PC([-0.2, 0], \mathbb{R}^3)$, *and*

$$A = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ -0.2 & 0.15 & -0.3 \\ 0 & 0.24 & 0.1 \end{bmatrix}, \quad f(t, X(t), X(t+s)) = \begin{bmatrix} 0.25x_1(t-0.1) \\ 0.5\sqrt{|x_2(t-0.2)x_3(t-0.2)|} \\ 0.25\cos(t)x_2(t) \end{bmatrix},$$

and

$$B_k = \begin{bmatrix} -0.7 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & -0.72 \end{bmatrix}, \quad C_k = \begin{bmatrix} 0.15 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}.$$

Using the notation in Theorem 3.3, we have $\tau = 0.2$ and $t_k - t_{k-1} = 0.2$. Choose $\alpha = 0.2010$ such that $\alpha > t_k - t_{k-1}$.

Let $V(t, X(t)) = \|X(t)\|^2$. Then condition (i) in Theorem 3.3 holds with $c_1 = c_2 = 1$ and $p = 2$. And we obtain

$$\begin{aligned} D^+V(t, X(t)) &= (X')^T X + X^T X' \\ &= X^T(A^T + A)X + 2f^T X \\ &\leq \lambda_{max}(A^T + A)\|X\|^2 + \tfrac{1}{8}\|X(t+s)\|^2 + \tfrac{17}{16}\|X\|^2 \\ &\leq (0.5884 + \tfrac{17}{16} + \tfrac{1}{8}e^{\lambda(2\alpha+\tau)})V(t, X), \end{aligned}$$

whenever $V(t+s, X(t+s)) \leq e^{\lambda(2\alpha+\tau)} V(t, X(t))$. Choose $\lambda = 2.5$ and we get $c = 0.5884 + \frac{17}{16} + \frac{1}{8} e^{\lambda(2\alpha+\tau)} = 2.21$. Thus $\lambda \geq c > 0$ and condition (ii) holds with $q = e^{\lambda(2\alpha+\tau)} = 4.5042$.

Furthermore, we obtain that

$$\begin{aligned} V(t_k, X(t_k)) &= \|X(t_k)\|^2 = \|(I+B_k)X(t_k^-) + C_k X(t_k^- - 0.1)\|^2 \\ &= X^T(t_k^-)(I+B_k)^T(I+B_k)X(t_k^-) + 2X^T(t_k^- - 0.1)C_k^T(I+B_k)X(t_k^-) \\ &\quad + X^T(t_k^- - 0.1)C_k^T C_k X(t_k^- - 0.1) \\ &\leq 0.1350\|X(t_k^-)\|^2 + 0.08\|X(t_k^- - 0.1)\|^2, \end{aligned}$$

this implies that condition (iii) of Theorem 3.3 holds with $d_k = 0.1350$ and $e_k = 0.08$ and also we know that condition (iv) holds with since $\frac{1}{\lambda}\ln(\frac{1}{d_k + e_k e^{\lambda\tau}}) = 0.5284$ and

$$t_{k+1} - t_k = 0.2 < \min\left\{\alpha,\ 0.5284 - \alpha\right\} = 0.2010.$$

Thus by Theorem 3.3, we obtain that the trivial solution of (4.3) is globally exponentially stable with convergence rate 1.25.

By applying the 4-step, 2nd-order Runge-Kutta method with step size 0.01, the numerical simulation of the system of delay differential equations with delayed impulses (4.3) with the initial function $\phi(s) = (\sin(s), e^{-s}, 1-2s)^T$ for $s \in [-0.2, 0]$ is given in Figure 3, the graph of solution of the corresponding system without impulse is given in Figure 4.

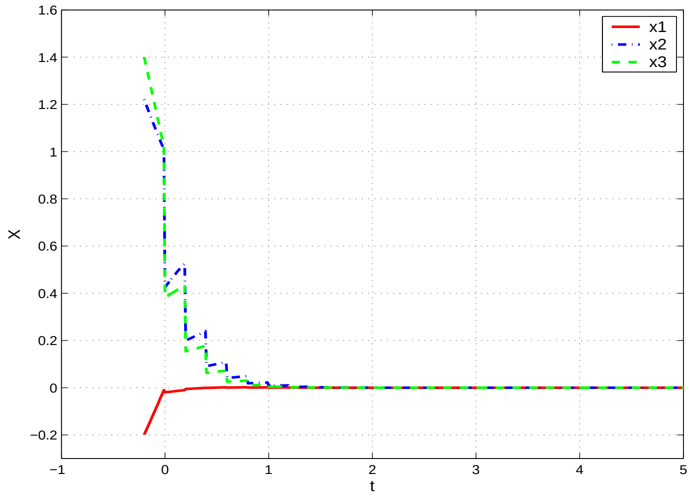

Figure 3: Numerical simulation of Example 4.2, stabilized system with delayed impulses.

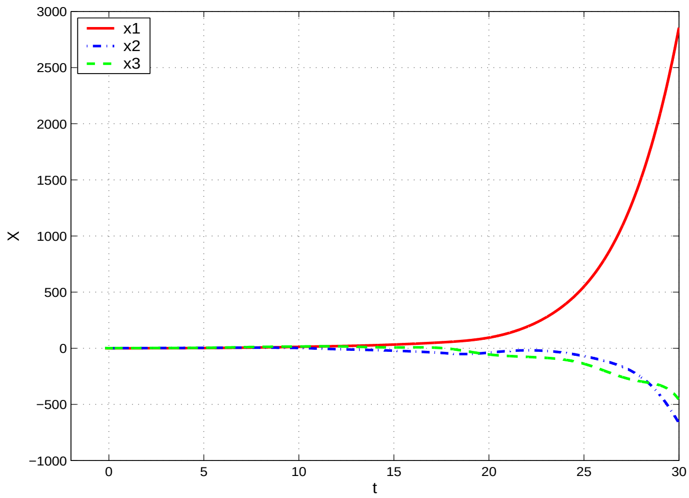

Figure 4: Numerical simulation of Example 4.2, unstable nonimpulsive system.

Remark 4.1. We note that the linear part $X'(t) = AX(t)$ in the above example is unstable since all eigenvalues of A have positive real parts ($l_1 = 0.143$, $l_2 = 0.1035 + 0.3342i$, $l_3 = 0.1035 - 0.3342i$). As shown in Figure 4, the corresponding nonlinear system without impulses is unstable, however Figure 3 shows that it can be exponentially stabilized by impulses.

Remark 4.2. The stability theorems in [10]-[12], [15]-[18] can not apply to the above examples because their proposed Lyapunov functions or functionals do not deal with time delays at the impulsive moments.

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