EVANESCENT SOLUTIONS FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The problem of existence of the solutions for ordinary differential equations vanishing at $\pm\infty$ is considered.

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1. INTRODUCTION

There exists a large classical theory concerning the asymptotic behavior of the solutions of the "perturbed" equation

$$\dot{x} = Ax + F\left(t, x\right) \tag{1}$$

determined by the behavior of the homogenous equation

$$\dot{x} = Ax. \tag{2}$$

An early account is found in the book of Bellman [4], with progressive treatments in Coppel [6], Hale [8] , Kartsatos [11], Yoshizawa [18] etc.

In addition to stability problems, special attention has been devoted to the boundedness of solutions on $\mathbb{R}_+ = [0, +\infty[$ or on $\mathbb{R}$; although numerous results have been obtained, this field is not exhausted. During the last few years, very interesting results concerning the existence of bounded solutions have been established; we mention, in particular, the ones of Mawhin, Ortega, Tineo and Ward ([12], [13], [14], [15], [16], [17]).

A behavior stronger than the boundedness is the one when the solutions admit finite limits on the boundary of the definition domain interval; in particular $+\infty$ when this interval is $\mathbb{R}_+$ or $\pm\infty$ when this interval is $\mathbb{R}$. The solutions of this type are called **convergent** and their existence has been the object of many works (see e.g. [1], [9], [10]). In the same direction, during the last several years inverstigators have considered the problem of the existence of solutions satisfying boundary conditions of type $x(+\infty) = x(-\infty)$, where $x(\pm\infty) := \lim_{t\to\pm\infty} x(t)$.

Such a problem will be considered in the present paper; more precisely, we shall prove the existence of solutions for an equation of type (1), satisfying the boundary condition

$$x(+\infty) = x(-\infty) = 0. \tag{3}$$

A solution of a differential functional equation, satisfying the condition (3) is called **evanescent solution.**

It is well-known that the problem of the existence of evanescent solutions on $\mathbb{R}_+$ is closely related to the problem of the asymptotic stability; in this sense an interesting result is contained in [5]. The results of this note concern the existence of solutions for the problem $(1), (3)$ and yield certain generalizations of the results contained in [14], related to the existence of the bounded solutions of the equation (1).

The second section of the paper is devoted to the notations and the main classical results. In the third section a general existence theorem for the linear equation with continuous perturbations is given. This result is used in the next section for the $n-$th order nonresonant linear equation with constant coefficients and continuous perturbations. Finally, in the fifth section the case of certain nonlinear perturbations is considered.

2. NOTATIONS AND PRELIMINARY RESULTS

In what follows A will be a constant matrix $n \times n$ and $f : \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ will be continuous functions. Consider the following spaces

$$\begin{aligned}
C \ &: \ = \{x : \mathbb{R} \to \mathbb{R}^n,\ x \text{ continuous and bounded}\} \\
C_0 \ &: \ = \{x : \mathbb{R} \to \mathbb{R},\ x \in C,\ x(+\infty) = x(-\infty) = 0\} \\
C_0^n \ &: \ = \left\{y : \mathbb{R} \to \mathbb{R},\ y \text{ of class } C^n,\ y^{(p)}(\pm\infty) = 0,\ p \in \overline{0, n-1}\right\} \\
P_0 \ &: \ = \{y : \mathbb{R} \to \mathbb{R},\ y \text{ continuous and admits an evanescent primitive}\}.
\end{aligned}$$

C is a Banach space with the norm

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} |x(t)|,$$

where $|\cdot|$ represents a norm in $\mathbb{R}^n$; C_0 is a subspace of C.

C_0^n endowed with the norm

$$\|y\|_n := \sum_{j=1}^{n} \sup_{t\in\mathbb{R}} \left|y^j(t)\right|,$$

it becomes a Banach space.

Notice that $C_0 \not\subset P_0$ and $P_0 \not\subset C_0$, as shown respectively by $p(t) = \frac{1}{1+t^2}$ and $p(t) = \mathrm{sgn}\ (t) \sin t^2$, $P(t) = \left(\int_0^t p(s)\ ds - \int_0^\infty \sin s^2\ ds\right)'$.

Denote by $(a_j)_{j\in\overline{1,n}}$ the eigenvalues of the matrix A.

Definition 1. *The matrix A is called* **nonresonant** *iff no a_j lies on the imaginary axis.*

This classical result is due to Perron (see e.g. [7], p. 150 and [8], p. 22).

Proposition 1. *Suppose that no eigenvalue of A lies on the imaginary axis (i.e. A is nonresonant). Then, for every $f \in C$, the equation*

$$\dot{x} = Ax + f(t) \tag{4}$$

has a unique bounded solution x. This solution satisfies the inequality

$$|x(t)| \le \frac{k \cdot \|f\|_\infty}{a}, \tag{5}$$

where $k > 0$ is a constant and $a \in]0, \min_{1\le j\le n} |a_j|]$. This solution is given by the equality

$$x(t) = \int_{-\infty}^{t} P_- e^{A(t-s)} f(s)\ ds - \int_{t}^{+\infty} P_+ e^{A(t-s)} f(s)\ ds, \tag{6}$$

where P_-, P_+ are two supplementary projectors in $\mathbb{R}^n$, commuting with A.

Furthermore,

$$\left|e^{At} P_- x\right| \le k e^{-at} |P_- x|,\ t \ge 0, \tag{7}$$

$$\left|e^{At} P_+ x\right| \le k e^{at} |P_+ x|,\ t \le 0, \tag{8}$$

with $k > 0$ and $x \in \mathbb{R}^n$.

In the special case of a nonhomogenous scalar linear differential equation,

$$L[y](t) = h(t), \tag{9}$$

associated to the linear differential operator with constant coefficients a_j,

$$L[y](t) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y, \tag{10}$$

the above result implies the existence of a unique bounded solution for every bounded function h, if and only if no zero of the **characteristic polynomial**

$$L(\lambda) := \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 \tag{11}$$

lies on the imaginary axis (i.e. L is **nonresonant**).

In this case, the estimate (5) can be written under the form

$$\|y\|_n \leq \frac{k\|h\|_\infty}{a}. \tag{12}$$

Remark 1. A **evanescent** solution for (9) is a solution y such that $y^{(j)}(\pm\infty) = 0$, $j \in \overline{0, n-1}$.

3. LINEAR EQUATIONS WITH CONTINUOUS FORCING TERM

The first result on the existence of a evanescent solution is the following.

Theorem 1. *Suppose that A is nonresonant. Then for every evanescent function f, the equation* (1) *has an unique evanescent solution and this solution satisfies* (5).

Proof. The unique bounded solution of (1) is given by (6). It remains to prove that this solution is evanescent.

By using (7) one gets

$$\left|\int_{-\infty}^{t} P_- e^{A(t-s)} f(s)\ ds\right| \leq ke^{-at}\int_{-\infty}^{t} e^{as}|P_- f(s)|\ ds. \tag{13}$$

The integral appearing in the right side of the inequality (13) is a nondecreasing real function; if in addition it is bounded, then the right side of (13) tends to 0 as $t \to +\infty$. If this integral is not bounded, then it will tend to $+\infty$ as $t \to +\infty$.

By L'Hospital rule one deduces

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{e^{at}}\int_{-\infty}^{t}e^{as}\left|P_{-}f\left(s\right)\right|\,ds &= \lim_{t\to+\infty}\frac{e^{at}\left|P_{-}f\left(t\right)\right|}{ae^{at}}= \\ &= \lim_{t\to+\infty}\frac{\left|P_{-}f\left(t\right)\right|}{a}=0.\end{aligned}$$

Analogously, from

$$\left|\int_{t}^{+\infty}P_{+}e^{A(t-s)}f\left(s\right)\,ds\right|\leq ke^{at}\int_{t}^{+\infty}e^{-as}\left|P_{+}f\left(s\right)\right|\,ds$$

one obtains

$$\lim_{t\to-\infty}\int_{t}^{+\infty}P_{+}e^{A(t-s)}f\left(s\right)\,ds=0.$$

Therefore, for x given by (6) one has

$$\lim_{t\to+\infty}x\left(t\right)=0.$$

A similar reasoning shows that

$$\lim_{t\to-\infty}x\left(t\right)=0,$$

which ends the proof. □

Corrolary 1. *If L is nonresonant, then for every evanescent h, the equation* (9) *has an unique evanescent solution y and this solution satisfies* (12) *.*

4. LINEAR EQUATIONS OF $n-$ ORDER WITH CONTINUOUS FORCING TERM

Another result is contained in the following theorem.

Theorem 2. *Suppose that $\lambda=0$ is a simple zero for* (11) *and* (11) *has no other zero on the imaginary axis. Then the equation* (9) *has an unique evanescent solution if and only if $h\in P_0$.*

Proof. If $n=1$, by our assumption $L\left[y\right]=y'$ and the result is obvious. Assume now that $n\geq 2$. We first prove the necessity. Let y be an evenescent solution for (9); by integrating the both members of (9) one gets

$$y^{(n-1)}\left(t\right)+...+a_1y\left(t\right)=\int_0^t h\left(s\right)\,ds+y^{(n-1)}\left(0\right)+...+a_1y\left(0\right), \qquad (14)$$

since, by our assumtion, $a_0 = 0$. Then the primitive of h

$$H(t) = \int_0^t h(s)\ ds + y^{(n-1)}(0) + ... + a_1 y(0)$$

is evenescent.

We now prove the sufficiency. Let $h \in P_0$ and consider the equation

$$z^{(n-1)} + a_{n-1} z^{(n-2)} + ... + a_1 z = H(t), \tag{15}$$

when H is the (unique) evanescent primitive of h.

By the assumptions, every zero of the characteristic polynomial associated to (15) has no zero real part and the equation (15) an unique solution z such that

$$z^{(j)}(\pm\infty) = 0,\ j \in \overline{0, n-2}. \tag{16}$$

But, from (15) it results that

$$z^{(n-1)}(\pm\infty) = 0.$$

By (15) one obtains

$$z^{(n)} + a_{n-1} z^{(n-1)} + ... + a_1 z' = h(t),$$

and the proof is complete. □

Corrolary 2. *The equation*

$$y''(t) + cy'(t) = h(t),\ c \neq 0 \tag{17}$$

has a unique evanescent solution if and only if $h \in P_0$.

Theorem 3. *Assume that* L *is nonresonant. Then the equation* (9) *has a evanescent solution if and only if* $h = \varphi + \psi$, *when* $\varphi \in P_0$ *and* $\psi \in C_0$.

Proof.

The necessity. Let y be a evanescent solution for (9). Then

$$\begin{aligned}\varphi(t) &= y^{(n)}(t),\\ \psi(t) &= a_{n-1} y^{(n-1)}(t) + ... + a_1 y'(t) + a_0 y(t).\end{aligned}$$

The sufficiency. Let

$$\Lambda := \frac{d^n}{dt^n} + b_{n-1}\frac{d^{n-1}}{dt^{n-1}} + ... + b_1 \frac{d}{dt}$$

be a linear differential operator satisfying the conditions of theorem 2; then the equation

$$\Lambda[y](t) = \varphi(t)$$

has an evenescent solution u. If one changes the variable

$$y := z + u,$$

then z is the unique evanescent solution of the equation

$$L[z](t) = \Lambda[u](t) - L[u](t) + \psi(t). \tag{18}$$

This last equation has an unique evanescent solution since the right side of (18) is a evanescent function. □

Corrolary 3. *Let $p \in C$ be given and let $b > 0$ and $c \neq 0$ or $b < 0$. Then the equation*

$$y'' + cy' + by = p(t)$$

has a evanescent solution if and only if $p \in P_0 + C_0$.

5. NONLINEAR PERTURBATION OF A NONRESONANT EQUATION

In [13] the problem of the existence of bounded solutions for an equation of type

$$L[y](t) = h(t, y(t)),$$

where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function and L is nonresonant, is considered.

In this section the problem of the existence of the evanescent solutions for the equation

$$L[y](t) = h\left(t, y(t), y'(t), ..., y^{(n-1)}(t)\right), \tag{19}$$

where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function and L is nonresonant, is considered; the method of proof will be different by the one used in [13].

Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Set

$$M := \sup_{\mathbb{R}^{n+1}} |h(t, y_1, ..., y_n)|, \ \rho = \frac{kM}{a}, \tag{20}$$

where k, a are the constants appearing in (12) and

$$B_\rho = \{(y_1, ..., y_n)\,,\ \ |y_i| \leq \rho\}\,.$$

Theorem 4. *Suppose that L is nonresonant and the limits*

$$\lim_{t\to\pm\infty} h\,(t, y_1, ..., y_n) = 0 \tag{21}$$

exist and are uniform on B_ρ. Then the equation (19) *admits at least a evanescent solution.*

Proof. Consider the Fréchet space

$$C_c := \{x : \mathbb{R} \to \mathbb{R}^n,\ x \text{ continuous}\}\,,$$

endowed with the seminorms family

$$|x|_m := \sup_{t\in[-m,m]} \{|x\,(t)|\}\,.$$

Set

$$S := \{x \in C_c,\ \ |x\,(t)| \leq \rho,\ t \in \mathbb{R}\}\,.$$

Obviously, S is a closed convex and bounded set in C_c. As usual, we transform the equation (9) under the form

$$\dot{x} = Ax + F\,(t, x)\,. \tag{22}$$

Define on S the operator $H : S \to C_c$ by the equality

$$(Hx)\,(t) := \int_{-\infty}^{t} P_- e^{A(t-s)} F\,(s, x\,(s))\,ds - \int_{t}^{+\infty} P_+ e^{A(t-s)} F\,(s, x\,(s))\,ds. \tag{23}$$

We shall apply to S the Schauder's fixed point theorem on the set $S \subset C_c$.
By Proposition 1, the boundedness of F and (20) it follows

$$HS \subset S, \tag{24}$$

which shows in addition that the family HS is uniformly bounded on the compacts of $\mathbb{R}$.

Since

$$y = Hx$$

and

$$\dot{y}(t) = Ay(t) + F(t, x(t)),$$

there results

$$|\dot{y}(t)| \leq \|A\| \rho + M$$

and so the family HS is equi-continuous on the compact of $\mathbb{R}$ (in fact on $\mathbb{R}$).

Finally, the continuity of H results from hypotheses in an elementary way.

Therefore, H admits at least a fixed point $x \in S$; since for this x one has

$$F(\cdot, x(\cdot)) \in C_0,$$

the conclusion of the theorem follows by Proposition 1. $\square$

Corrolary 4. *Let L be nonresonant, $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous and bounded function and $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous and evanescent and $p : \mathbb{R} \to \mathbb{R}$ be continuous. Then the equation*

$$L[y](t) + \alpha(t) g\left(y(t), y'(t), ..., y^{(n-1)}(t)\right) = p(t) \tag{25}$$

admits evanescent solutions if and only if $p \in P_0 + C_0$.

Proof. Let u be a evanescent solution for the equation

$$L[u](t) = p(t), \tag{26}$$

which exists from Theorem 3. Setting $y = z + u$, our problem is reduced to finding a evanescent solution for

$$L[z](t) = -\alpha(t) g\left(u(t) + z(t), u'(t) + z'(t), ..., u^{(n-1)}(t) + z^{(n-1)}(t)\right)$$

and the existence of such solution follows from Theorem 4. $\square$

Corollary 5. *Let $p, q : \mathbb{R} \to \mathbb{R}$ be two continuous functions and $\alpha : \mathbb{R} \to \mathbb{R}$ be a evanescent function. Let also $b, c \in \mathbb{R}$ be with $b < 0$ or $b > 0$ and $c \neq 0$. Then the equation*

$$y''(t) + by'(t) + cy(t) + \alpha(t) g(y(t)) = p(t) \tag{27}$$

admits evanescent solutions if and only if $p \in P_0 + C_0$.

A similar result can be obtained for the equation

$$L[y](t) + g(y(t)) = p(t), \tag{28}$$

or, more generally, for the equation (22), where $a \equiv 1$.

Theorem 5. *Suppose that:*
i) L *is nonresonant;*
ii) $g : \mathbb{R} \to \mathbb{R}$ *is a continuous and bounded function satisfying the conditions*

$$g(0) = 0 \tag{29}$$

$$|g(y_1) - g(y_2)| \leq L|y_1 - y_2|, \ (\forall)\ y_i \in \mathbb{R},\ |y_i| \leq \rho,\ i \in \overline{1,2}; \tag{30}$$

iii) $p : \mathbb{R} \to \mathbb{R}$ *is a function in* C_0;
iv) *the following inequality holds*

$$kL < a. \tag{31}$$

Then the equation (25) *admits a unique evanescent solution.*

The proof is reduced to an application of Banach's theorem to operator H on the closed ball in C_0 having the center in 0 and radius ρ. □

From this theorem it follows

Corollary 6. *Let* $b, c \in \mathbb{R}$ *satisfying the same conditions as in Corollary 5. If* (29), (30), (31) *are fulfilled and* $p : \mathbb{R} \to \mathbb{R}$ *is a continuous function, then the equation*

$$y''(t) + cy'(t) + by(t) + g(y(t)) = p(t)$$

admits evanescent solutions if and only if $p \in P_0 + C_0$.

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