

Spectrum of one dimensional p-Laplacian Operator with indefinite weight

A. Anane¹, O. Chakrone¹ and M. Moussa²

¹Département de mathématiques, Faculté des Sciences,
Université Mohamed I^{er}, Oujda. Maroc.
e-mail: anane@sciences.univ-oujda.ac.ma
chakrone@sciences.univ-oujda.ac.ma

²Département de mathématiques, Faculté des Sciences,
Université Ibn Tofail, Kénitra. Maroc.
e-mail: moussa@univ-ibntofail.ac.ma

Abstract

This paper is concerned with the nonlinear boundary eigenvalue problem

$$-(|u'|^{p-2}u')' = \lambda m|u|^{p-2}u \quad u \in I =]a, b[, \quad u(a) = u(b) = 0,$$

where $p > 1$, λ is a real parameter, m is an indefinite weight, and a, b are real numbers. We prove there exists a unique sequence of eigenvalues for this problem. Each eigenvalue is simple and verifies the strict monotonicity property with respect to the weight m and the domain I , the k -th eigenfunction, corresponding to the k -th eigenvalue, has exactly $k - 1$ zeros in (a, b) . At the end, we give a simple variational formulation of eigenvalues.

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1 Introduction

The spectrum of the p -Laplacian operator with indefinite weight is defined as the set $\sigma_p(-\Delta_p, m)$ of $\lambda := \lambda(m, I)$ for which there exists a nontrivial (weak) solution $u \in W_0^{1,p}(\Omega)$ of problem

$$(\mathcal{V}\mathcal{P}_{(m,\Omega)}) \quad \begin{cases} -\Delta_p u = \lambda m|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, Δ_p : is the p -Laplacian operator, defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, in a bounded domain $\Omega \subset \mathbb{R}^N$, and $m \in M(\Omega)$ is an indefinite weight, with

$$M(\Omega) := \{m \in L^\infty(\Omega) / \operatorname{meas}\{x \in \Omega, m(x) > 0\} \neq 0\}.$$

The values $\lambda(m, \Omega)$ for which there exists a nontrivial solution of $(\mathcal{V}\mathcal{P}_{(m, \Omega)})$ are called eigenvalues and the corresponding solutions are the eigenfunctions. We will denote $\sigma_p^+(-\Delta_p, m)$ the set of all positive eigenvalues, and by $\sigma_p^-(-\Delta_p, m)$ the set of negative eigenvalues.

For $p = 2$ ($\Delta_p = \Delta$ Laplacian operator) it is well known (cf [4, 7, 8]) that,

- $\sigma_2^+(-\Delta, m) = \{\mu_k(m, \Omega), k = 1, 2, \dots\}$, with $0 < \mu_1(m, \Omega) < \mu_2(m, \Omega) \leq \mu_3(m, \Omega) \leq \dots \rightarrow +\infty$, $\mu_k(m, \Omega)$ repeated according to its multiplicity.
- The k -th eigenfunction corresponding to $\mu_k(m, \Omega)$, has at most k nodal domains.
- The eigenvalues $\mu_k(m, \Omega)$, $k \geq 1$, verify the strict monotonicity property (SMP in brief), i.e if $m, m' \in M(\Omega)$, $m(x) \leq m'(x)$ a.e in Ω and $m(x) < m'(x)$ in some subset of nonzero measure, then $\mu_k(m, \Omega) > \mu_k(m', \Omega)$.
- Equivalence between the SMP and the unique continuation one.

For $p \neq 2$ (nonlinear problem), it is well known that the critical point theory of Ljusternik-Schnirelmann (cf [15]), provides a sequence of eigenvalues for those problems. Whether or not this sequence, denoted $\lambda_k(m, \Omega)$, constitutes the set of all eigenvalues is an open question when $N \geq 1$, $m \neq 1$, and $p \neq 2$. The principal results for the problem seems to be given in (cf [1, 2, 3, 5, 6, 9, 10, 11, 12, 13]), where is shown that there exists a sequence of eigenvalues of $(\mathcal{V}\mathcal{P}_{(m, \Omega)})$ given by,

$$\lambda_n(m, \Omega) = \inf_{K \in \mathcal{B}_n} \max_{v \in K} \frac{\int_{\Omega} |\nabla v|^p dx}{\int_{\Omega} m|v|^p dx} \quad (1)$$

$\mathcal{B}_n = \{K, \text{symmetrical compact, } 0 \notin K, \text{ and } \gamma(K) \geq n\}$, γ is the genus function, or equivalently,

$$\frac{1}{\lambda_n(m, \Omega)} = \sup_{K \in \mathcal{B}_n} \min_{v \in K} \frac{\int_{\Omega} m|v|^p dx}{\int_{\Omega} |\nabla v|^p dx} \quad (2)$$

which can be written simply,

$$\frac{1}{\lambda_n(m, \Omega)} = \sup_{K \in \mathcal{A}_n} \min_{v \in K} \int_{\Omega} m|v|^p dx \quad (3)$$

$\mathcal{A}_n = \{K \cap S, K \in \mathcal{B}_n\}$. S is the unit sphere of $W_0^{1,p}(\Omega)$ endowed with the usual norm ($\|v\|_{1,p}^p = \int_{\Omega} |\nabla v|^p dx$), the equation (2) is the generalized Rayleigh quotient for the problem $(\mathcal{V}\mathcal{P}_{(m, \Omega)})$. The sequence is ordered as $0 < \lambda_1(m, \Omega) < \lambda_2(m, \Omega) \leq \lambda_3(m, \Omega) \leq \dots \rightarrow +\infty$. The first eigenvalue $\lambda_1(m, \Omega)$ is of special importance. We give some of its properties which will be of interest for us (cf [1]). First, $\lambda_1(m, \Omega)$ is given by,

$$\frac{1}{\lambda_1(m, \Omega)} = \sup_{v \in S} \int_{\Omega} m|v|^p dx = \int_{\Omega} m|\phi_1|^p dx \quad (4)$$

$\phi_1 \in S$ is any eigenfunction corresponding to $\lambda_1(m, \Omega)$, for this reason $\lambda_1(m, \Omega)$ is called the principal eigenvalue, also we know that $\lambda_1(m, \Omega) > 0$, simple (i.e if v and u are two

eigenfunctions corresponding to $\lambda_1(m, \Omega)$ then $v = \alpha u$ for some $\alpha \in \mathbb{R}$), isolate (i.e there is no eigenvalue in $]0, a[$ for some $a > \lambda_1(m, \Omega)$), finally it is the unique eigenvalue which has an eigenfunction with constant sign. We denote $\phi_1(x)$ the positive eigenfunction corresponding to $\lambda_1(m, \Omega)$, $\phi_1(x)$ verifies the strong maximum principle (cf [17]), $\frac{\partial \phi_1}{\partial n}(x) < 0$, for x in $\partial\Omega$ satisfying the interior ball condition.

In [14] Otani considers the case $N = 1$, $m(x) \equiv 1$, and proves that $\sigma_p(-\Delta_p, 1) = \{\mu_k(m, I), k = 1, 2, \dots\}$, the sequence can be ordered as $0 < \mu_1(m, \Omega) < \mu_2(m, \Omega) < \mu_3(m, \Omega) < \dots \rightarrow +\infty$, the k -th eigenfunction has exactly $k - 1$ zeros in $I = (a, b)$. In [10] Elbert proved the same results in the case $N = 1$, $m(x) \geq 0$ and continuous, the author gives an asymptotic relation of eigenvalues.

In this paper we consider the general case, $N = 1$ and $m(x)$ can change sign and is not necessarily continuous. We prove that $\sigma_p^+(-\Delta_p, m) = \{\lambda_k(m, I), k = 1, 2, \dots\}$, the sequence can be ordered as $0 < \lambda_1(m, \Omega) < \lambda_2(m, \Omega) < \lambda_3(m, \Omega) < \dots \lambda_k(m, I) \rightarrow +\infty$ as $k \rightarrow +\infty$, the k -th eigenfunction has exactly $k - 1$ zeros in $I = (a, b)$. The eigenvalues verify the SMP with respect to the weight m and the domain I .

In the next section we denote by: $M(I) := \{m \in L^\infty(I) / meas\{x \in I, m(x) > 0\} \neq 0\}$, $m|_J$ the restriction of m on J for a subset J of I , $Z(u) = \{t \in I / u(t) = 0\}$, a nodal domain ω of u is a component of $I \setminus Z(u)$, where $(u, \lambda(m, I))$ is a solution of $(\mathcal{V}, \mathcal{P}_{(m, I)})$. $\tilde{u}|_\omega$ is the extension, by zero, on I of $u|_\omega$

2 Results and technical Lemmas

We first state our main results

Theorem 1 *Assume that $N=1$ ($\Omega =]a, b[= I$), $m \in M(I)$ such that $m \not\equiv 1$ and $p \neq 2$, we have*

1. *Every eigenfunction corresponding to the k -th eigenvalue $\lambda_k(m, I)$, has exactly $k-1$ zeros.*
2. *For every k , $\lambda_k(m, I)$ is simple and verifies the strict monotonicity property with respect to the weight m and the domain I .*
3. *$\sigma_p^+(-\Delta_p, m) = \{\lambda_k(m, I), k = 1, 2, \dots\}$, for any $m \in M(I)$. The eigenvalues are ordered as $0 < \lambda_1(m, \Omega) < \lambda_2(m, \Omega) < \lambda_3(m, \Omega) < \dots \lambda_k(m, I) \rightarrow +\infty$ as $k \rightarrow +\infty$.*

Corollary 1 *For any integer n , we have the simple variational formulation,*

$$\frac{1}{\lambda_n(m, I)} = \sup_{F \in \mathcal{F}_n} \min_{F \cap S} \int_a^b m|v|^p dx \quad (5)$$

$\mathcal{F}_n = \{F / F \text{ is a } n \text{ dimensional subspace of } W_0^{1,p}(I)\}$.

For the proof of Theorem 1 we need some technical Lemmas.

Lemma 1 Let $m, m' \in M(I)$, $m(x) \leq m'(x)$, then for any n , $\lambda_n(m', I) \leq \lambda_n(m, I)$

Proof Making use of equation (2), we obtain immediately $\lambda_n(m', I) \leq \lambda_n(m, I)$.

Lemma 2 Let $(u, \lambda(m, I))$ be a solution of $(\mathcal{V}\mathcal{P}_{(m,I)})$, $m \in M(I)$, then $m/\omega \in M(\omega)$ for any nodal domain ω of u .

Proof Let ω be a nodal domain of u and multiply $(\mathcal{V}\mathcal{P}_{(m,I)})$ by u/ω so that we obtain

$$0 < \int_{\omega} |u'|^p dx = \lambda(m, I) \int_{\omega} m |u|^p dx. \quad (6)$$

This completes the proof. ■

Lemma 3 The restriction of a solution $(u, \lambda(m, I))$ of problem $(\mathcal{V}\mathcal{P}_{(m,I)})$, on a nodal domain ω , is an eigenfunction of problem $(\mathcal{V}\mathcal{P}_{(m/\omega,\omega)})$, and we have $\lambda(m, I) = \lambda_1(m/\omega, \omega)$.

Proof Let $v \in W_0^{1,p}(\omega)$ and let \tilde{v} be the extension by zero of v on Ω . It is obvious that $\tilde{v} \in W_0^{1,p}(\Omega)$. Multiply $(\mathcal{V}\mathcal{P}_{(m,\Omega)})$ by \tilde{v}

$$\int_{\omega} |u'|^{p-2} u' v' dx = \lambda(m, I) \int_{\omega} m |u|^{p-2} u v dx \quad (7)$$

for all $v \in W_0^{1,p}(\omega)$. Hence the restriction of u in ω is a solution of problem $(\mathcal{V}\mathcal{P}_{(m/\omega,\omega)})$ with constant sign. We then have $\lambda(m, \Omega) = \lambda_1(m/\omega, \omega)$, which completes the proof. ■

Lemma 4 Each solution $(u, \lambda(m, I))$ of the problem $(\mathcal{V}\mathcal{P}_{(m,I)})$ has a finite number of zeros.

Proof This Lemma plays an essential role in our work. We start by showing that u has a finite number of nodal domains. Assume that there exists a sequence I_k , $k \geq 1$, of nodal domains (intervals), $I_i \cap I_j = \emptyset$ for $i \neq j$. We deduce by Lemmas 3 and 1, respectively, that

$$\lambda(m, I) = \lambda_1(m, I_k) \geq \lambda_1(C, I_k) = \frac{\lambda_1(1, I_k)}{C} = \frac{\lambda_1(1,]0, 1])}{C(\text{meas}(I_k))^p}, \quad (8)$$

where $C = \|m\|_{\infty}$.

From equation (8) we deduce $(\text{meas}(I_k)) \geq (\frac{\lambda_1(1,]0, 1])}{\lambda(m, I) C})^{\frac{1}{p}}$, for all k , so

$$\text{meas}(I) = \sum_{k \geq 1} (\text{meas}(I_k)) = +\infty.$$

This yields a contradiction.

Let $\{I_1, I_2, \dots, I_k\}$ be the nodal domains of u . Put $I_i =]a_i, b_i[$, where $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots < a_k < b_k \leq b$. It is clear that the restriction of u on $]a, b_1[$ is a nontrivial eigenfunction with constant sign corresponding to $\lambda(m, I)$. The maximum principle (cf [17]) yields $u(t) > 0$ for all $t \in]a, b_1[$, so $a = a_1$. By a similar argument we prove that $b_1 = a_2, b_2 = a_3, \dots, b_k = b$, which completes the proof. ■

Lemma 5 (cf [16]) Let u be a solution of problem $(\mathcal{V}\mathcal{P}_{(m,\Omega)})$ and $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ then $u \in C^{1,\alpha}(\Omega) \cap C^1(\bar{\Omega})$ for some $\alpha \in (0, 1)$.

3 Proof of main results

Proof of Theorem 1

For $n = 1$, we know that $\lambda_1(m, I)$ is simple, isolate and the corresponding eigenfunction has constant sign. Hence it has no zero in (a, b) and it remains to prove the SMP.

Proposition 1 $\lambda_1(m, I)$ verifies the strict monotonicity property with respect to weight m and the domain I . i.e If $m, m' \in M(I)$, $m(x) \leq m'(x)$ and $m(x) < m'(x)$ in some subset of I of nonzero measure then,

$$\lambda_1(m', I) < \lambda_1(m, I) \tag{9}$$

and, if J is a strict sub interval of I such that $m_{/J} \in M(J)$ then,

$$\lambda_1(m, I) < \lambda_1(m_{/J}, J). \tag{10}$$

Proof Let $m, m' \in M(I)$ as in Proposition 1 and recall that the principal eigenfunction $\phi_1 \in S$ corresponding to $\lambda_1(m, I)$ has no zero in I ; i.e $\phi_1(t) \neq 0$ for all $t \in I$. By (3), we get

$$\frac{1}{\lambda_1(m, I)} = \int_I m |\phi_1|^p dx < \int_I m' |\phi_1|^p dx \leq \sup_{v \in S} \int_I m' |v|^p dx = \frac{1}{\lambda_1(m', I)}. \tag{11}$$

Then inequality (9) is proved. To prove inequality (10), let J be a strict sub interval of I and $m_{/J} \in M(J)$. Let $u_1 \in S$ be the (principal) positive eigenfunction of $(\mathcal{V}, \mathcal{P}_{(m, J)})$ corresponding to $\lambda_1(m_{/J}, J)$, and denote by \tilde{u}_1 the extension by zero on I . Then

$$\frac{1}{\lambda_1(m_{/J}, J)} = \int_J m |u_1|^p dx = \int_I m |\tilde{u}_1|^p dx < \sup_{v \in S} \int_I m |v|^p dx = \frac{1}{\lambda_1(m, I)}. \tag{12}$$

The last strict inequality holds from the fact that \tilde{u}_1 vanishes in I/J so can't be an eigenfunction corresponding to the principal eigenvalue $\lambda_1(m, I)$. ■

For $n = 2$ we start by proving that $\lambda_2(m, I)$ has a unique zero in (a, b) .

Proposition 2 There exists a unique real $c_{2,1} \in I$, for which we have $Z(u) = \{c_{2,1}\}$ for any eigenfunction u corresponding to $\lambda_2(m, I)$. For this reason, we will say that $c_{2,1}$ is the zero of $\lambda_2(m, I)$.

Proof Let u be an eigenfunction corresponding to $\lambda_2(m, I)$. u changes sign in I . Consider $I_1 =]a, c[$ and $I_2 =]c', b[$ two nodal domains of u , by Lemma 3, $\lambda_1(m_{/I_1}, I_1) = \lambda_2(m, I) = \lambda_1(m_{/I_2}, I_2)$. Assume that $c < c'$, choose $d \in]c, c'[$ and put $J_1 =]a, d[$, $J_2 =]d, b[$; hence $J_1 \cap J_2 = \emptyset$, and for $i = 1, 2$, $I_i \subset J_i$ strictly, and $m_{/J_i} \in M(J_i)$. Making use of Lemma 3, by (10), we get

$$\lambda_1(m_{/J_1}, J_1) < \lambda_1(m_{/I_1}, I_1) = \lambda_2(m, I) \tag{13}$$

and

$$\lambda_1(m_{/J_2}, J_2) < \lambda_1(m_{/I_2}, I_2) = \lambda_2(m, I). \tag{14}$$

Let $\phi_i \in S$ be an eigenfunction corresponding to $\lambda_1(m_{/J_i}, J_i)$, by (4) we have for $i = 1, 2$

$$\frac{1}{\lambda_1(m, J_i)} = \int_{J_i} m |\phi_i|^p dx \tag{15}$$

$\tilde{\phi}_i$ is the extension by zero of ϕ_i on I . Consider the two dimensional subspace $F = \langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle$ and put $K_2 = F \cap S \subset W_0^{1,p}(I)$. Obviously $\gamma(K_2) = 2$ and we remark that for $v = \alpha\tilde{\phi}_1 + \beta\tilde{\phi}_2$, $\|v\|_{1,p} = 1 \iff |\alpha|^p + |\beta|^p = 1$. Hence by (3), (13), (14) and (15) we obtain,

$$\begin{aligned} \frac{1}{\lambda_2(m,I)} &\geq \min_{v \in K_2} \int_I m|v|^p dx \\ &= \min_{v = \alpha\tilde{\phi}_1 + \beta\tilde{\phi}_2 \in K_2} (|\alpha|^p \int_{J_1} m|\phi_1|^p + |\beta|^p \int_{J_2} m|\phi_2|^p) \\ &= |\alpha_0|^p \int_{J_1} m|\phi_1|^p + |\beta_0|^p \int_{J_2} m|\phi_2|^p \\ &= \frac{|\alpha_0|^p}{\lambda_1(m/J_1, J_1)} + \frac{|\beta_0|^p}{\lambda_1(m/J_2, J_2)} \\ &> \frac{|\alpha_0|^p + |\beta_0|^p}{\lambda_2(m,I)} \\ &= \frac{1}{\lambda_2(m,I)}, \end{aligned}$$

a contradiction; hence $c = c'$. On the other hand, let v be another eigenfunction corresponding to $\lambda_2(m, I)$. Denote by d its unique zero in (a, b) . Assume, for example, that $c < d$. By Lemma 3 and relation (10), we get

$$\lambda_2(m, I) = \lambda_1(m/J_{[a,d]},]a, d]) < \lambda_1(m/J_{[a,c]},]a, c]) = \lambda_2(m, I). \quad (16)$$

This is a contradiction so $c = d$. We have proved that every eigenfunction corresponding to $\lambda_2(m, I)$ has one, and only one, zero in (a, b) , and that the zero is the same for all eigenfunctions, which completes the proof of the Proposition. ■

Lemma 6 $\lambda_2(m, I)$ is simple, hence $\lambda_2(m, I) < \lambda_3(m, I)$.

Proof Let u and v be two eigenfunctions corresponding to $\lambda_2(m, I)$. By Lemma 3 the restrictions of u and v on $]a, c_{2,1}[$ and $]c_{2,1}, b[$ are eigenfunctions corresponding to $\lambda_1(m/J_{]a, c_{2,1}[},]a, c_{2,1}[)$ and $\lambda_1(m/J_{]c_{2,1}, b[},]c_{2,1}, b[)$, respectively. Making use of the simplicity of the first eigenvalue, we get $u = \alpha v$ in $]a, c_{2,1}[$ and $u = \beta v$ in $]c_{2,1}, b[$. But both of u and v are eigenfunctions, so then by Lemma 5, there are $C^1(I)$. The maximum principle (cf [17]) tell us that $u'(c_{2,1}) \neq 0$, so $\alpha = \beta$. Finally, by the simplicity of $\lambda_2(m, I)$ and the theorem of multiplicity (cf [15]) we conclude that $\lambda_2(m, I) < \lambda_3(m, I)$. ■

Proposition 3 $\lambda_2(m, I)$ verifies the SMP with respect to the weight m and the domain I .

Proof Let $m, m' \in M(I)$ such that, $m(x) \leq m'(x)$ a.e in I and $m(x) < m'(x)$ in some subset of nonzero measure. Let $c_{2,1}$ and $c'_{2,1}$ be the zeros of $\lambda_2(m, I)$ and $\lambda_2(m', I)$ respectively. We distinguish three cases :

1. $c_{2,1} = c'_{2,1} = c$, then $meas(\{x \in I / m(x) < m'(x) \cap]a, c[\}) \neq 0$, or $meas(\{x \in I / m(x) < m'(x) \cap]c, b[\}) \neq 0$, by Lemma 3 and (9) we obtain

$$\lambda_2(m', I) = \lambda_1(m'_{/]a, c[},]a, c]) < \lambda_1(m_{/]a, c[},]a, c]) = \lambda_2(m, I), \quad (17)$$

or

$$\lambda_2(m', I) = \lambda_1(m'_{/]c, b[},]c, b]) < \lambda_1(m_{/]c, b[},]c, b]) = \lambda_2(m, I). \quad (18)$$

2. $c_{2,1} < c'_{2,1}$, by Lemmas 1, 3 and (10), we get

$$\begin{aligned} \lambda_2(m', I) = \lambda_1(m'_{/]a, c'_{2,1}[},]a, c'_{2,1}[) &\leq \lambda_1(m_{/]a, c'_{2,1}[},]a, c'_{2,1}[) \\ &< \lambda_1(m_{/]a, c_{2,1}[},]a, c_{2,1}[) = \lambda_2(m, I). \end{aligned} \quad (19)$$

3. $c'_{2,1} < c_{2,1}$, as before, by Lemmas 1, 3 and (10), we have

$$\begin{aligned} \lambda_2(m', I) = \lambda_1(m'_{/]c'_{2,1}, b[},]c'_{2,1}, b[) &\leq \lambda_1(m_{/]c'_{2,1}, b[},]c'_{2,1}, b[) \\ &< \lambda_1(m_{/]c_{2,1}, b[},]c_{2,1}, b[) = \lambda_2(m, I). \end{aligned} \quad (20)$$

For the SMP with respect to the domain, put $J =]c, d[$ a strict sub interval of I with $m_{/J} \in M(J)$, and denote $c'_{2,1}$ the zero of $\lambda_2(m_{/J}, J)$. As in the SMP with respect to the weight, three cases are distinguished:

1. $c_{2,1} = c'_{2,1} = l$, then $]c, l[$ is a strict sub interval of $]a, l[$ or $]l, d[$ is a strict sub interval of $]l, b[$. By Lemma 3 and (10), we get

$$\lambda_2(m, I) = \lambda_1(m_{/]a, l[},]a, l[) < \lambda_1(m_{/]c, l[},]c, l[) = \lambda_2(m_{/J}, J) \quad (21)$$

or

$$\lambda_2(m, I) = \lambda_1(m_{/]l, b[},]l, b[) < \lambda_1(m_{/]l, d[},]l, d[) = \lambda_2(m_{/J}, J). \quad (22)$$

2. $c_{2,1} < c'_{2,1}$, again by Lemma 3 and (10), we obtain

$$\lambda_2(m, I) = \lambda_1(m_{/]c_{2,1}, b[},]c_{2,1}, b[) < \lambda_1(m_{/]c'_{2,1}, b[},]c'_{2,1}, b[) = \lambda_2(m_{/J}, J). \quad (23)$$

3. $c'_{2,1} < c_{2,1}$, for the same reason as in the last case, we get

$$\lambda_2(m, I) = \lambda_1(m_{/]a, c_{2,1}[},]a, c_{2,1}[) < \lambda_1(m_{/]c, c'_{2,1}[},]c, c'_{2,1}[) = \lambda_2(m_{/J}, J). \quad (24)$$

The proof is complete. ■

Lemma 7 *If any eigenfunction u corresponding to some eigenvalue $\lambda(m, I)$ is such that $Z(u) = \{c\}$ for some real number c , then $\lambda(m, I) = \lambda_2(m, I)$.*

Proof We shall prove that $c = c_{2,1}$. Assume, for example, that $c < c_{2,1}$. By Lemma 1 and (10) we get

$$\lambda(m, I) = \lambda_1(m_{/]c, b[},]c, b[) < \lambda_1(m_{/]c_{2,1}, b[},]c_{2,1}, b[) = \lambda_2(m, I). \quad (25)$$

On the other hand,

$$\lambda_2(m, I) = \lambda_1(m_{/]a, c_{2,1}[},]a, c_{2,1}[) < \lambda_1(m_{/]a, c[},]a, c[) = \lambda(m, I), \quad (26)$$

a contradiction. Hence, $c = c_{2,1}$ and $\lambda(m, I) = \lambda_2(m, I)$. The proof is complete. ■

For $n > 2$, we use a recurrence argument. Assume that, for any k , $1 \leq k \leq n$, that the following hypothesis:

1. **H.R.1** For any eigenfunction u corresponding to the k -th eigenvalue $\lambda_k(m, I)$, there exists a unique $c_{k,i}$, $1 \leq i \leq k - 1$, such that $Z(u) = \{c_{k,i}, 1 \leq i \leq k - 1\}$.
2. **H.R.2** $\lambda_k(m, I)$ is simple.
3. **H.R.3** $\lambda_1(m, I) < \lambda_2(m, I) < \dots < \lambda_{n+1}(m, I)$.
4. **H.R.4** If $(u, \lambda(m, I))$ is a solution of $(\mathcal{V}, \mathcal{P}_{(m, I)})$ such that $Z(u) = \{c_i, 1 \leq i \leq k-1\}$, then $\lambda(m, I) = \lambda_k(m, I)$.
5. **H.R.5** $\lambda_k(m, I)$ verifies the SMP with respect to the weight m and the domain I .

Holds, and prove them for $n + 1$.

Proposition 4 *There exists a unique family $\{c_{n+1,i}, 1 \leq i \leq n\}$ such that $Z(u) = \{c_{n+1,i}, 1 \leq i \leq n\}$, for any eigenfunction u corresponding to $\lambda_{n+1}(m, I)$.*

Proof Let u be an eigenfunction corresponding to $\lambda_{n+1}(m, I)$. By H.R.3 and H.R.4, u has at least n zeros. According to Lemma 4, we can consider the $n + 1$ nodal domains of u , $I_1 =]a, c_1[$, $I_2 =]c_1, c_2[$, \dots , $I_n =]c_{n-1}, c_n[$, $I_{n+1} =]c, b[$. We shall prove that $c = c_n$. Remark that the restrictions of u on $]a, c_{i+1}[$, $1 \leq i \leq n - 1$, are eigenfunctions with i zeros, by H.R.4 $\lambda_{n+1}(m, I) = \lambda_i(m_{/]a, c_{i+1}[},]a, c_{i+1}[)$. Assume that $c_n < c$, choose d in $]c_n, c[$ and put, $J_1 =]a, d[$, $J_2 =]d, b[$. Remark that $J_1 \cap J_2 = \emptyset$, $]a, c_n[$ is a strict sub interval of $J_1 \subset I$, and $]c, b[$ is a strict sub interval of $J_2 \subset I$. It is clear that $m_{/J_i} \in M(J_i)$ for $i = 1, 2$, by H.R.4 and H.R.5. We have

$$\lambda_n(m_{/J_1}, J_1) < \lambda_n(m_{/]a, c_n[},]a, c_n]) = \lambda_{n+1}(m, I), \quad (27)$$

and

$$\lambda_1(m_{/J_2}, J_2) < \lambda_1(m_{/]c, b[},]c, b]) = \lambda_{n+1}(m, I). \quad (28)$$

Denote by $(\phi_{n+1}, \lambda_1(m_{/J_2}, J_2))$ a solution of $(\mathcal{V}, \mathcal{P}_{(m, J_2)})$, $(v, \lambda_n(m_{/J_1}, J_1))$ a solution of $(\mathcal{V}, \mathcal{P}_{(m, J_1)})$, ϕ_i , $1 \leq i \leq n$ the restrictions of v on I_i , and $\tilde{\phi}_i$, $1 \leq i \leq n$, their extensions, by zero, on I . Let $F_{n+1} = \langle \tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_{n+1} \rangle$ and $K_{n+1} = F_{n+1} \cap S$, then $\gamma(K_{n+1}) = n + 1$. We obtain by (3) and the same proof as in Proposition 2

$$\frac{1}{\lambda_{n+1}(m, I)} \geq \min_{K_{n+1}} \int_I m|v|^p dx > \frac{1}{\lambda_{n+1}(m, I)}, \quad (29)$$

a contradiction, so $c = c_n$. On the other hand, let v be an eigenfunction corresponding to $\lambda_{n+1}(m, I)$. Denote by d_1, d_2, \dots, d_n the zeros of v . If $d_1 \neq c_1$, then $\lambda_{n+1}(m, I) = \lambda_1(m_{/]a, d_1[},]a, d_1]) \neq \lambda_1(m_{/]a, c_1[},]a, c_1]) = \lambda_{n+1}(m, I)$, so $d_1 = c_1$, by the same argument we conclude that $d_i = c_i$ for all $1 \leq i \leq n$. ■

Lemma 8 $\lambda_{n+1}(m, I)$ is simple, hence. $\lambda_{n+1}(m, I) < \lambda_{n+2}(m, I)$.

Proof Let u and v be two eigenfunctions corresponding to $\lambda_{n+1}(m, I)$. The restrictions of u and v on $]a, c_{n+1,1}[$ and $]c_{n+1,1}, b[$ respectively, are eigenfunctions corresponding to $\lambda_1(m_{/]a, c_{n+1,1}[},]a, c_{n+1,1}[)$ and $\lambda_n(m_{/]c_{n+1,1}, b[},]c_{n+1,1}, b[)$. By H.R.2 and H.R.4 we have $u = \alpha v$ in $]a, c_{n+1,1}[$ and $u = \beta v$ in $]c_{n+1,1}, b[$. On the other hand, u and v are $C^1(I)$ and $u'(c_{n+1,1}) \neq 0$, so $\alpha = \beta$. From the simplicity of $\lambda_{n+1}(m, I)$ and theorem of multiplicity we conclude that $\lambda_{n+1}(m, I) < \lambda_{n+2}(m, I)$. ■

Proposition 5 $\lambda_{n+1}(m, I)$ verifies the SMP with respect to the weight m and the domain I .

Proof Let $m, m' \in M(I)$, such that $m(x) \leq m'(x)$ with $m(x) < m'(x)$ in some subset of nonzero measure and $(c'_{n+1,i})_{1 \leq i \leq n}$ the zeros of $\lambda_{n+1}(m')$ three cases are distinguished,

1. $c_{n+1,1} = c'_{n+1,1} = c$, one of the subsets is of nonzero measure,

$$\{x \in I / m(x) < m'(x)\} \cap]a, c[\quad \text{and} \quad \{x \in I / m(x) < m'(x)\} \cap]c, b[.$$

By Lemma 3 and (9), we have

$$\lambda_{n+1}(m', I) = \lambda_1(m'_{/]a, c[},]a, c[) < \lambda_1(m_{/]a, c[},]a, c[) = \lambda_{n+1}(m, I) \quad (30)$$

or

$$\lambda_{n+1}(m', I) = \lambda_n(m'_{/]c, b[},]c, b[) < \lambda_n(m_{/]c, b[},]c, b[) = \lambda_{n+1}(m, I). \quad (31)$$

2. $c_{n+1,1} < c'_{n+1,1}$, by Lemmas 1, 3 and (10) we have

$$\begin{aligned} \lambda_{n+1}(m', I) &= \lambda_1(m'_{/]a, c'_{n+1,1}[},]a, c'_{n+1,1}[) \\ &\leq \lambda_1(m_{/]a, c'_{n+1,1}[},]a, c'_{n+1,1}[) \\ &< \lambda_1(m_{/]a, c_{n+1,1}[},]a, c_{n+1,1}[) = \lambda_{n+1}(m, I). \end{aligned} \quad (32)$$

3. $c'_{n,1} < c_{n,1}$, from the same reason as before, we get

$$\begin{aligned} \lambda_{n+1}(m', I) &= \lambda_n(m'_{/]c'_{n+1,1}, b[},]c'_{n+1,1}, b[) \\ &\leq \lambda_n(m_{/]c'_{n+1,1}, b[},]c'_{n+1,1}, b[) \\ &< \lambda_n(m_{/]c_{n+1,1}, b[},]c_{n+1,1}, b[) = \lambda_{n+1}(m, I). \end{aligned} \quad (33)$$

By similar argument as in proof of Proposition 3, we prove the SMP with respect to the domain I . ■

Lemma 9 If $(u, \lambda(m, I))$ is a solution of $(\mathcal{V}\mathcal{P}_{(m, I)})$ such that $Z(u) = \{d_1, d_2, \dots, d_n\}$, then $\lambda(m, I) = \lambda_{n+1}(m, I)$.

Proof It is sufficient to prove that $d_i = c_{n+1,i}$ for all $1 \leq i \leq n$. If $c_{n+1,1} < d_1$ then, by Lemma 1, (10), H.R.4 and H.R.5,

$$\begin{aligned} \lambda(m, I) = \lambda_1(m_{/]a, d_1[},]a, d_1[) &< \lambda_1(m_{/]a, c_{n+1,1}[},]a, c_{n+1,1}[) \\ &= \lambda_{n+1}(m, I) \\ &= \lambda_n(m_{/]c_{n+1,1}, b[},]c_{n+1,1}, b[) \\ &< \lambda_n(m_{/]d_1, b[},]d_1, b[) \\ &= \lambda(m, I), \end{aligned} \quad (34)$$

a contradiction. If $d_1 < c_{n+1,1}$ then, by Lemma 1, (10), H.R.4 and H.R.5,

$$\begin{aligned} \lambda_{n+1}(m, I) = \lambda_1(m_{/]a, c_{n+1}[},]a, c_{n+1}[) &< \lambda_1(m_{/]a, d_1[},]a, d_1[) \\ &= \lambda(m, I) \\ &= \lambda_n(m_{/]d_1, b[},]d_1, b[) \\ &< \lambda_n(m_{/]c_{n+1,1}, b[},]c_{n+1,1}, b[) \\ &= \lambda_{n+1}(m, I), \end{aligned} \tag{35}$$

a contradiction. The proof is then complete, which completes the proof of Theorem 1. ■

Proof of Corollary 1. Since for $F \in \mathcal{F}_n$, the compact $F \cap S \in \mathcal{A}_n$, by (3) we have:

$$\sup_{F \in \mathcal{F}_n} \min_{v \in F \cap S} \int_{\Omega} m|v|^p dx \leq \frac{1}{\lambda_n(m, \Omega)}. \tag{36}$$

On the other hand, for a n dimensional subspace F of $W_0^{1,p}(I)$, the compact set $K = F \cap S \in \mathcal{A}_n$. Let u be an eigenfunction corresponding to $\lambda_n(m, I)$ and put

$$F = \langle \tilde{\phi}_1(]a, c_{n,1}[), \tilde{\phi}_1(]c_{n,1}, c_{n,2}[), \dots, \tilde{\phi}_1(]c_{n,1,n}, b[) \rangle,$$

to conclude $F \cap S \in \mathcal{A}_n$. By an elementary computation as in Proposition 2, one can show that

$$\frac{1}{\lambda_n(m, I)} = \min_{F \cap S} \int_I m|v|^p dx. \tag{37}$$

Then combine (36) with (37) to get (5). Which completes the proof. ■

3.1 Remark

The spectrum of p -Laplacian, with indefinite weight, in one dimension, is entirely determined by the sequence $(\lambda_n(m, I))_{n \geq 1}$ if $m(x) \geq 0$ a.e in I . Therefore, if $m(x) < 0$ in some subset $J \subset I$ of nonzero measure, replace m by $-m$; by Theorem 1, since $-m \in M(I)$ we conclude that, the negative spectrum $\sigma_p^-(-\Delta_p, m) = -\sigma_p^+(-\Delta_p, -m)$ of this operator is constituted by a sequence of eigenvalues $\lambda_{-n}(m, I) = -\lambda_n(-m, I)$. Thus the spectrum of the operator is,

$$\sigma_p(-\Delta_p, m) = \sigma_p^+(-\Delta_p, m) \cup \sigma_p^-(-\Delta_p, m).$$

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