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MULTIPLICITY RESULTS FOR A CLASS OF $p(x)$-KIRCHHOFF TYPE EQUATIONS WITH COMBINED NONLINEARITIES

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ABSTRACT. Using the mountain pass theorem combined with the Ekeland variational principle, we obtain at least two distinct, non-trivial weak solutions for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities. We also show that the similar results can be obtained in the case when the domain has cylindrical symmetry.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded regular domain. In this paper, we are interested in the multiplicity of solutions for the $p(x)$-Kirchhoff type equation

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $p \in C_+(\overline{\Omega})$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, satisfying some certain conditions, λ is a parameter.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883, see [15]. This equation is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, elliptic problems involving p-Kirchhoff type operators have been studied in many papers, we refer to [2, 3, 4, 17, 18, 21, 22], in which the authors have used different methods to get the existence of solutions for (1.1) in the case when $p(x) = p$ is a constant.

If $p : \overline{\Omega} \to \mathbb{R}$ is a continuous function, problem (1.1) has been firstly studied by variational methods in [7, 8]. The $p(x)$-Laplacian possesses more complicated nonlinearities than

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p-Laplacian, for example it is not homogeneous. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. In [7], the authors studied problem (1.1) in the special case $M(t) = a + bt$. By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, they established in [7] the existence of infinitely many distinct positive solutions whose $W^{1,p(x)}(\Omega)$-norms and L^∞-norms tend to zero under suitable hypotheses about nonlinearity. In [8], the authors considered the problem in the case when $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous and non-descreasing function, satisfying the following conditions:

($\mathbf{M_1'}$) There exists $M_0 > 0$ such that $M(t) \geq M_0$ for all $t \geq 0$;

($\mathbf{M_2'}$) There exists $\theta \in (0, 1)$ such that $\widehat{M}(t) \geq (1 - \theta)M(t)t$ for all $t \geq 0$, where $\widehat{M}(t) = \int_0^t M(s)ds$.

Regarding the nonlinearity, they required f to verify the Ambrosetti-Rabinowitz type condition, i.e., there exist $\mu > \frac{p^+}{1-\theta}$ and $T > 0$ such that

$$0 < \mu F(x, t) \leq f(x, t)t \text{ for all } t \geq T \text{ and a.e. } x \in \Omega. \tag{1.3}$$

Using the mountain pass theorem in [1], the authors obtained at least one weak solution for (1.1). Also in [8], the authors considered the case when f verifies the condition

$$f(x, t) \geq C|t|^{\gamma(x)-1}, \quad t \to 0, \tag{1.4}$$

where $p^+ < \gamma^- \leq \gamma^+ < \frac{p^-}{1-\theta}$. Using the fountain theorem and the dual fountain theorem, the authors obtained a sequence of weak solutions $\{\pm u_m\}$ with negative energy. In [5, 6], the authors studied the multiplicity of solutions for problem (1.1) using the condition ($\mathbf{M_1'}$) and the three critical points theorem by B. Ricceri.

Motivated by the papers [7, 8] and the ideas introduced in [10], the goal of this paper is to study the multiplicity of weak solutions for problem (1.1) with combined nonlinearities. More exactly, we assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

($\mathbf{M_1}$) There exist $m_2 \geq m_1 > 0$, $\delta_2 \geq \delta_1 > 1$ such that

$$m_1 t^{\delta_1 - 1} \leq M(t) \leq m_2 t^{\delta_2 - 1}$$

for all $t \in \mathbb{R}^+$;

($\mathbf{M_2}$) For all $t \in \mathbb{R}^+$, it holds that

$$\widehat{M}(t) \geq M(t)t$$

where $\widehat{M}(t) = \int_0^t M(s)ds$.

Using the mountain pass theorem and the Ekeland variational principle, we prove that problem (1.1) has at least two distinct, non-trivial weak solutions under suitable conditions on the nonlinear term f. It should be noticed that we do not require the condition $(\mathbf{M_1'})$ as in [5, 6, 7, 8], for example $(\mathbf{M_1'})$ is not satisfied when $M(t) = t^{\delta-1}$ for $\delta > 1$, $t > 0$. We also show that the similar result can be established in the case when the domain Ω has cylindrical symmetry. This comes from the ideas introduced by W. Wang [23], and developed by J. Gao et al. [13, 14]. Due to the special structure of the domain, we can get the solutions of problem (1.1) with critical and supercritical gowth. In this situation, problem (1.1) is called a Hénon type problem.

In this paper, we consider the problem (1.1) in the particular case

$$f(x,u) = \lambda\Big(a(x)|u|^{\alpha(x)-2}u + b(x)|u|^{\beta(x)-2}u\Big),$$

where $p, \alpha, \beta \in C(\overline{\Omega})$ with

$$1 < \alpha^- \leq \alpha^+ < \delta_1 p^- < \delta_2 p^+ < \beta^- \leq \beta^+ < \min\left\{N, \frac{Np^-}{N-p^-}\right\}, \tag{1.5}$$

with δ_1, δ_2 are given by the hypothesis $(\mathbf{M_1})$ and the following conditions hold:

- $(\mathbf{A})$ $a : \overline{\Omega} \to \mathbb{R}$, satisfies $a \in L^{\alpha_0(x)}(\Omega)$ and $\alpha_0 \in C_+(\overline{\Omega})$, such that $\frac{Np(x)}{Np(x)-\alpha(x)(N-p(x))} < \alpha_0(x) < \frac{p(x)}{p(x)-\alpha(x)}$ for all $x \in \overline{\Omega}$;
- $(\mathbf{B})$ $b : \overline{\Omega} \to \mathbb{R}$, satisfies $b \in L^{\beta_0(x)}(\Omega)$ and $\beta_0 \in C_+(\overline{\Omega})$, such that $\frac{p(x)}{p(x)-\beta(x)} < \beta_0(x) < \frac{p(x)}{Np(x)-\beta(x)(N-p(x))}$ for all $x \in \overline{\Omega}$.

Then problem (1.1) becomes

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda\Big(a(x)|u|^{\alpha(x)-2}u + b(x)|u|^{\beta(x)-2}u\Big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Definition 1.1. We say that $u \in X = W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1.6) if and only if

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx - \lambda\int_\Omega a(x)|u|^{\alpha(x)-2}uvdx - \lambda\int_\Omega b(x)|u|^{\beta(x)-2}uvdx = 0$$

for any $v \in X$.

The first result of this paper can be described as follows.

Theorem 1.2. *Assume that the conditions (1.5) and* $(\mathbf{M_1})$-$(\mathbf{M_2})$, $(\mathbf{A})$, $(\mathbf{B})$ *are satisfied, then there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *problem (1.6) has at least two distinct, non-trivial weak solutions.*

Next, we consider the domain $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$, $\Omega_1 \subset \mathbb{R}^m$ $(m \geq 1)$ a bounded regular domain, and $\Omega_2 \subset \mathbb{R}^k$ $(k \geq 2)$ a ball of radius R, centered at the origin. In this case, we assume that $c : \overline{\Omega} \to \mathbb{R}$ is a non-negative Hölder continuous function, satisfying the following conditions

$(\mathbf{C_1})$ $c : \Omega \to \mathbb{R}$ is radially symmetric with respect to $x_2 \in \Omega_2$, and satisfying $c(x_1, 0) = 0$;
$(\mathbf{C_2})$ $l_c > 0$, where

$$l_c = \sup \left\{ \lambda > 0 : \quad \frac{|c(x)|}{|x_2|^\lambda} < \infty, \quad x \in \Omega \right\}.$$

More precisely, we consider the following $p(x)$-Kirchhoff type problem

$$\begin{cases} -M \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) = \lambda \Big(a(x)|u|^{\alpha(x)-2} u + c(x)|u|^{\gamma(x)-2} u \Big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where the function c verifies the conditions $(\mathbf{C_1})$ and $(\mathbf{C_2})$, and $p, a, \alpha, \gamma : \overline{\Omega} \to \mathbb{R}$ are continuous functions, $p, \gamma \in S(\Omega) := \{u : \overline{\Omega} \to \mathbb{R} : u$ is real measurable function and $u(x_1, x_2) = u(x_1, |x_2|)\}$, satisfying

$$1 < \alpha^- \leq \alpha^+ < \delta_1 p^- < \delta_2 p^+ < \gamma^- \leq \gamma^+ < \min \left\{ N, \frac{Np^-}{N - p^-} \right\} + \tau, \tag{1.8}$$

with δ_1, δ_2 are given by the hypothesis $(\mathbf{M_1})$ and τ is a positive real number defined by Proposition 2.4.

Due to the cylindrical symmetry of the domain Ω, we can deal with problem (1.1) in the supercritical case. To this purpose, we introduce the following space

$$X_s = W_{0,s}^{1,p(x)}(\Omega) = W_0^{1,p(x)}(\Omega) \cap S(\Omega) = \left\{ u \in W_0^{1,p(x)}(\Omega) : \quad u(x_1, x_2) = u(x_1, |x_2|) \right\},$$

which is a closed subspace of $W_0^{1,p(x)}(\Omega)$.

Definition 1.3. We say that $u \in X_s$ is a weak solution of problem (1.7) if and only if

$$M \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx - \lambda \int_\Omega a(x)|u|^{\alpha(x)-2} uv dx - \lambda \int_\Omega c(x)|u|^{\gamma(x)-2} uv dx = 0$$

for any $v \in X_s$.

Our result concerning problem (1.7) can be described as follows.

Theorem 1.4. *Assume that the conditions (1.8) and* $(\mathbf{M_1})$-$(\mathbf{M_2})$*,* $(\mathbf{A})$*,* $(\mathbf{C_1})$*,* $(\mathbf{C_2})$ *are satisfied, then there exists* $\lambda^{**} > 0$ *such that for any* $\lambda \in (0, \lambda^{**})$*, problem (1.7) has at least two distinct, non-trivial weak solutions.*

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book of Musielak [20] and the papers of Kováčik and Rákosník [16] and Fan et al. [11, 12]. Set

$$C_+(\overline{\Omega}) := \{h : \quad h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \text{ and } h^- = \inf_{x \in \Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \text{ a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \quad \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\Big| \int_\Omega uv dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{(p')^-} \Big) |u|_{p(x)} |v|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} \, dx.$$

Proposition 2.1 (see [12])**.** *If* $u \in L^{p(x)}(\Omega)$ *and* $p^+ < \infty$ *then the following relations hold*

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ *while*

$$|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ *and*

$$|u_n - u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

Proposition 2.2 (see [19])**.** *Let* p *and* q *be measurable functions such that* $p \in L^\infty(\Omega)$ *and* $1 \le p(x)q(x) \le \infty$ *for a.e.* $x \in \Omega$*. Let* $u \in L^{q(x)}(\Omega)$*,* $u \ne 0$*. Then the following relations hold*

$$|u|_{p(x)q(x)}^{p^+} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|_{p(x)q(x)}^{p^-} \tag{2.4}$$

provided $|u|_{p(x)} \le 1$ *while*

$$|u|_{p(x)q(x)}^{p^-} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|_{p(x)q(x)}^{p^+} \tag{2.5}$$

provided $|u|_{p(x)} \ge 1$*. In particular, if* $p(x) = p$ *is a constant, then*

$$\left||u|^{p}\right|_{q(x)} = |u|_{pq(x)}^{p}. \tag{2.6}$$

Next, we define the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 2.3 (see [12])**.** *The space* $\left(W_0^{1,p(x)}(\Omega), \|.\|\right)$ *is a separable and Banach space. Moreover, if* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous, where* $p^*(x) = \frac{Np(x)}{N-p(x)}$ *if* $p(x) < N$ *or* $p^*(x) = \infty$ *if* $p(x) > N$*.*

Now, we consider the weighted variable exponent Lebesgue spaces. Let $\sigma : \Omega \to \mathbb{R}$ be a measurable real function such that $\sigma(x) > 0$ for a.e. $x \in \Omega$. We define

$$L_{\sigma(x)}^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is a measurable function such that } \int_\Omega \sigma(x)|u(x)|^{p(x)}dx < \infty \right\}$$

with the norm

$$|u|_{p(x),\sigma(x)} = \inf\left\{ \mu > 0 : \quad \int_\Omega \sigma(x)\left|\frac{u(x)}{\mu}\right|^{p(x)}dx \le 1 \right\}.$$

The space $L_{\sigma(x)}^{p(x)}(\Omega)$ endowed with the above norm is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. In [13], the authors proved the following result which helps us in proving Theorem 1.4.

Proposition 2.4 (see [13, Theorem 4.1]). *Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$, where $\Omega_1 \subset \mathbb{R}^m$, $m \geq 1$ is a bounded regular domain, and $\Omega_2 \subset \mathbb{R}^k$, $(k \geq 2)$ is a ball of radius R, centered at the origin. Assume that $p, \gamma : \overline{\Omega}$ are continuous functions, $p(x) < N$ for all $x \in \overline{\Omega}$, $p, \gamma \in S(\Omega)$, the function $c : \overline{\Omega} \to \mathbb{R}$ satisfies the conditions* ($\mathbf{C_1}$) *and* ($\mathbf{C_2}$)*. Then, there exists a positive constant $\tau = \tau(c, p, m, k)$ such that the embedding $X_s = W_{0,s}^{1,p(x)}(\Omega)$ into $L^{\gamma(x)}(\Omega, c(x))$ is compact and continuous with $p(x) < \gamma(x) < p^*(x) + \tau$ for all $x \in \overline{\Omega}$.*

3. Proofs of the main results

Problems (1.6) and (1.7) will be studied using variational methods. We first prove Theorem 1.2 in details, the proof of Theorem 1.4 is similar. Let us associate with problem (1.6) the functional energy $J_1 : X := W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$J_1(u) = \Phi(u) - \lambda \Psi_1(u), \tag{3.1}$$

where

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right), \quad \Psi_1(u) = \int_\Omega \frac{a(x)}{\alpha(x)} |u|^{\alpha(x)} dx + \int_\Omega \frac{b(x)}{\beta(x)} |u|^{\beta(x)} dx, \tag{3.2}$$

where $\widehat{M}(t) = \int_0^t M(s)ds$. The functional J_1 associated with problem (1.6) is well defined and of C^1 class on X. Moreover, we have

$$\begin{aligned} J_1'(u)(v) &= M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx \\ &\quad - \lambda \int_\Omega a(x)|u|^{\alpha(x)-2} uv dx - \lambda \int_\Omega b(x)|u|^{\beta(x)-2} uv dx \\ &= \Phi'(u)(v) - \lambda \Psi_1'(u)(v) \end{aligned}$$

for all $u, v \in X$. Thus, weak solutions of problem (1.1) are exactly the ciritical points of the functional J. Due to the conditions ($\mathbf{M_1}$) and (1.5), we can show that J_1 is weakly lower semi-continuous in X. The following lemma plays an important role in our arguments.

Lemma 3.1. *The following assertions hold:*

(**i**) *There exist $\lambda^* > 0$ and $\rho, r > 0$ such that for any $\lambda \in (0, \lambda^*)$, we have*

$$J_1(u) \geq r, \quad \forall u \in X \text{ with } \|u\| = \rho;$$

(**ii**) *There exists $\varphi \in X$, $\varphi \neq 0$, such that*

$$\lim_{t \to \infty} J_1(t\varphi) = -\infty;$$

(**iii**) *There exists $\psi \in X$ such that $\psi \geq 0$, $\psi \neq 0$ and*

$$J_1(t\psi) < 0$$

for all $t > 0$ small enough.

Proof. (**i**) By (1.5) and the conditions (**A**) and (**B**), the embeddings from X to the weighted spaces $L^{\alpha(x)}(\Omega, a(x))$ and $L^{\beta(x)}(\Omega, b(x))$ are compact, see [19, Theorems 2.7, 2.8]. Then, there exist two positive constants c_1 and c_2 such that

$$\int_\Omega a(x)|u|^{\alpha(x)}dx \leq c_1\Big(\|u\|^{\alpha^+} + \|u\|^{\alpha^-}\Big) \tag{3.3}$$

and

$$\int_\Omega b(x)|u|^{\beta(x)}dx \leq c_2\Big(\|u\|^{\beta^+} + \|u\|^{\beta^-}\Big) \tag{3.4}$$

for all $u \in X$. Hence, for any $u \in X$ with $\|u\| = 1$, we get

$$\begin{aligned} J_1(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) - \lambda\int_\Omega \frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx - \lambda\int_\Omega \frac{b(x)}{\beta(x)}|u|^{\beta(x)}dx \\ &\geq \frac{m_1}{\delta_1}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^{\delta_1} - \lambda\frac{1}{\alpha^+}\int_\Omega a(x)|u|^{\alpha(x)}dx - \lambda\frac{1}{\beta^+}\int_\Omega b(x)|u|^{\beta(x)}dx \\ &\geq \frac{m_1}{\delta_1(p^+)^{\delta_1}} - \lambda\frac{2c_1}{\alpha^-} - \lambda\frac{2c_2}{\beta^-}. \end{aligned} \tag{3.5}$$

By (3.5), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ we get $J_1(u) \geq r > 0$ for all $u \in X$ with $\|u\| = 1$.

(**ii**) Let $\varphi \in C_0^\infty$, $\varphi \neq 0$ and $t > 1$. By ($\mathbf{M_1}$), there exists $c_3 > 0$ such that

$$\begin{aligned} J_1(t\varphi) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}dx\right) - \lambda\int_\Omega \frac{a(x)}{\alpha(x)}|t\varphi|^{\alpha(x)}dx - \lambda\int_\Omega \frac{b(x)}{\beta(x)}|t\varphi|^{\beta(x)}dx \\ &\leq \frac{m_2}{\delta_2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}dx\Big)^{\delta_2} - \lambda\frac{t^{\beta^-}}{\beta^+}\int_\Omega b(x)|\varphi|^{\beta(x)}dx \\ &\leq \frac{m_2}{\delta_2(p^-)^{\delta_2}}t^{\delta_2 p^+}\Big(\int_\Omega |\nabla\varphi|^{p(x)}dx\Big)^{\delta_2} - \lambda\frac{t^{\beta^-}}{\beta^+}\int_\Omega b(x)|\varphi|^{\beta(x)}dx \end{aligned}$$

Since $\delta_2 p^+ < \beta^-$, we get $\lim_{t\to\infty} J_1(t\varphi) = -\infty$ as $t \to \infty$.

(**iii**) Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$, $\psi \neq 0$, $t \in (0, 1)$. By ($\mathbf{M_1}$), we have

$$\begin{aligned} J_1(t\psi) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\psi|^{p(x)}dx\right) - \lambda\int_\Omega \frac{a(x)}{\alpha(x)}|t\psi|^{\alpha(x)}dx - \lambda\int_\Omega \frac{b(x)}{\beta(x)}|t\psi|^{\beta(x)}dx \\ &\leq \frac{m_2}{\delta_2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\psi|^{p(x)}dx\Big)^{\delta_2} - \lambda\frac{t^{\alpha^+}}{\alpha^+}\int_\Omega a(x)|\psi|^{\alpha(x)}dx \\ &\leq \frac{m_2}{\delta_2(p^-)^{\delta_2}}t^{\delta_2 p^-}\Big(\int_\Omega |\nabla\psi|^{p(x)}dx\Big)^{\delta_2} - \lambda\frac{t^{\alpha^+}}{\alpha^+}\int_\Omega a(x)|\psi|^{\alpha(x)}dx < 0 \end{aligned}$$

for all $t < \delta^{\frac{1}{\delta_2 p^- - \alpha^+}}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda\delta_2(p^-)^{\delta_2}\int_\Omega a(x)|\psi|^{\alpha(x)}dx}{m_2\alpha^+\Big(\int_\Omega |\nabla\psi|^{p(x)}dx\Big)^{\delta_2}}\right\}.$$

The proof of Lemma 3.1 is complete. □

Lemma 3.2. *The functional J_1 satisfies the Palais-Smale condition in X.*

Proof. Let $\{u_m\} \subset X$ be a sequence such that

$$J_1(u_m) \to \overline{c} > 0, \quad J_1'(u_m) \to 0 \text{ in } X^*, \tag{3.6}$$

where X^* is the dual space of X.

We first prove that $\{u_m\}$ is bounded in X. Indeed, we assume the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_m\}$, we may assume that $\|u_m\| \to \infty$ as $m \to \infty$. Thus we may consider that $\|u_m\| > 1$ for any m. Using $(\mathbf{M_1})$, $(\mathbf{M_2})$ we deduce from (3.6) that

$$\begin{aligned}
&\overline{c} + 1 + \|u_m\| \geq J_1(u_m) - \frac{1}{\beta^-} J_1'(u_m)(u_m) \\
&\quad = \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)} dx\right) - \lambda \int_\Omega \frac{a(x)}{\alpha(x)} |u_m|^{\alpha(x)} dx - \lambda \int_\Omega \frac{b(x)}{\beta(x)} |u_m|^{\beta(x)} dx \\
&\qquad - \frac{1}{\beta^-} M\left(\int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)} dx\right) \int_\Omega |\nabla u_m|^{p(x)} dx \\
&\qquad + \frac{\lambda}{\beta^-} \int_\Omega a(x) |u_m|^{\alpha(x)} dx + \frac{\lambda}{\beta^-} \int_\Omega b(x) |u_m|^{\beta(x)} dx \\
&\quad \geq \Big(\frac{1}{p^+} - \frac{1}{\beta^-}\Big) M\left(\int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)} dx\right) \int_\Omega |\nabla u_m|^{p(x)} dx + \lambda\Big(\frac{1}{\beta^-} - \frac{1}{\alpha^-}\Big) \int_\Omega a(x) |u_m|^{\alpha(x)} dx \\
&\quad \geq \frac{\beta^- - p^+}{(p^+)^{\delta_1} \beta^-} \Big(\int_\Omega |\nabla u_m|^{p(x)} dx\Big)^{\delta_1} + \lambda\Big(\frac{1}{\beta^-} - \frac{1}{\alpha^-}\Big) \int_\Omega a(x) |u_m|^{\alpha(x)} dx \\
&\quad \geq \frac{\beta^- - p^+}{(p^+)^{\delta_1} \beta^-} \|u_m\|^{\delta_1 p^-} - \lambda\Big(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\Big)\Big(\|u_m\|^{\alpha^+} + \|u_m\|^{\alpha^-}\Big).
\end{aligned}$$

Since $\alpha^- < \alpha^+ < \delta_1 p^-$ and $\beta^- > \delta_2 p^+ > p^+$, the sequence $\{u_m\}$ is bounded in X. Thus, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in X. Then $\{\|u_m - u\|\}$ is bounded. By (1.5) and the conditions $(\mathbf{A})$ and $(\mathbf{B})$, the embeddings from X to the weighted spaces $L^{\alpha(x)}(\Omega, a(x))$ and $L^{\beta(x)}(\Omega, b(x))$ are compact. Then, using the Hölder inequalities, Propositions 2.1-2.3, we have

$$\begin{aligned}
\Big|\int_\Omega a(x)|u_m|^{\alpha(x)-2} u_m (u_m - u) dx\Big| &\leq \int_\Omega a(x) |u_m|^{\alpha(x)-1} |u_m - u| dx \\
&\leq c_3 \Big|\Big(a(x)|u_m|^{\alpha(x)}\Big)^{\frac{\alpha(x)-1}{\alpha(x)}}\Big|_{\alpha'(x)} |u_m - u|_{a(x),\alpha(x)} \\
&\leq c_4 \Big|a(x)|u_m|^{\alpha(x)}\Big|_{L^1(\Omega)}^{\frac{\alpha^+-1}{\alpha^+}} |u_m - u|_{a(x),\alpha(x)} \\
&\leq c_5 |u_m|_{a(x),\alpha(x)}^{\frac{\alpha^+-1}{\alpha^+}} |u_m - u|_{a(x),\alpha(x)} \\
&\leq c_6 \|u_m\|^{\frac{\alpha^+-1}{\alpha^+}} |u_m - u|_{a(x),\alpha(x)},
\end{aligned} \tag{3.7}$$

which tends to 0 as $m \to \infty$, $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1$ for a.e. $x \in \Omega$, c_i, $i = 3, 4, 5, 6$ are positive constants.

Similarly, we get

$$\lim_{m\to\infty} \int_\Omega b(x)|u_m|^{\beta(x)-2} u_m(u_m - u)dx = 0. \tag{3.8}$$

On the other hand, by (3.6), we have

$$\lim_{m\to\infty} J_1'(u_m)(u_m - u) = 0. \tag{3.9}$$

From (3.7), (3.8) and (3.9), we get

$$\lim_{m\to\infty} \Phi'(u_m)(u_m - u) = 0. \tag{3.10}$$

Since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}dx \to t_0 \geq 0 \text{ as } m \to \infty.$$

If $t_0 = 0$ then $\{u_m\}$ converges strongly to $u = 0$ in X and the proof is finished. If $t_0 > 0$ then since the function M is continuous, we get

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}dx\Big) \to M(t_0) \text{ as } m \to \infty.$$

Thus, by $(\mathbf{M_1})$, for sufficiently large m, we have

$$0 < c_7 \leq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}dx\Big) \leq c_8. \tag{3.11}$$

From (3.10), (3.11), it follows that

$$\lim_{m\to\infty} \int_\Omega |\nabla u_m|^{p(x)-2}(\nabla u_m - \nabla u)dx = 0.$$

Thus, $\{u_m\}$ converges strongly to u in X and the functional J_1 satisfies the Palais-Smale condition. □

Proof Theorem 1.2. By Lemmas 3.1 and 3.2, all assumptions of the mountain pass theorem in [1] are satisfied. Then we deduce u_1 as a non-trivial critical point of the functional J with $J_1(u_1) = \overline{c}$ and thus a non-trivial weak solution of problem (1.6).

We now prove that there exists a second weak solution $u_2 \in X$ such that $u_2 \neq u_1$. Indeed, by (3.5), the functional J is bounded from below on the unit ball $\overline{B}_1(0)$.

Applying the Ekeland variational principle in [9] to the functional $J_1 : \overline{B}_1(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_1(0)$ such that

$$J_1(u_\epsilon) < \inf_{u\in\overline{B}_1(0)} J_1(u) + \epsilon,$$

$$J_1(u_\epsilon) < J_1(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

By Lemma 3.1, we have

$$\inf_{u\in\partial B_1(0)} J_1(u) \geq r > 0 \text{ and } \inf_{u\in\overline{B}_1(0)} J_1(u) < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u\in\partial B_1(0)} J_1(u) - \inf_{u\in\overline{B}_1(0)} J_1(u).$$

Then, $J_1(u_\epsilon) < \inf_{u\in\partial B_1(0)} J_1(u)$ and thus, $u_\epsilon \in B_1(0)$.

Now, we define the functional $I_1 : \overline{B}_1(0) \to \mathbb{R}$ by $I_1(u) = J_1(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_1 and thus

$$\frac{I_1(u_\epsilon + tv) - I_1(u_\epsilon)}{t} \geq 0$$

for all $t > 0$ small enough and all $v \in B_1(0)$. The above information shows that

$$\frac{J_1(u_\epsilon + tv) - J_1(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\left\langle J_1'(u_\epsilon), v\right\rangle \geq -\epsilon\|v\|.$$

It should be noticed that $-v$ also belongs to $B_1(0)$, so replacing v by $-v$, we get

$$\left\langle J_1'(u_\epsilon), -v\right\rangle \geq -\epsilon\| - v\|$$

or

$$\left\langle J_1'(u_\epsilon), v\right\rangle \leq \epsilon\|v\|,$$

which helps us to deduce that $\|J_1'(u_\epsilon)\|_{X^*} \leq \epsilon$.

Therefore, there exists a sequence $\{u_m\} \subset B_1(0)$ such that

$$J_1(u_m) \to \underline{c} = \inf_{u\in\overline{B}_1(0)} J_1(u) < 0 \text{ and } J_1'(u_m) \to 0 \text{ in } X^* \text{ as } m \to \infty. \tag{3.12}$$

From Lemma 3.2, the sequence $\{u_m\}$ converges strongly to u_2 as $m \to \infty$. Moreover, since $J_1 \in C^1(X, \mathbb{R})$, by (3.12) it follows that $J_1(u_2) = \underline{c}$ and $J_1'(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.6).

Finally, we point out the fact that $u_1 \neq u_2$ since $J_1(u_1) = \overline{c} > 0 > \underline{c} = J_1(u_2)$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.4. With the similar argument of the proof of Theorem 1.2, we associate with problem (1.7) the energy functional $J_2 : X_s = W_{0,s}^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$J_2(u) = \Phi(u) - \lambda\Psi_2(u), \tag{3.13}$$

where

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right), \quad \Psi_2(u) = \int_\Omega \frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx + \int_\Omega \frac{c(x)}{\gamma(x)}|u|^{\gamma(x)}dx, \tag{3.14}$$

where $\widehat{M}(t) = \int_0^t M(s)ds$. Then functional J_2 is well defined and of C^1 class on X_s. Moreover, we have

$$\begin{aligned} J_2'(u)(v) &= M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx \\ &\quad - \lambda\int_\Omega a(x)|u|^{\alpha(x)-2}uvdx - \lambda\int_\Omega c(x)|u|^{\gamma(x)-2}uvdx \\ &= \Phi'(u)(v) - \lambda\Psi_2'(u)(v) \end{aligned}$$

for all $u, v \in X_s$. Thus, weak solutions of problem (1.1) are exactly the ciritical points of the functional J_2.

From the proof of Theorem 1.2 and Proposition 2.4, using the mountain pass theorem combined with the Ekeland variational principle, we can prove that problem (1.7) has at least two distinct non-trivial weak solutions in X_s. □

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