SOME REMARKS ON A FIXED POINT THEOREM OF KRASNOSELSKII

Cezar AVRAMESCU

Abstract

Using a particular locally convex space and Schaefer's theorem, a generalization of Krasnoselskii's fixed point Theorem is proved. This result is further applied to certain nonlinear integral equation proving the existence of a solution on $\mathbb{R}_+ = [0, +\infty)$.

Key words and phrases: Fixed point theorem, Nonlinear integral equations.

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1. Introduction

Two main results of fixed point theory are Schauder's and Banach's theorems (also called contraction mapping principle). Krasnoselskii combined them into the following result (see [5], [8], [9], [10]).

Theorem K. *Let M be a closed convex non-empty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B maps M into X, such that the following hypotheses are fulfilled:*

(i) $Ax + By \in M$, $(\forall)$ $x, y \in M$;

(ii) *A is continuous and AM is contained in a compact set;*

(iii) *B is a contraction with constant $\alpha < 1$.*

Then, there is a $x \in M$, with $Ax + Bx = x$. □

The proof is based on the fact that from hypothesis (iii) it results that the mapping $I - B : M \to (I - B) M$ is a homeomorphism. Therefore, the proof is reduced to showing that the operator

$$U := (I - B)^{-1} A$$

admits fixed points. However, it is easily seen that the operator U satisfies the hypotheses of the Schauder's fixed point theorem. This is a captivating result and it has a number of interesting applications. In recent years much attention has been paid to this result. T.A. Burton (see [2]) remarks that in practice it is difficult to check hypothesis (i) and he proposes replacing it by the condition

(i')
$$(x = Bx + Ay,\ y \in M) \Longrightarrow x \in M.$$

In particularly, if

$$M := \{x \in X,\ \ \|x\| \leq r\},$$

the hypothesis (i') is fulfilled if the following conditions hold

$$\begin{array}{rcl} AM & \subset & M, \\ \|x\| & \leq & \|(I - B)\, x\|,\ \ (\forall)\ x \in M. \end{array}$$

Following the improvement of hypothesis (i), Burton and Kirk (see [3]) prove the following variant of Theorem K.

Theorem K'. *Let X be a Banach space, A, $B : X \to X$, B a contraction with $\alpha < 1$ and A a compact operator.*

Then either

(a) *$x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax$ has a solution for $\lambda = 1$*

or

(b) *the set $\left\{x \in X,\ x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax,\ \lambda \in (0,1)\right\}$ is unbounded.* □

We mention that through compact operator one understands a continuous operator which transforms bounded sets into relatively compact sets.

The proof of Theorem K' is based on the remark that $\lambda B\left(\frac{x}{\lambda}\right)$, $\lambda \in (0,1)$ is a contraction, too, with the same contraction constant α and therefore

$$x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax \Longleftrightarrow x = \lambda\,(I - B)^{-1}\, Ax$$

and it uses the following fundamental result due to Schaefer (see [8]).

Theorem S. *Let E be a linear locally convex space and let $H : B \to B$ be a compact operator. Then either*

(α) *the equation $x = \lambda Hx$ has a solution for $\lambda = 1$*

or

(β) *the set $\{x \in X,\ x = \lambda Hx,\ \lambda \in (0,1)\}$ is unbounded.* □

In [3] one uses the variant of Schauder's theorem in E, a normed space (see [9]) and one takes $H = (I - B)^{-1}\, Ax$.

In a recent note, B.C. Dhage (see [4]) recall that the condition that B to be a contraction is only sufficient to ensure the existence and the continuity of the operator $(I-B)^{-1}$; this also happens in the case when an iteration A^p is a contraction. Actually, this property that $I-B$ to be a homeomorphism on the rank is a property available in metric spaces, without any reference at the linearity of the space (see [6], [7] or [10]). In this direction, Dhage proves the following result.

Theorem K". *Let $(X, \|\cdot\|)$ be a Banach space, A, B be two operators such that:*

(A) *A is a compact operator;*

(B) *B is linear and bounded and there exists a $p \in \mathbb{N}^*$ such that*

$$\|B^p x - B^p y\| \leq \Phi(\|x-y\|), \quad (\forall) \ x, \ y \in X, \tag{1.1}$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function such that $\Phi(r) < r$, $(\forall)$ $r > 0$.

Then either

1) *the equation $\lambda Ax + Bx = x$ has a solution for $\lambda = 1$*

or

2) *the set $\{x \in X, \ \lambda Ax + Bx = x, \ (\forall) \ \lambda \in (0,1)\}$ is unbounded.* □

Recall that an operator which satisfies (1.1) is named **nonlinear contraction**; this condition ensures the existence and the continuity of $(I-B)^{-1}$. Simultaneously, from the linearity of B it results that $\lambda B\left(\frac{x}{\lambda}\right) = B(x)$ and the proof follows the same way as in Theorem K'.

2. Some remarks on the Dhage's result

Obviously, in a certain sense, the Dhage's result is more general than the one of Burton and Kirk. To illustrate this thing, Dhage considers the equation

$$x(t) = q(t) + \int_0^{\mu(t)} v(t,s)\, x(\theta(s))\, ds + \int_0^{\sigma(t)} k(t,s)\, g(s, x(\eta(s)))\, ds \tag{2.1}$$

in the space

$$X = \{x : [0,1] \to \mathbb{R}, \ x \text{ bounded and measurable}\}$$

endowed with the norm

$$\|x\| := \sup_{t \in [0,1]} |x(t)|. \tag{2.2}$$

At this point, the operator B is given by

$$(Bx)(t) = \int_0^{\mu(t)} v(t,s)\, x(\theta(s))\, ds. \tag{2.3}$$

The assumed hypotheses are:

$$\begin{array}{l} v \text{ is continuous on the set } \{(s,t),\ 0 \le s \le t \le 1\}, \\ \mu,\ \theta : [0,1] \to [0,1] \text{ are continuous and } \mu(t) \le t,\ \theta(t) \le t. \end{array} \tag{2.4}$$

One verifies immediately that

$$\|B^n x - B^n y\| \le \frac{V^n}{n!} \|x-y\|, \tag{2.5}$$

where

$$V = \sup\{|v(t,s)|,\ 0 \le s \le t \le 1\},$$

hence it results that B^n is contraction, for n large enough.

The reality is that B is contraction, but not with respect to the norm (2.2); it is a contraction with respect to an equivalent norm, i.e.

$$\|x\|_\lambda := \sup_{t\in[0,1]} \left\{ |x(t)|\, e^{-\lambda t} \right\},\ \lambda > 0. \tag{2.6}$$

Indeed, since $\theta(t) \le t$, it follows that

$$|x(\theta(t)) - y(\theta(t))|\, e^{-\lambda t} \le |x(\theta(t)) - y(\theta(t))|\, e^{-\lambda\theta(t)} \le \|x-y\|_\lambda$$

and we have

$$\begin{aligned} |(Bx)(t) - (By)(t)| &\le V \int_0^{\sigma(t)} |x(\theta(s)) - y(\theta(s))|\, e^{-\lambda s} \cdot e^{\lambda s} ds \le \\ &\le V \|x-y\|_\lambda \int_0^t e^{\lambda s} ds = \frac{V}{\lambda} \|x-y\|_\lambda \left(e^{\lambda t} - 1\right) < \\ &< \frac{V}{\lambda} \|x-y\|_\lambda\, e^{\lambda t}. \end{aligned}$$

Therefore,

$$|(Bx)(t) - (By)(t)|\, e^{-\lambda t} \le \frac{V}{\lambda} \|x-y\|_\lambda, (\forall)\, t \in [0,1]$$

and so

$$\|Bx - By\|_\lambda \le \frac{V}{\lambda} \|x-y\|_\lambda. \tag{2.7}$$

By taking $\lambda > V$, it results that B is contraction.

Evidently, to prove the compactity of the operator A, one may use the norm (2.2) as well as (2.6). Let us remark in addition that the existence of the operator $(I-B)^{-1}$ does not depend on the norm considered in X and if it is continuous with respect to a norm, it will be continuous with respect to any equivalent norm.

Observe that the operator B is compact, too. Indeed, the continuity of B follows by the fact that it is a contraction. Being continuous with respect to the norm $\|\cdot\|_{\lambda}$, it is still continuous with respect to the norm $\|\cdot\|$. By the continuity of the operator B, it results

$$\left|(Bx)(t)-(Bx)(t')\right| \le \left| \int_0^{\mu(t)} v(t,s)\left|x(\theta(s))\right| ds - \int_0^{\mu(t')} v(t',s)\left|x(\theta(s))\right| ds \right|$$

$$\le \int_0^{\mu(t)} \left|v(t,s)-v(t',s)\right| \left|x(\theta(s))\right| ds + \tag{2.8}$$

$$+ \left| \int_{\mu(t')}^{\mu(t)} \left|v(t',s)\right| \left|x(\theta(s))\right| ds \right|.$$

If $\|x\| \le r$, then

$$\left|(Bx)(t)-(Bx)(t')\right| \le r \int_0^1 \left|v(t,s)-v(t',s)\right| ds + V_2 \left|\mu(t)-\mu(t')\right|,$$

hence, the uniform continuity of the functions v and μ gives us

$$\begin{array}{rcl} (\forall)\ \epsilon & > & 0,\ (\exists)\ \delta=\delta(\epsilon),\ (\forall)\ x,\ y,\ \|x\|\le r,\ \|y\|\le r, \\ (\forall)\ t,\ t' & \in & [0,1],\ |t-t'|<\delta,\ \left|(Bx)(t)-(Bx)(t')\right|<\epsilon. \end{array} \tag{2.9}$$

By (2.8) and (2.9) it results, based on the Ascoli-Arzelà Theorem, that B is a compact operator on each set $\{x\in X,\ \|x\|\le r\}$. Since B is, as A, a compact operator, one can get the Dhage's result, by using the topological degree.

In his work, Dhage proves that in certain hypotheses on the functions g, k, σ, η, the operator A is compact and that there is a positive number r such that if x fulfills the equality

$$x = Bx + \lambda Ax,\ \text{ for a } \lambda\in(0,1),$$

then

$$\|x\| \leq r. \tag{2.10}$$

Consider in X the open and bounded set

$$\Omega := \{x \in X, \ \ \|x\| < 2r\}.$$

Define on $\overline{\Omega} \times [0,1]$ the operator

$$H(x, \lambda) = Bx + \lambda Ax.$$

Obviously, $H(x, \lambda)$ is a homotopy. From the above, it follows that

$$x \neq H(x, \lambda), \ \ x \in \partial\Omega, \ \ \lambda \in (0, 1), \tag{2.11}$$

where $\partial\Omega$ denotes the boundary of Ω.

If $x = H(x, 1)$, $x \in \partial\Omega$, then Tx is a solution for the equation (2.1); it remains to study the case $x \neq H(x, 1)$, $x \in \partial\Omega$.

Obviously,

$$(x = H(x, 0)) \Longleftrightarrow (x = 0).$$

Therefore, we have

$$x \neq H(x, \lambda), \ \ x \in \partial\Omega, \ \ \lambda \in [0, 1]. \tag{2.12}$$

Using the invariance property of the topological degree with respect to a homotopy, we have

$$\deg(I - H(\cdot, 1), \Omega, 0) = \deg(I - H(\cdot, 0), \Omega, 0). \tag{2.13}$$

But

$$\deg(I - H(\cdot, 1), \Omega, 0) = \deg(I - B, \Omega, 0) = \pm 1,$$

since B is linear, compact, injective and $0 \in \Omega$, from a well known property of the topological degree. By (2.13) one gets

$$\deg(I - H(\cdot, 1), \Omega, 0) \neq 0$$

and, hence $H(\cdot, 1)$ has at least one fixed point. Obviously, each x for which $x = H(x, 1)$ represents a solution for the equation (2.1).

For further details regarding the topological degree, we recommend [10].

3. A theorem of Krasnoselskii type

If we are interested about the existence of solutions on a noncompact interval for a concrete problem we cannot always use always the theorems K, K' and K", since the spaces of continuous functions on noncompact interval cannot be organized always as Banach spaces. We are forced to use spaces more general than the Banach spaces, for example the Fréchet spaces.

Because the terminology is diverse, we are forced to enumerate some fundamental definitions and properties.

We call **Fréchet space** each linear metrizable and complete space. One of the most convenient ways to build a Fréchet space is the one based on the notion of **seminorm**.

Let X be a linear space; recall that a seminorm on X is a mapping $|\cdot| : X \to [0, +\infty)$ having all the properties of a norm except that $|x| = 0$ does not always imply that $x = 0$.

Suppose that we have a numerable family of seminorms on X, $|\cdot|_n$; we say that this family is **sufficient** iff

$$(\forall)\ \ x \in X,\ \ x \neq 0,\ \ (\exists)\ \ n \in \mathbb{N}^*, |x|_n \neq 0. \tag{3.1}$$

Every space $(X, |\cdot|_n)$, endowed with a numerable and sufficient family of seminorms can be organized as a metric space, by setting the metric

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x-y|_n}{1 + |x-y|_n}. \tag{3.2}$$

If $(X, |\cdot|_n)$ is complete in the metric (3.2), then it will be called Fréchet.

Recall that the convergence determined by the metric (3.2) can be characterized more precisely with the seminorms, i.e.

$$(x_n \to x) \Longleftrightarrow \left((\forall)\ \ n \in \mathbb{N}^*,\ \lim_{m \to \infty} |x_m - x|_n = 0 \right).$$

We mention that two families of seminorms $|\cdot|_n$, $\|\cdot\|_n$ are called **equivalent** iff they define the same metric topology. Obviously, if $(E, |\cdot|_n)$ is complete, it will remain complete with respect to each equivalent family of seminorms.

We remark that for every family of seminorms $|\cdot|_n$, there is an equivalent family of seminorms, **ordered** in the sense that

$$(\forall)\ \ n \in \mathbb{N}^*,\ \ (\forall)\ \ x \in X,\ \ |x|_n \leq |x|_{n+1}\,.$$

It is easy to characterize continuity of a mapping and the compactity of a set through the notion of seminorms.

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $(X, |\cdot|_n))$ be a Fréchet space. Let us consider $M \subset X$.

Definition 3.1. *Let $U : M \to X$ be an operator. We call U an $\alpha-$**contraction** on M iff*

$$(\forall)\, n \in \mathbb{N}^*, (\exists)\, \alpha_n \in [0,1), (\forall)\, x, y \in X, |Ux - Uy|_n \leq \alpha_n |x-y|_n. \quad (3.3)$$

Theorem B (Banach)**.** *Let $(X, |\cdot|_n)$ be a Fréchet space and let $M \subset X$ be a closed subset.*

Every $\alpha-$contraction mapping on M, $U : M \to M$ admits a unique fixed point. □

The proof is an immediate consequence of the fact that, if U is a $\alpha-$contraction, then for d given by (3.2), we have

$$d(Ux, Uy) \leq kd(x,y), \ \ (\forall)\ x,\ y \in X, \quad (3.4)$$

where

$$k := \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n} < 1.$$

We remark that U may not be a contraction in $(X, |\cdot|_n)$, but it can be contraction in a space $(X, \|\cdot\|_n)$, endowed with a family of seminorms equivalent with the initial one.

The Theorems K, K', K" can be extended to the case when X is a Fréchet space; we will state only the extension of the Burton-Kirk' Theorem K'.

Theorem K"'. *Let $(X, |\cdot|_n)$ be a Fréchet space and let A, $B : X \to X$ be two operators; set $U_\lambda x := \lambda B(x/\lambda) + \lambda Ax$.*

Suppose that the following hypothesis are fulfilled:

(A) *A is a compact operator;*

(B) *B is a contraction operator with respect to a family of seminorms $\|\cdot\|_n$ equivalent with the family $|\cdot|_n$;*

(C) *the set*

$$\{x \in X,\ x = U_\lambda x,\ \lambda \in (0,1)\}$$

is bounded.

Then there is $x \in X$ such that

$$x = Ax + Bx. \quad (3.5)$$

□

The proof of this theorem is immediate. Indeed, hypothesis (B) ensures us the existence and the continuity of the operator $(I-B)^{-1}$. By applying to the operator $x \to \lambda (I-B)^{-1} Ax$ the Theorem S, from hypothesis (C) the conclusion follows, since $U_1 x = Bx + Ax$.

4. An example

We would now like to apply the Theorem K"'. To this end, we take the Dhage's example in a more general framework.

In what follows, $\mathbb{R}_+ := [0, +\infty)$, $v(t,s)$ and $k(t,s)$ are continuous on the set $\{(s,t),\ 0 \leq s \leq t < \infty\}$ and quadratic $d \times d$ matrices, $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $q : \mathbb{R}_+ \to \mathbb{R}^d$ are continuous functions, the functions μ, θ, σ, $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy the conditions $\mu(t) \leq t$, $\sigma(t) \leq t$, $\eta(t) \leq t$, $(\forall)\ t \geq 0$.

Consider the equation

$$x(t) = q(t) + \int_0^{\mu(t)} v(t,s)\, x(\theta(s))\, ds + \int_0^{\sigma(t)} k(t,s)\, g(s, x(\eta(s)))\, ds, \quad (4.1)$$

$t \in \mathbb{R}_+$.

Theorem 4.1. *Assume that the following hypotheses are fulfilled:*

(i)

$$|g(t,x)| \leq \varphi(t)\, \psi(|x|),\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^d,$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and* $\psi : \mathbb{R}_+ \to (0, +\infty)$ *is a continuous and nondecreasing function;*

(ii)

$$\int_{(\cdot)}^{+\infty} \frac{ds}{s + \psi(s)} = +\infty.$$

Then, the equation (4.1) *admits solutions.*

We set as a fundamental space, the space

$$X = C_c := \left\{ x : \mathbb{R}_+ \to \mathbb{R}^d,\ x \text{ continuous} \right\},$$

endowed with the ordered, sufficient and numerable family of seminorms

$$|x|_n := \sup_{t \in [0,n]} \{|x(t)|\}, \quad (4.2)$$

where for $x = (x_i)_{i \in \overline{1,d}} \in \mathbb{R}^d$ we denoted

$$|x| = \max \left\{ |x_i|,\ i \in \overline{1,d} \right\}.$$

The space $(C_c, |\cdot|_n)$ is a Fréchet space.
For a quadratic matrix $d \times d$, $C = (c_{ij})_{i,j\in\overline{1,d}}$ we set

$$|C| = \max_{i\in\overline{1,d}} \sum_{j=1}^{d} |c_{ij}| .$$

Consider in C_c the operators A, $B : C_c \to C_c$, defined by

$$\begin{aligned} (Ax)(t) &= q(t) + \int_0^{\sigma(t)} k(t,s)\, g(s, x(\eta(s)))\, ds, \\ (Bx)(t) &= \int_0^{\mu(t)} v(t,s)\, x(\theta(s))\, ds. \end{aligned}$$

We set

$$\begin{aligned} K_n &= \sup\{|k(t,s)|\,,\ 0 \le s \le t \le n\}\,, \\ V_n &= \sup\{|v(t,s)|\,,\ 0 \le s \le t \le n\}\,. \end{aligned}$$

We show firstly that B is a contraction mapping; to this aim, we consider in C_c the family of seminorms

$$\|x\|_n := \sup\left\{ |x(t)|\, e^{-h_n t},\ t \in [0,n]\,,\ h_n > 0 \right\}, \tag{4.3}$$

which is equivalent with the family (4.2), since

$$e^{-nh_n} |x|_n \le \|x\|_n \le |x|_n\,,\ (\forall)\ x \in C_c, (\forall)\ n \ge 1.$$

Repeating the reasoning from the Section 2, but now on the whole interval $[0,n]$, we get an inequality of type (2.7) , i.e.

$$\|Bx - By\|_n \le \frac{V_n}{h_n} \|x - y\|_n\,. \tag{4.4}$$

Choosing h_n such that $h_n > V_n$, it follows from (4.4) that B is contraction and so there exists $(I - B)^{-1} : C_c \to C_c$ and it is continuous.

Let us show that the operator $A : C_c \to C_c$ is compact; to this aim, we must prove that it is continuous and it transforms every bounded set into a relatively compact set.

Let us prove firstly the continuity. Let x_m, $x \in C_c$ be such that $x_m \to x$ in C_c, i.e.

$$\begin{aligned} (\forall)\ n &\ge 1,\ (\forall)\ \epsilon > 0,\ (\exists)\ m_0 = m_0(\epsilon, n)\,,\ (\forall)\ m \ge m_0, \\ |x_m - x|_n &< \epsilon. \end{aligned} \tag{4.5}$$

Let $n \geq 1$ be fixed; we have

$$|(Ax_m)(t) - (Ax)(t)| \leq \int_0^{\sigma(t)} |k(t,s)| \, |g(s, x_m(\eta(s))) - g(s, x(\eta(s)))| \, ds,$$

and so, for $t \in [0, n]$, we get

$$|(Ax_m)(t) - (Ax)(t)| \leq K_n \int_0^n |g(s, x_m(\eta(s))) - g(s, x(\eta(s)))| \, ds. \quad (4.6)$$

But the convergence of a sequence implies the boundedness; hence there is a number $L_n > 0$ such that

$$|x_m(t)| \leq L_n, \;\; |x(t)| \leq L_n, \;\; (\forall) \;\; t \in [0, n], \;\; n \geq 1.$$

But the function g is uniformly continuous on the compact set

$$\left\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \;\; t \in [0, n], \;\; |x| \leq L_n\right\}.$$

Taking into account that

$$|x_m(\eta(\cdot)) - x(\eta(\cdot))|_n \leq |x_m(\cdot) - x(\cdot)|_n,$$

it results that

$$|g(t, x_m(\eta(t))) - g(t, x(\eta(t)))| \leq \frac{\epsilon}{nK_n}, \;\; (\forall) \;\; m \geq m_0.$$

Then, from (4.6) it follows

$$|Ax_m - Ax|_n \leq \epsilon, \;\; (\forall) \;\; m \geq m_0.$$

The continuity of A is proved.

For proving the compactity of A it remains to show that this operator maps bounded sets of C_c into relatively compact sets of this space.

Recall that $M \subset C_c$ is bounded if and only if $(\forall)$ $n \in \mathbb{N}^*$, $(\exists)$ $r_n > 0$, $(\forall)$ $x \in M$, $|x|_n \leq r_n$; and $M = \{x(t)\} \subset C_c$ is relatively compact if and only if $(\forall)$ $n \geq 1$, the family $\left\{x(t)\,|_{[0,n]}\right\}$ is equi continuous and uniformly bounded on $[0, n]$.

Let $x \in M$, M bounded, i.e.

$$(\forall) \;\; n \in \mathbb{N}^*, \;\; (\exists)\, r_n > 0, \;\; (\forall) \;\; x \in M, \;\; |x|_n \leq r_n.$$

One has, for $t \in [0,n]$,

$$|(Ax)(t)| \leq \sup_{t\in[0,n]} |q(t)| + nK_nG_n,$$

where

$$G_n = \sup\{|g(t,x)|, \ t \in [0,n], \ |x| \leq r_n\}$$

and so

$$|Ax|_n \leq nK_nG_n + \sup\{|q(t)|, \ t \in [0,n]\}.$$

It remains to prove the equi continuity of the set $\{Ax, \ x \in M\}$; to this aim we shall adapt the method applied in Section 2 to the operator B on $[0,1]$ and to the operator A on $[0,n]$.

For applying the Theorem K"', we must check hypothesis (C).

So, let us consider $x \in C_c$, such that

$$\begin{aligned} x(t) &= \lambda q(t) + \int_0^{\mu(t)} v(t,s)\, x(\theta(s))\, ds + \\ &+ \lambda \int_0^{\sigma(t)} k(t,s)\, g(s, x(\eta(s)))\, ds, \end{aligned} \tag{4.7}$$

for an $\lambda \in (0,1)$; by (4.7) and taking into account that $\lambda \in (0,1)$, it follows that

$$|x(t)| \leq Q_n + V_n \int_0^{\mu(t)} |x(\theta(s))|\, ds + K_n\Phi_n \int_0^{\sigma(t)} \varphi(|x(\eta(s))|)\, ds, \tag{4.8}$$

$t \in [0,n]$, where

$$Q_n := \sup\{|q(t)|, \ t \in [0,n]\}, \ \Phi_n := \sup\{\varphi(t), \ t \in [0,n]\}.$$

We set

$$w_n(t) := \sup\{|x(s)|, \ 0 \leq s \leq t \leq n\}.$$

Clearly,

$$|x(t)| \leq w_n(t), \ \ (\forall) \ t \in [0,n]. \tag{4.9}$$

On the other hand, there is $t^* \in [0,t]$, such that

$$w_n(t) = |x(t^*)|$$

and $w_n(t)$ is increasing on $[0,n]$.

Taking into account that $|x(\theta(s))| \leq w(s)$, $|x(\eta(s))| \leq s$ and the monotonicity of ψ, from (4.8) one gets

$$\begin{aligned} w(t) = |x(t^*)| &\leq Q_n + V_n \int_0^{\mu(t^*)} w(s)\,ds + \\ &+ K_n \Phi_n \int_0^{\sigma(t^*)} \psi(w(s))\,ds \leq \\ &\leq Q_n + C_n \int_0^t [w(s) + \psi(w(s))]\,ds, \end{aligned} \tag{4.10}$$

$(\forall)\, t \in [0, n]$, where

$$C_n := \max\{V_n,\ K_n \Phi_n, Q_n\}.$$

Set

$$u_n(t) = Q_n + C_n \int_0^t [w(s) + \psi(w(s))]\,ds,\ t \in [0, n].$$

We have

$$w_n(t) \leq u_n(t),\ t \in [0, n]$$

and

$$\dot{u}_n(t) = C_n [w_n(s) + \psi(w_n(s))] \leq C_n [u_n(t) + \psi(u_n(t))],\ t \in [0, n].$$

Therefore,

$$\int_0^t \frac{\dot{u}_n(s)}{u_n(s) + \psi(u_n(s))} ds = \int_{u(0)}^{u_n(t)} \frac{ds}{s + \psi(s)} \leq C_n,\ t \in [0, n]. \tag{4.11}$$

Since $u_n(0) = Q_n$, we get definitively

$$\int_{Q_n}^{u_n(t)} \frac{ds}{s + \psi(s)} \leq C_n,\ t \in [0, n]. \tag{4.12}$$

Consider now the strictly increasing function

$$F_n(t) := \int_{Q_n}^t \frac{ds}{s + \psi(s)},\ t \geq Q_n.$$

From

$$F_n([Q_n, +\infty)) = [0, +\infty),$$

it follows that there is an unique $r_n > 0$ such that

$$F_n(r_n) = C_n.$$

Since u_n is strictly increasing, it follows that

$$\left(\int_{Q_n}^{u_n(t)} \frac{ds}{s + \psi (s)} \leq C_n \right) \Longleftrightarrow (u_n (t) \leq r_n) .$$

But

$$|x (t)| \leq w_n (t) \leq u_n (t) , \; t \in [0, n] ,$$

so

$$|x (t)| \leq r_n, \; t \in [0, n]$$

and

$$|x|_n \leq r_n, \; n \geq 1,$$

which ends the proof.

5. Final remarks

It would be interesting to study the case when $I - B$ is not injective. In this case, in certain conditions, $(I - B)^{-1}$ could be seen as a multivalued operator and one could try to apply to the multivalued operator $(I - B)^{-1} A$ one of the numerous interesting results looking for the existence of fixed points for multivalued operators.

A great part of the existence problems are related to the equations of type $Lx = Nx$, where L is a linear operator and N is an arbitrary operator; one can study the problem when such an equation can be written, in an equivalent manner, under the form $x = Ax + Bx$. □

References

[1] C. Avramescu, C. Vladimirescu, *Some remarks on the fixed point theorem of Krasnoselskii* (to appear).

[2] T.A. Burton, *A fixed point theorem of Krasnoselskii*, Appl. Math. Lett. **11**(1998), pp. 85-88.

[3] T.A. Burton and C. Kirk, *A fixed point theorem of Krasnoselskii's type*, Math. Nachr., **189**(1998), pp. 23-31.

[4] B.C. Dhage, *On a fixed point of Krasnoselskii-Schaeffer type*, Electronic Journal of Qualitative Theory of Differential Equations, No. **6**(2002), pp. 1-9.

[5] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Cambridge University Press, New York, 1964.

[6] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, New York, 1978.

[7] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[8] H. Schaeffer, *Über die Methode der a priori-Schranken*, Math. Ann. **129**(1955), pp. 415-416.

[9] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1980.

[10] E. Zeidler, *Nonlinear functional analysis and its applications,* I. *Fixed- point theorems*, Springer-Verlag, Berlin, 1993.

Cezar Avramescu
Department of Mathematics
University of Craiova,
13 A.I. Cuza Street, 1100 Craiova, ROMANIA
E-mail address: cezaravramescu@hotmail.com