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# Positive almost periodic solutions for a predator-prey Lotka-Volterra system with delays*

Yuan Ye
School of Graduate, Yunnan University
Kunming, Yunnan 650091
People's Republic of China

**Abstract**

In this paper, by using Mawhin's continuation theorem of coincidence degree theory, sufficient conditions for the existence of positive almost periodic solutions are obtained for the predator-prey Lotka-Volterra competition system with delays

$$\begin{cases} \dfrac{\mathrm{d}u_i(t)}{\mathrm{d}t} = u_i(t)\Big[a_i(t) - \sum\limits_{l=1}^{n} a_{il}(t)u_l(t-\sigma_{il}(t)) - \sum\limits_{j=1}^{m} b_{ij}(t)v_j(t-\tau_{ij}(t))\Big], i = 1, \dots, n, \\ \dfrac{\mathrm{d}v_j(t)}{\mathrm{d}t} = v_j(t)\Big[-r_j(t) + \sum\limits_{l=1}^{n} d_{jl}(t)u_l(t-\delta_{jl}(t)) - \sum\limits_{h=1}^{m} e_{jh}(t)v_h(t-\theta_{jh}(t))\Big], j = 1, \dots, m, \end{cases}$$

where $a_i, r_j, a_{il}, b_{ij}, d_{jl}, e_{jh} \in C(\mathbb{R}, (0, \infty)), \sigma_{il}, \tau_{ij}, \delta_{jl}, \theta_{jh} \in C(\mathbb{R}, \mathbb{R})(i, l = 1, \dots, n, j, h = 1, \dots, m)$ are almost periodic functions.

**Keywords:** Predator-prey Lotka-Volterra system; Almost periodic solutions; Coincidence degree; Delays.
**MSC2010:** 34K14; 92D25.

## 1 Introduction

Proposed by Lotka [1] and Volterra [2], the well-known Lotka-Volterra models concerning ecological population modeling have been extensively investigated in the literature. In recent years, it has also been found with successful and interesting applications in epidemiology, physics, chemistry, economics, biological science and other areas (see [3-5]). Owing to their theoretical and practical significance, the Lotka-Volterra systems have been studied extensively [6-17].

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Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically or almost periodically varying environment are considered as important selective forces on systems in a fluctuating environment. Therefore, on the one hand, models should take into account both the seasonality of the periodically changing environment and the effects of time delays [6-11, 13, 14, 17-27]. However, on the other hand, in fact, it is more realistic to consider almost periodic system than periodic system.

There are many works on the study of the Lotka-Volterra type periodic systems that have been developed in [6-9, 11, 17, 19, 21, 24]. But, relatively few papers have been published on the existence of almost periodic solutions for the Lotka-Volterra type almost periodic systems. Recently, by using the definition of almost periodic function, the contraction mapping, fixed point theory, appropriate Lyapunov functionals and almost periodic functional hull theory some authors have done many good works in theory on almost periodic systems [10, 26, 28-30]. Motivated by above, in this paper, we are concerned with the following predator-prey Lotka-Volterra system with delays

$$\begin{cases} \dfrac{\mathrm{d}u_i(t)}{\mathrm{d}t} = u_i(t)\Big[a_i(t) - \sum\limits_{l=1}^{n} a_{il}(t)u_l(t-\sigma_{il}(t)) - \sum\limits_{j=1}^{m} b_{ij}(t)v_j(t-\tau_{ij}(t))\Big], i = 1,\dots,n, \\ \dfrac{\mathrm{d}v_j(t)}{\mathrm{d}t} = v_j(t)\Big[-r_j(t) + \sum\limits_{l=1}^{n} d_{jl}(t)u_l(t-\delta_{jl}(t)) - \sum\limits_{h=1}^{m} e_{jh}(t)v_h(t-\theta_{jh}(t))\Big], j = 1,\dots,m, \end{cases} \tag{1.1}$$

where $a_i, r_j, a_{il}, b_{ij}, d_{jl}, e_{jh} \in C(\mathbb{R}, (0,\infty)), \sigma_{il}, \tau_{ij}, \delta_{jl}, \theta_{jh} \in C(\mathbb{R}, \mathbb{R}) (i, l = 1,\dots,n, j, h = 1,\dots,m)$ are almost periodic functions.

Our main purpose of this paper is by using the coincidence degree theory [30] to study the existence of positive almost periodic solutions of (1.1). Our result obtained in this paper is completely new and our methods used in this paper can be used to study the existence of positive almost periodic solutions to other types of Lotka-Volterra systems with delays.

# 2 Preliminaries

Let $X, Y$ be normed vector spaces, $L : \mathrm{Dom}\, L \subset X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim\mathrm{Ker}\, L = \mathrm{codim}\mathrm{Im}\, L < +\infty$ and $\mathrm{Im}\, L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero and there exists continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}\, L = \mathrm{Ker}\, L$, $\mathrm{Ker}\, Q = \mathrm{Im}\, L = \mathrm{Im}\,(I-Q)$, it follows that the mapping $L_{\mathrm{Dom}\, L \cap \mathrm{Ker} P} : (I-P)X \to \mathrm{Im}\, L$ is invertible. We denote the inverse of that mapping by $K_P$. If $\Omega$ is an open bounded subset of $X$, then the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\mathrm{Im}\, Q$ is isomorphic to $\mathrm{Ker}\, L$, there exists an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$.

We introduce the Mawhin's continuation theorem [30] as follows.

**Lemma 2.1** ([30]). *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is $L$-compact on $\bar{\Omega}$. Assume that*

(1) *$Ly \neq \lambda N y$ for every $y \in \partial\Omega \cap \mathrm{Dom}\, L$ and $\lambda \in (0,1)$;*

(2) *$QNy \neq 0$ for every $y \in \partial\Omega \cap \operatorname{Ker} L$;*

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

*Then $Ly = Ny$ has at least one solution in $\operatorname{Dom} L \cap \bar{\Omega}$.*

For convenience, we denote $AP(\mathbb{R}, \mathbb{R}^n)$ is the set of all vector valued, almost periodic functions on $\mathbb{R}$ and for $f \in AP(\mathbb{R}, \mathbb{R}^n)$ we denote by

$$\Lambda(f) = \left\{ \lambda \in \mathbb{R} : \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s) e^{-i\lambda s} \, \mathrm{d}s \neq 0 \right\}$$

and

$$\operatorname{mod}(f) = \left\{ \sum_{j=1}^{m} n_j \lambda_j : n_j \in \mathbb{Z},\, m \in \mathbb{N},\, \lambda_j \in \Lambda(f),\, j = 1, 2, \ldots, m \right\}$$

the set of Fourier exponents and the module of $f$, respectively. Suppose that $f(t, \phi)$ is almost periodic in $t$, uniformly with respect to $\phi \in S$. $E\{f, \varepsilon, S\}$ denotes the set of $\varepsilon$-almost periods for $f$ with respect to $S \subset C([-\sigma, 0], \mathbb{R}^n)$, $l(\varepsilon, S)$ denotes the length of the inclusion interval and $M(f) = \lim\limits_{T \to \infty} \frac{1}{T} \int_0^T f(s) \, \mathrm{d}s$ denotes the mean value of $f$.

The following lemma will paly an important role in the proof of our main result.

**Lemma 2.2.** *If $f \in C(\mathbb{R}, \mathbb{R})$ is almost periodic, $t_0 \in \mathbb{R}$. For any $\varepsilon > 0$ and inclusion length $l(\varepsilon)$, $\forall t_1, t_2 \in [t_0, t_0 + l(\varepsilon)]$. Then for all $t \in \mathbb{R}$, the following hold*

$$f(t) \leq f(t_1) + \int_{t_0}^{t_0 + l(\varepsilon)} |f'(s)| \, \mathrm{d}s + \varepsilon \tag{2.1}$$

*and*

$$f(t) \geq f(t_2) - \int_{t_0}^{t_0 + l(\varepsilon)} |f'(s)| \, \mathrm{d}s - \varepsilon. \tag{2.2}$$

*Proof.* For any $t \in \mathbb{R}$, there exists $\tau \in E\{f, \varepsilon\}$ such that $t \in [t_0 - \tau, t_0 - \tau + l(\varepsilon)]$. Thus, $t + \tau \in [t_0, t_0 + l(\varepsilon)]$. So we can obtain

$$\begin{aligned} f(t) - f(t_1) = \int_{t_1}^{t} f'(s) \, \mathrm{d}s &= \int_{t_1}^{t+\tau} f'(s) \mathrm{d}s + \int_{t+\tau}^{t} f'(s) \, \mathrm{d}s \\ &\leq \int_{t_1}^{t+\tau} |f'(s)| \, \mathrm{d}s + |f(t+\tau) - f(t)| \\ &\leq \int_{t_0}^{t_0 + l(\varepsilon)} |f'(s)| \, \mathrm{d}s + \varepsilon. \end{aligned}$$

Hence, (2.1) holds.

Similarly, we also have

$$f(t) - f(t_2) = \int_{t_2}^{t} f'(s) \, \mathrm{d}s = \int_{t_2}^{t+\tau} f'(s) \mathrm{d}s + \int_{t+\tau}^{t} f'(s) \, \mathrm{d}s$$

$$\geq -\int_{t_2}^{t+\tau} |f^{'}(s)|\,\mathrm{d}s - |f(t+\tau) - f(t)|$$
$$\geq -\int_{t_0}^{t_0+l(\varepsilon)} |f^{'}(s)|\,\mathrm{d}s - \varepsilon.$$

Thus, (2.2) holds. The proof is complete. ∎

Set
$$\mathbb{X} = \mathbb{Y} = V_1 \oplus V_2,$$
where
$$V_1 = \big\{z = (x_1, \ldots, x_n, y_1, \ldots, y_m)^T \in AP(\mathbb{R}, \mathbb{R}^n) : \mathrm{mod}(y) \subset \mathrm{mod}(\Pi)\ \forall \mu_0 \in \Lambda(z) \text{ satisfies } |\mu_0| \geq \alpha\big\}$$
and
$$V_2 = \big\{z = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))^T \equiv (k_1, \ldots, k_{n+m})^T,\ (k_1, \ldots, k_{n+m})^T \in \mathbb{R}^n\big\},$$
where $\Pi = (\Pi_1, \ldots, \Pi_{n+m})^T$,
$$\Pi_i(t,\phi) = a_i(t) - \sum_{l=1}^{n} a_{il}(t) e^{\varphi_l(-\sigma_{il}(t))} - \sum_{j=1}^{m} b_{ij}(t) e^{\psi_j(-\tau_{ij}(t))},\ i = 1, 2, \ldots, n,$$
$$\Pi_{n+j}(t,\phi) = -r_j(t) + \sum_{l=1}^{n} d_{jl}(t) e^{\varphi_l(-\delta_{jl}(t))} + \sum_{h=1}^{m} e_{jh}(t) e^{\psi_h(-\theta_{jh}(t))},\ j = 1, 2, \ldots, m,$$
$\phi = (\varphi_1, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_n)^T \in C([-\sigma, 0], \mathbb{R}^n), \sigma = \max\limits_{\substack{1\leq i,m\leq n\\ 1\leq j,h\leq m}} \sup\limits_{t\in\mathbb{R}} \{\sigma_{il}(t), \tau_{ij}(t), \delta_{jl}(t), \theta_{jh}(t)\}$
and $\alpha$ is a given positive constant. Define the norm
$$\|z\| = \sup_{t\in\mathbb{R}} |z(t)| = \sup_{t\in\mathbb{R}} \max_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \{|x_i(t)|, |y_j(t)|\},\ \ z \in \mathbb{X}\,(\text{or } \mathbb{Y}).$$

# 3 Main results

By making the substitution
$$u_i(t) = \exp\{x_i(t)\},\ \ v_j(t) = \exp\{y_j(t)\},\ \ i = 1, \ldots, n,\ j = 1, \ldots, m.$$
Eq.(1.1) is reformulated as
$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = a_i(t) - \sum\limits_{l=1}^{n} a_{il}(t) e^{x_l(t-\sigma_{il}(t))} - \sum\limits_{j=1}^{m} b_{ij}(t) e^{y_j(t-\tau_{ij}(t))},\ i = 1, \ldots, n, \\ \dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} = -r_j(t) + \sum\limits_{l=1}^{n} d_{jl}(t) e^{x_l(t-\delta_{jl}(t))} - \sum\limits_{h=1}^{m} e_{jh}(t) e^{y_h(t-\theta_{jh}(t))},\ j = 1, \ldots, m. \end{cases} \quad (3.1)$$

**Lemma 3.1.** *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with the norm $\|\cdot\|$.*

*Proof.* If $\{z_n\} \subset V_1$ and $z_n$ converges to $z_0$, then it is easy to show that $z_0 \in AP(\mathbb{R}, \mathbb{R}^n)$ with $\mathrm{mod}(z_0) \subset \mathrm{mod}(\Pi)$. Indeed, for all $|\lambda| < \alpha$ we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T z_n(s)e^{-i\lambda s}\,\mathrm{d}s = 0.$$

Thus

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T z_0 e^{-i\lambda s}\,\mathrm{d}s = 0,$$

which implies that $z_0 \in V_1$. One can easily see that $V_1$ is a Banach space endowed with the norm $\|\cdot\|$. The same can be concluded for the spaces $\mathbb{X}$ and $\mathbb{Y}$. The proof is complete. ∎

**Lemma 3.2.** *Let $L : \mathbb{X} \to \mathbb{Y}$ such that $Lz = \frac{\mathrm{d}z}{\mathrm{d}t}$. Then $L$ is a Fredholm mapping of index zero.*

*Proof.* Clearly, $\mathrm{Ker}\, L = V_2$. It remains to prove that $\mathrm{Im}\, L = V_1$. Suppose that $\phi \in \mathrm{Im}\, L \subset \mathbb{Y}$. Then, there exist $\phi_{V_1} = (\phi_1^{(1)}, \ldots, \phi_1^{(n+m)})^T \in V_1$ and $\phi_{V_2} = (\phi_2^{(1)}, \ldots, \phi_2^{(n+m)})^T \in V_2$ such that

$$\phi = \phi_{V_1} + \phi_{V_2}.$$

From the definitions of $\phi(t)$ and $\phi_{V_1}(t)$, one can deduce that $\int^t \phi(s)\,\mathrm{d}s$ and $\int^t \phi_{V_1}(s)\,\mathrm{d}s$ are almost periodic functions and thus $\phi_{V_2}(t) \equiv (0, 0, \ldots, 0)^T := \mathbf{0}$, which implies that $\phi(t) \in V_1$. Thus, $\mathrm{Im} L \subset V_1$. On the other hand, if $\varphi(t) = (\varphi_1(t), \ldots, \varphi_{n+m}(t))^T \in V_1 \backslash \{\mathbf{0}\}$ then we have $\int_0^t \varphi(s)\,\mathrm{d}s \in AP(\mathbb{R}, \mathbb{R}^n)$. Indeed, if $\lambda \neq 0$ then we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\int_0^t \varphi(s)\,\mathrm{d}s\right] e^{-i\lambda t}\,\mathrm{d}t = \frac{1}{i\lambda} \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(s)e^{-i\lambda t}\,\mathrm{d}s.$$

It follows that

$$\Lambda\left[\int_0^t \varphi(s)\,\mathrm{d}s - M\left(\int_0^t \varphi(s)\,\mathrm{d}s\right)\right] = \Lambda(\varphi).$$

Thus

$$\int_0^t \varphi(s)\,\mathrm{d}s - M\left(\int_0^t \varphi(s)\,\mathrm{d}s\right) \in V_1 \subset \mathbb{X}.$$

Note that $\int_0^t \varphi(s)\,\mathrm{d}s - M(\int_0^t \varphi(s)\,\mathrm{d}s)$ is the primitive of $\varphi(t)$ in $\mathbb{X}$, so we have $\varphi(t) \in \mathrm{Im}\, L$. Hence, $V_1 \subset \mathrm{Im}\, L$, which completes the proof of our claim. Therefore, $\mathrm{Im}\, L = V_1$.

Furthermore, one can easily show that $\mathrm{Im}\, L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}\, L = n = \mathrm{codimIm}\, L$. Therefore, $L$ is a Fredholm mapping of index zero. The proof is complete. ∎

**Lemma 3.3.** *Let $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$, $Q : \mathbb{Y} \to \mathbb{Y}$ such that*

$$Nz = (E_1 z, \ldots, E_{n+m} z)^T,\ z = (x_1, \ldots, x_n, y_1, \ldots, y_m)^T \in \mathbb{X},$$

*where $(E_k z)(t) = \Pi_k(t, z),\ t \in \mathbb{R}, z \in \mathbb{X}, k = 1, \ldots, n+m$ and*

$$Pz = M(z),\ z \in \mathbb{X},\ Qz = M(z),\ z \in \mathbb{Y}.$$

*Then $N$ is $L$-compact on $\bar{\Omega}$, where $\Omega$ is any open bounded subset of $\mathbb{X}$.*

*Proof.* The projections $P$ and $Q$ are continuous such that

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Im} L = \operatorname{Ker} Q.$$

It is clear that

$$(I-Q)V_2 = \{\mathbf{0}\} \quad \text{and} \quad (I-Q)V_1 = V_1.$$

Therefore

$$\operatorname{Im}(I-Q) = V_1 = \operatorname{Im} L.$$

In view of

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q),$$

we can conclude that the generalized inverse (of $L$) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - M\bigg(\int_0^t z(s)\,\mathrm{d}s\bigg).$$

Thus

$$QNz = (F_1 z, \ldots, F_{n+m} z)^T$$

and

$$K_P(I-Q)Nz = G[z(t)] - QG[z(t)],$$

where $G[z]$ is defined by

$$G[z(t)] = \int_0^t [Nz(s) - QNz(s)]\,\mathrm{d}s$$

and

$$F_k z = M(E_k z) = M(\Pi_k(t,z)),\ k = 1, \ldots, n+m.$$

$QN$ and $(I-Q)N$ are obviously continuous. Now we claim that $K_P$ is also continuous. By our hypothesis, for any $\varepsilon < 1$ and any compact set $S \subset C([-\sigma,0],\mathbb{R}^n)$, where $\sigma = \max\limits_{\substack{1\le i,l\le n\\ 1\le j,h\le m}} \sup\limits_{t\in\mathbb{R}} \{\sigma_{il}(t), \tau_{ij}(t), \delta_{jl}(t), \theta_{jh}(t)\}$, let $l(\varepsilon,S)$ be the inclusion interval of $E\{F,\varepsilon,S\}$. Suppose that $\{z_k(t)\} \subset \operatorname{Im}L = V_1$ and $z_k(t)$ uniformly converges to $z_0(t)$. Since $\int_0^t z_k(s)\,\mathrm{d}s \in \mathbb{Y}\,(n = 0,1,2,\ldots)$, there exists $\rho\,(0 < \rho < \varepsilon)$ such that $E\{F,\rho,S\} \subset E\{\int_0^t z_n(s)\,\mathrm{d}s, \varepsilon\}$. Let $l(\rho,S)$ be the inclusion interval of $E\{F,\rho,S\}$ and $l = \max\{l(\rho,S), l(\varepsilon,S)\}$. It is easy to see that $l$ is the inclusion interval of both $E\{\Pi,\varepsilon,S\}$ and $E\{\Pi,\rho,S\}$. Hence, for all $t \notin [0,l]$, there exists $\tau_t \in E\{F,\rho,S\} \subset E\{\int_0^t z_k(s)\,\mathrm{d}s, \varepsilon\}$ such that $t + \tau_t \in [0,l]$. Therefore, by the definition of almost periodic functions we observe that

$$\begin{aligned}\bigg\|\int_0^t z_k(s)\,\mathrm{d}s\bigg\| &= \sup_{t\in\mathbb{R}}\bigg|\int_0^t z_k(s)\,\mathrm{d}s\bigg| \\ &\le \sup_{t\in[0,l]}\bigg|\int_0^t z_k(s)\,\mathrm{d}s\bigg| + \sup_{t\notin[0,l]}\bigg|\bigg(\int_0^t z_k(s)\,\mathrm{d}s - \int_0^{t+\tau_t} z_k(s)\,\mathrm{d}s\bigg) + \int_0^{t+\tau_t} z_k(s)\,\mathrm{d}s\bigg|\end{aligned}$$

$$
\begin{aligned}
&\le 2 \sup_{t\in[0,l]} \left| \int_0^t z_k(s)\,\mathrm{d}s \right| + \sup_{t\notin[0,l]} \left| \int_0^t z_k(s)\,\mathrm{d}s - \int_0^{t+\tau_t} z_k(s)\,\mathrm{d}s \right| \\
&\le 2 \int_0^l |z_k(s)|\,\mathrm{d}s + \varepsilon. \qquad (3.2)
\end{aligned}
$$

By applying (3.2), we conclude that $\int_0^t z(s)\,\mathrm{d}s\,(z \in \operatorname{Im} L)$ is continuous and consequently $K_P$ and $K_P(I-Q)Nz$ are also continuous.

From (3.2), we also have that $\int_0^t z(s)\,\mathrm{d}s$ and $K_P(I-Q)Nz$ are uniformly bounded in $\bar{\Omega}$. In addition, we can easily conclude that $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)Nz$ is equicontinuous in $\bar{\Omega}$. Hence by the Arzelà-Ascoli theorem, we can immediately conclude that $K_P(I-Q)N(\bar{\Omega})$ is compact. Thus $N$ is $L$-compact on $\bar{\Omega}$. The proof is complete. ∎

**Theorem 3.1.** *If the following condition is satisfied:*

$(H)$ *The system of linear algebraic equations*

$$
\begin{cases}
M(a_i) = \sum\limits_{l=1}^{n} M(a_{il})x_l + \sum\limits_{j=1}^{m} M(b_{ij})y_j,\ i = 1,\dots,n, \\
M(r_j) = \sum\limits_{l=1}^{n} M(d_{jl})x_l - \sum\limits_{h=1}^{m} M(e_{jh})y_h,\ j = 1,\dots,m
\end{cases} \qquad (3.3)
$$

*has a unique solution* $(x_1^*,\dots,x_n^*,y_1^*,\dots,y_m^*)^T \in \mathbb{R}^{n+m}$ *with* $x_i^* > 0, y_j^* > 0,\ i = 1,\dots,n, j = 1,\dots,m.$

*Then Eq.*(1.1) *has at least one positive almost periodic solution.*

*Proof.* In order to apply Lemma 2.1, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ the same as those in Lemma 3.1 and the mappings $L, N, P, Q$ the same as those defined in Lemmas 3.2 and 3.3, respectively. Thus, we can obtain that $L$ is a Fredholm mapping of index zero and $N$ is a continuous operator which is $L$-compact on $\bar{\Omega}$. It remains to search for an appropriate open and bounded subset $\Omega$.

Corresponding to the operator equation

$$
Lz = \lambda N z,\ \ \lambda \in (0,1), \quad \text{where} \quad z = (x_1,\dots,x_n,y_1,\dots,y_m)^T,
$$

we have

$$
\begin{cases}
\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \lambda\Big[a_i(t) - \sum\limits_{l=1}^{n} a_{il}(t)e^{x_l(t-\sigma_{il}(t))} - \sum\limits_{j=1}^{m} b_{ij}(t)e^{y_j(t-\tau_{ij}(t))}\Big],\ i = 1,\dots,n, \\
\dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} = \lambda\Big[-r_j(t) + \sum\limits_{l=1}^{n} d_{jl}(t)e^{x_l(t-\delta_{jl}(t))} - \sum\limits_{h=1}^{m} e_{jh}(t)e^{y_h(t-\theta_{jh}(t))}\Big],\ j = 1,\dots,m.
\end{cases} \qquad (3.4)
$$

Suppose that $z \in \mathbb{X}$ is a solution of (3.4) for a certain $\lambda \in (0,1)$. For any $t_0 \in \mathbb{R}$, we can choose a point $\tilde{\tau} - t_0 \in [l,2l] \cap E\{\Pi,\rho,S\}$, where $\rho\,(0<\rho<\varepsilon)$ satisfies $E\{\Pi,\rho\} \subset E\{z,\varepsilon\}$. Integrating (3.4) from $t_0$ to $\tilde{\tau}$, we get

$$
\lambda \int_{t_0}^{\tilde{\tau}} \Big[\sum_{l=1}^{n} a_{il}(s)e^{x_l(s-\sigma_{il}(s))} + \sum_{j=1}^{m} b_{ij}(s)e^{y_j(s-\tau_{ij}(s))}\Big]\mathrm{d}s
$$

$$\leq \lambda \int_{t_0}^{\tilde{\tau}} a_i(s)\,\mathrm{d}s + \left| \int_{t_0}^{\tilde{\tau}} \dot{x}_i(s)\,\mathrm{d}s \right| \leq \lambda \int_{t_0}^{\tilde{\tau}} a_i(s)\,\mathrm{d}s + \varepsilon,\ i = 1, \ldots, n, \tag{3.5}$$

$$\lambda \int_{t_0}^{\tilde{\tau}} \left[ \sum_{l=1}^{n} d_{jl}(s) e^{x_l(s-\delta_{jl}(s))} - \sum_{h=1}^{m} e_{jh}(s) e^{y_h(s-\theta_{jh}(s))} \right] \mathrm{d}s$$
$$\leq \lambda \int_{t_0}^{\tilde{\tau}} r_j(s)\,\mathrm{d}s + \left| \int_{t_0}^{\tilde{\tau}} \dot{y}_j(s)\,\mathrm{d}s \right| \leq \lambda \int_{t_0}^{\tilde{\tau}} r_j(s)\,\mathrm{d}s + \varepsilon,\ j = 1, \ldots, m \tag{3.6}$$

and

$$\lambda \int_{t_0}^{\tilde{\tau}} \left[ \sum_{l=1}^{n} d_{jl}(s) e^{x_l(s-\delta_{jl}(s))} - \sum_{h=1}^{m} e_{jh}(s) e^{y_h(s-\theta_{jh}(s))} \right] \mathrm{d}s$$
$$\geq \lambda \int_{t_0}^{\tilde{\tau}} r_j(s)\,\mathrm{d}s - \left| \int_{t_0}^{\tilde{\tau}} \dot{y}_j(s)\,\mathrm{d}s \right| \geq \lambda \int_{t_0}^{\tilde{\tau}} r_j(s)\,\mathrm{d}s - \varepsilon,\ j = 1, \ldots, m. \tag{3.7}$$

Hence, from (3.4) and (3.5), we obtain

$$\int_{t_0}^{\tilde{\tau}} |\dot{x}_i(s)|\,\mathrm{d}s \leq \lambda \int_{t_0}^{\tilde{\tau}} a_i(s)\,\mathrm{d}s + \lambda \int_{t_0}^{\tilde{\tau}} \left[ \sum_{l=1}^{n} a_{il}(s) e^{x_l(s-\sigma_{il}(s))} + \sum_{j=1}^{m} b_{ij}(s) e^{y_j(s-\tau_{ij}(s))} \right] \mathrm{d}s$$
$$\leq 2\lambda \int_{t_0}^{\tilde{\tau}} a_i(s)\,\mathrm{d}s + \varepsilon \leq 2 \int_{t_0}^{\tilde{\tau}} a_i(s)\,\mathrm{d}s + 1 := C_i,\ i = 1, \ldots, n.$$

Therefore, for $\tilde{\tau} \geq t_0 + l$, we have

$$\int_{t_0}^{t_0+l} |\dot{x}_i(t)|\,\mathrm{d}t \leq C_i,\ i = 1, \ldots, n.$$

Similarly, from (3.4), (3.6) and (3.7), we can obtain

$$\int_{t_0}^{\tilde{\tau}} |\dot{y}_j(s)|\,\mathrm{d}s \leq 2 \int_{t_0}^{\tilde{\tau}} r_j(s)\,\mathrm{d}s + 1 := C_{n+j},\ j = 1, \ldots, m.$$

Thus, since $\tilde{\tau} \geq t_0 + l$, one has

$$\int_{t_0}^{t_0+l} |\dot{y}_j(t)|\,\mathrm{d}t \leq C_{n+j},\ j = 1, \ldots, m.$$

Denote

$$\bar{\theta} = \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} x_i(t), \quad \underline{\theta} = \min_{1\leq i\leq n} \inf_{t\in\mathbb{R}} x_i(t),\ i = 1, \ldots, n.$$

In view of (3.4), we obtain

$$M(a_i) = M\Big( \sum_{l=1}^{n} a_{il}(t) e^{x_l(t-\sigma_{il}(t))} + \sum_{j=1}^{m} b_{ij}(t) e^{y_j(t-\tau_{ij}(t))} \Big),\ i = 1, \ldots, n. \tag{3.8}$$

From (3.8), one has

$$M(a_i) \geq \Big(\sum_{l=1}^{n} M(a_{il}) + \sum_{j=1}^{m} M(b_{ij})\Big)e^{\underline{\theta}},\ i = 1, \ldots, n,$$

or

$$\underline{\theta} \leq \min_{1\leq i\leq n}\Big\{\ln\frac{M(a_i)}{\sum\limits_{l=1}^{n} M(a_{il}) + \sum\limits_{j=1}^{m} M(b_{ij})}\Big\} := B.$$

Consequently, by Lemma 2.2, for any $\varepsilon > 0$, there exist $\xi_\varepsilon^i$ and $\zeta_\varepsilon^j$ such that

$$\begin{aligned} x_i(t) &\leq x_i(\xi_\varepsilon^i) + \int_{t_0}^{t_0+l} |\dot{x}_i(t)|\,\mathrm{d}t < (\underline{\theta} + \varepsilon) + C_i \\ &< B + 1 + C_i,\ i = 1, \ldots, n \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} y_j(t) &\leq y_j(\zeta_\varepsilon^i) + \int_{t_0}^{t_0+l} |\dot{y}_j(t)|\,\mathrm{d}t < (\underline{\theta} + \varepsilon) + C_{n+j} \\ &< B + 1 + C_{n+j},\ j = 1, \ldots, m. \end{aligned} \tag{3.10}$$

Similarly, we get

$$M(a_i) \leq \Big(\sum_{l=1}^{n} M(a_{il}) + \sum_{j=1}^{m} M(b_{ij})\Big)e^{\bar{\theta}},\ i = 1, \ldots, n,$$

so

$$\bar{\theta} \geq \max_{1\leq i\leq n}\Big\{\ln\frac{M(a_i)}{\sum\limits_{l=1}^{n} M(a_{il}) + \sum\limits_{j=1}^{m} M(b_{ij})}\Big\} := C.$$

By Lemma 2.2, for any $\varepsilon > 0$, there exist $\eta_\varepsilon^i$ and $\varsigma_\varepsilon^j$ such that

$$\begin{aligned} x_i(t) &\geq x_i(\eta_\varepsilon^i) - \int_{t_0}^{t_0+l} |\dot{x}_1(t)|\,\mathrm{d}t > (\bar{\theta} - \varepsilon) - C_i \\ &\geq C - C_i - 1,\ i = 1, \ldots, n \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} y_j(t) &\geq y_j(\varsigma_\varepsilon^i) - \int_{t_0}^{t_0+l} |\dot{y}_j(t)|\,\mathrm{d}t > (\bar{\theta} - \varepsilon) - C_{n+j} \\ &\geq C - C_{n+j} - 1,\ j = 1, \ldots, m. \end{aligned} \tag{3.12}$$

It follows from (3.9)-(3.12) that

$$||z|| \leq \max_{1\leq k\leq n+m}\big\{|B + (C_k + 1)|, |C - (C_k + 1)|\big\} := D.$$

Clearly, $D$ is independent of the choice of $\lambda$. Take $M = D + K$, where $K > 0$ is taken sufficiently large such that the unique solution $(x_1^*, \dots, y_1^*, \dots, y_n^*)^T$ of system (3.3) satisfies $\|(x_1^*, \dots, y_1^*, \dots, y_n^*)^T\| < M$. Next, take

$$\Omega = \{z = (x_1, \dots, x_n, y_1, \dots, y_m)^T \in \mathbb{X} : \|z\| < M\},$$

then it is clear that $\Omega$ satisfies the condition (1) of Lemma 2.1. When $z \in \partial\Omega \cap \operatorname{Ker} L$, then $z$ is a constant vector with $\|z\| = M$. Hence

$$QNz = (F_1 z, \dots, F_{n+m} y)^T \neq \mathbf{0},$$

which implies that condition (2) of Lemma 2.1 is satisfied. Furthermore, take $J : \operatorname{Im} Q \to \operatorname{Ker} L$ such that $J(z) = z$ for $z \in \mathbb{Y}$. In view of $(H)$, by a straightforward computation, we find

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Therefore, condition (3) of Lemma 2.1 holds. Hence, $Lz = Nz$ has at least one solution in $\operatorname{Dom} L \cap \bar{\Omega}$. In other words, Eq.(3.1) has at least one almost periodic solution $z(t)$, that is, Eq.(1.1) has at least one positive almost periodic solution $(u_1(t), \dots, u_n(t), v_1(t), \dots, v_m(t))^T$. The proof is complete. ∎

**Remark 3.1.** *Suppose that* (1.1) *is an $\omega$-periodic system. Take $X = Y = \{z \in C(\mathbb{R}, \mathbb{R}^{n+m}) : z(t + \omega) = z(t), t \in \mathbb{R}\}$ with the suprem norm, then $X, Y$ are Banach spaces. Similar to the proof of Theorem 2.1 in [6] and using the similar priori estimate method used in the proof of Theorem 3.1, one can easily get that*

*If the system of linear algebraic equations*

$$\begin{cases} \bar{a}_i = \sum_{l=1}^{n} \bar{a}_{il} x_l + \sum_{j=1}^{m} \bar{b}_{ij} y_j, \ i = 1, \dots, n, \\ \bar{r}_j = \sum_{l=1}^{n} \bar{d}_{jl} x_l - \sum_{h=1}^{m} \bar{e}_{jh} y_h, \ j = 1, \dots, m \end{cases}$$

*has a unique solution $(x_1^*, \dots, x_n^*, y_1^*, \dots, y_m^*)^T \in \mathbb{R}^{n+m}$ with $x_i^* > 0, y_j^* > 0$, $i = 1, \dots, n, j = 1, \dots, m$, where for a continuous $\omega$-periodic function $f$, we denote $\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) \mathrm{d}t$. Then* (1.1) *has at least one positive $\omega$-periodic solution.*

*To the best of the author's knowledge, this result is also a new one.*

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