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LOCAL ESTIMATES FOR MODIFIED RICCATI EQUATION IN THEORY OF HALF-LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we study the half-linear differential equation

$$\big(r(t)\Phi_p(x')\big)' + c(t)\Phi_p(x) = 0,$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$. Using modified Riccati technique and suitable local estimates for terms in modified Riccati equation we derive new characterization of principal solution and new nonoscillation criteria.

1. INTRODUCTION

In this paper we consider the equation

$$\mathcal{L}[x] := \big(r(t)\Phi_p(x')\big)' + c(t)\Phi_p(x) = 0, \tag{1}$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, $r \in C((t_0,\infty),\mathbb{R}^+)$, $c \in C((t_0,\infty),\mathbb{R})$ for some t_0. Under a solution of this equation we understand every continuously differentiable function x such that $r\Phi_p(x')$ is differentiable and (1) holds on (t_0,∞). This equation is called half-linear, since a constant multiple of any solution is also a solution of (1).

The asymptotic behavior of equation (1) is a subject of many papers. It turns out (see [5]) that equation (1) can be classified as oscillatory and nonoscillatory. Further, there is one significant solution of the nonoscillatory equation – the principal solution.

The aim of this paper is to continue some previous studies of nonoscillatory equations (especially [1, 7, 8]) and derive nonoscillation criteria for half-linear equations and new results related to the principal solutions. We use and refine the modified Riccati technique introduced in [3, 4] and show that global estimates used in these papers can be under

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some additional assumptions replaced by local versions. This results to new nonoscillation criteria and also a new test which can be used to detect whether a solution is principal or nonprincipal.

Throughout the paper we suppose that $h(t) \in C^1([t_0,\infty),(0,\infty))$ is a positive function such that $h'(t)$ has no zero in some neighborhood of infinity. Given equation (1) and the function $h(t)$, we define

$$G(t) = r(t)h(t)\Phi_p(h'(t)). \tag{2}$$

Note that the requirement $h'(t) \neq 0$ is natural in some sense in view of the following lemma.

Lemma A ([5, Lemma 4.1.1])**.** *Let $c(t) \neq 0$ for large t and x be a solution of nonoscillatory equation* (1). *Then either $x(t)x'(t) > 0$ or $x(t)x'(t) < 0$ for large t.*

The paper is organized as follows. In the next section we introduce basic facts related to the modified Riccati technique and derive local estimates to the nonlinear term in modified Riccati equation. Section 3 contains a short introduction to the principal solution and a new criterion which allows to detect a solution of nonoscillatory equation as principal. In Section 4 we derive new nonoscillation criteria.

2. Preliminary results

It is well known (see e.g. [5, Chapter 1.1.4]) that the substitution $w = r\Phi_p\left(\frac{x'}{x}\right)$ converts (1) into the following Riccati type equation

$$\mathcal{R}[w] := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0. \tag{3}$$

The following lemma shows that the Riccati operator from this equation is closely related to the nonoscillatory equation (1), see e.g. [5, Theorem 2.2.1].

Lemma 1. *Equation* (1) *is nonoscillatory if and only if there exists a differentiable function w which satisfies the Riccati type inequality*

$$\mathcal{R}[w](t) \leq 0 \tag{4}$$

for large t.

Our results heavily depend on the following relationship between the Riccati type differential operator $\mathcal{R}[\cdot]$ defined in (3) and the so-called modified Riccati operator (the operator on the right hand side of (5) below).

Lemma 2 ([7, Lemma 2.2]). *Let h and w be differentiable functions and $v = h^p w - G$, then we have the identity*

$$h^p \mathcal{R}[w] = v' + h\mathcal{L}[h] + (p-1)r^{1-q}h^{-q}H(t,v), \tag{5}$$

where $H(t,v) = |v+G|^q - q\Phi_q(G)v - |G|^q \geq 0.$

The following estimate plays a crucial role in the proofs of our main results.

Lemma 3. *Let $v(t)$ and $G(t)$ be real functions defined on $[t_0, \infty)$, such that*

$$\lim_{t\to\infty} \frac{v(t)}{G(t)} = 0. \tag{6}$$

Let $\gamma \in (1,2)$ and $K > 0$ be real numbers. There exists $t_1 \geq t_0$ such that

$$H(t, v(t)) \leq K|G(t)|^q \left|\frac{v(t)}{G(t)}\right|^\gamma \tag{7}$$

for $t \geq t_1$.

Remark 1. The inequality

$$H(t, v(t)) \leq q\beta_{\gamma,p}|G(t)|^q \left|\frac{v(t)}{G(t)}\right|^\gamma \tag{8}$$

which holds for $p \geq 2$, $\gamma \in [q, 2]$, a convenient number $\beta_{\gamma,p}$ and every t has been proved in [7, Lemma 2.3]. In contrast to (8), inequality (7) holds only for restricted values of the quotient $v(t)/G(t)$ and from this poin of view it can be considered as a local version of (8). It shows that if we restrict ourselves to the case $\frac{v}{G} \to 0$ as $t \to \infty$, then we can drop some of the assumptions of [7, Lemma 2.3].

Proof of Lemma 3. From the defininiton the function H we have

$$\begin{aligned} H(t,v) &= |v+G|^q - q\Phi_q(G)v - |G|^q \\ &= q|G|^q \left(\frac{1}{q}\left|\frac{v}{G}+1\right|^q - \frac{v}{G} - \frac{1}{q}\right) \\ &= q|G|^q g\left(\frac{v}{G}\right) \end{aligned}$$

where $g(x) = \frac{|x+1|^q}{q} - x - \frac{1}{q}$. The function $g(x)$ satisfies

$$g(x) = (q-1)\frac{1}{2!}x^2 + O(x^3)$$

in an neighborhood of $x = 0$ and hence for every $\gamma \in (1, 2)$ and every $K > 0$ there exists x_0 such that

$$g(x) \leq \frac{K}{q}|x|^\gamma$$

for every x which satisfies $|x| \leq x_0$. This inequality together with (6) implies that there exists t_1 such that (7) holds for every $t \geq t_1$. □

The following lemma presents a local lower estimate for the function $H(t, v(t))$. Since there is a close correspondence

$$H(t, v) = qP(\Phi_q(G), v + G), \tag{9}$$

between the function H and the function P given by

$$P(a, b) := \frac{|a|^p}{p} - ab + \frac{|b|^q}{q}, \tag{10}$$

(see [7, Eq. (14)]) the estimates from Lemma 3 and 4 can be also treated as local estimates for the function P.

Lemma 4. *For every $\gamma \geq 2$ and every $K_0 \in (0, \infty)$ there exists a constant $K > 0$ such that if $G(t) = 0$ or $\left|\frac{v(t)+G(t)}{G(t)}\right| \leq K_0$, then*

$$H(t, v(t)) \geq qK|G(t)|^{q-\gamma}|v(t)|^\gamma. \tag{11}$$

Proof. Let $\gamma \geq 2$ and $K_0 \in (0, \infty)$ be arbitrary. If $G(t) = 0$ then (11) holds. Suppose that $G(t) \neq 0$ and $\left|\frac{v(t)+G(t)}{G(t)}\right| \leq K_0$. Using (9), (10) and the obvious fact

$$P(a, b) = |a|^p P\left(1, \frac{b}{\Phi_p(a)}\right)$$

we can write the function H in the form

$$H(t, v) = q|\Phi_q(G)|^p P\left(1, \frac{v}{G} + 1\right) = q|G|^q f\left(\frac{v}{G} + 1\right), \tag{12}$$

where

$$f(x) = P(1, x) = \frac{1}{p} - x + \frac{|x|^q}{q}.$$

It is easy to see that the function

$$\varphi(x) = \begin{cases} \frac{|x-1|^\gamma}{f(x)} & x \neq 1 \\ \lim_{x\to 1} \frac{|x-1|^\gamma}{f(x)} & x = 1 \end{cases}$$

is well defined, nonegative and continuous on $[-K_0, K_0]$ and there exists C such that $\varphi(x) \le C$ on $[-K_0, K_0]$. This shows that

$$f(x) \ge K|x-1|^\gamma$$

holds with $K = \frac{1}{C}$ and $|x| \le K_0$. Combining the above coputations we get (11). ☐

3. Principal solution of nonoscillatory equation

If a half-linear equation is nonoscillatory, then there is a solution of the associated Riccati equation which can be extended to some neighborhood of infinity. It has been shown in [9], that among all solutions of (3) which can be extended to infinity there exists the so-called *minimal solution* $\widetilde{w}$ with the following property: if $\widetilde{w}$ and w are two distinct solutions of (3) defined on $[T, \infty)$, then $w(t) > \widetilde{w}(t)$ for $t \in [T, \infty)$.

The *principal solution* $\widetilde{x}$ of (1) is defined as the solution which determines the minimal solution $\widetilde{w}$ of (3) via the substitution $\widetilde{w} = r\Phi_p(\widetilde{x}'/\widetilde{x})$, i.e.,

$$\widetilde{x}(t) = C \exp\left\{\int^t \Phi_q\big(\widetilde{w}(s)/r(s)\big)\, \mathrm{d}s\right\}.$$

This principal solution is unique up to a nonzero constant multiple.

In [7, Theorem 4.1] we proved the following theorem.

Theorem A. *Suppose that* (1) *is nonoscillatory and* $h(t)$ *is its positive solution which satisfies* $h'(t) \neq 0$ *for large* t.

(i) Let $p \ge 2$*. If* h *is a principal solution, then for every* $\gamma \in [q, 2]$

$$\int^\infty \frac{\mathrm{d}t}{r^{\gamma-1}(t) h^\gamma(t) |h'(t)|^{(p-1)(\gamma-q)}} = \infty \tag{13}$$

holds.

(ii) Let $p \in (1, 2]$*. If* (13) *holds for some* $\gamma \in [2, q]$*, then* h *is a principal solution.*

The following theorem shows that under some additional assumptions we can drop the restrictions $p \le 2$ and $\gamma \le q$ from the implication (ii) and we get the following statement which is in some sense close to the opposite implication of the statement (i) of Theorem A.

Theorem 1. *Suppose that* (1) *is nonoscillatory and* $h(t)$ *is its positive solution which satisfies* $h'(t) \neq 0$ *for large* t*. Further suppose that* $\int_t^\infty c(s)\,\mathrm{d}s \ge 0$, $\int_t^\infty c(s)\,\mathrm{d}s \not\equiv 0$ *for large* t *and* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$*. If* *there exists a real number $\gamma \geq 2$ such that* (13) *holds, then h is the principal solution.*

Proof. Suppose, by contradiction, that assumptions of the theorem hold and h is not principal. Denote $w_h := r\Phi_p(h'/h)$ the corresponding solution of (3). Since h is not principal, there exists $T > 0$ and a solution $\widetilde{w}$ of (3) such that $\widetilde{w}(t) < w_h(t)$ for $t \geq T$. Condition $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$ and the convergence of $\int^\infty c(s)\,\mathrm{d}s$ imply (see [5, Theorem 2.2.3 and Theorem 2.2.4]) that $\int^\infty r^{1-q}(t)|\widetilde{w}(t)|^q\,\mathrm{d}t < \infty$ and

$$\widetilde{w}(t) = \int_t^\infty c(s)\,\mathrm{d}s + (p-1)\int_t^\infty r^{1-q}(t)|\widetilde{w}(t)|^q\,\mathrm{d}s$$

for $t \geq T$. Since $\int_t^\infty c(s)\,\mathrm{d}s \geq 0$, we have $\widetilde{w}(t) \geq 0$ and hence $0 \leq \frac{\widetilde{w}(t)}{w_h(t)} < 1$. Consequently, consider the function $v = h^p\widetilde{w} - G = h^p(\widetilde{w} - w_h)$. It holds $v(t) < 0$ for $t \geq T$ and since $\mathcal{L}[h] = 0$, we see from identity (5) that v is a solution of the modified Riccati equation

$$v' + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v) = 0 \tag{14}$$

for $t \geq T$. We have $\frac{v}{G} = \frac{h^p\widetilde{w}}{G} - 1 = \frac{\widetilde{w}}{w_h} - 1$, i.e., $-1 \leq \frac{v}{G} < 0$ and $\left|\frac{v+G}{G}\right| \leq 1$ for $t \geq T$. Now, using (11), there exists $K > 0$ such that

$$H(t, v(t)) \geq qK|G(t)|^{q-\gamma}|v(t)|^\gamma, \quad t \geq T,$$

hence

$$\begin{aligned}(p-1)r^{1-q}(t)h^{-q}(t)H(t,v(t)) &\geq pKr^{1-q}(t)h^{-q}(t)|G(t)|^{q-\gamma}|v(t)|^\gamma \\ &= pKr^{1-\gamma}(t)h^{-\gamma}(t)|h'|^{(p-1)(q-\gamma)}|v(t)|^\gamma, \quad t \geq T.\end{aligned}$$

It follows from (14), that v is a solution of the inequality

$$v' + pKr^{1-\gamma}(t)h^{-\gamma}(t)|h'|^{(p-1)(q-\gamma)}|v(t)|^\gamma \leq 0,$$

i.e.,

$$-\frac{v'}{|v|^\gamma} \geq pKr^{1-\gamma}(t)h^{-\gamma}(t)|h'|^{(p-1)(q-\gamma)}$$

on $t \in [T, \infty)$. Integrating this inequality over $[T, t]$ we obtain

$$\frac{1}{(\gamma-1)|v(T)|^{\gamma-1}} - \frac{1}{(\gamma-1)|v(t)|^{\gamma-1}} \geq pK\int_T^t r^{1-\gamma}(s)h^{-\gamma}(s)|h'(s)|^{(p-1)(q-\gamma)}\,\mathrm{d}s.$$

Letting $t \to \infty$, we have

$$\frac{1}{(\gamma-1)|v(T)|^{\gamma-1}} \geq pK\int_T^\infty r^{1-\gamma}(s)h^{-\gamma}(s)|h'(s)|^{(p-1)(q-\gamma)}\,\mathrm{d}s.$$

This contradicts (13). □

Note that the case when $G(t) \to 0$ as $t \to \infty$ is delicate in some sense, since in this case the integral in (13) may fail to be divergent if γ is not sufficiently large. Hence the "usual" integral criteria to detect principality which deal with $\gamma = 2$ (see [1, Example 1 and Remark 2]) may fail.

Using the definition (2) of the function G we can write (13) in the form

$$\int^{\infty} |G(t)|^{1-\gamma} \frac{|h'(t)|}{h(t)} \,\mathrm{d}t = \infty. \tag{15}$$

The following simple corollary shows, that if the function $G(t)$ approaches zero sufficiently fast and the fraction $\frac{|h'(t)|}{h(t)}$ does not tend to zero faster than a power function, then h is a principal solution.

Corollary 1. *Suppose that* $\int_t^\infty c(s)\,\mathrm{d}s \geq 0$, $\int_t^\infty c(s)\,\mathrm{d}s \not\equiv 0$ *for large* t *and* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$. *Suppose that there exist a positive solution* h *of equation* (1), *real number* β *and positive real numbers* ε *and* K *such that*

$$h'(t) \neq 0 \quad \text{and} \quad \frac{|h'(t)|}{h(t)} \geq K t^{\beta} \quad \text{for large } t,$$
$$G(t) = O(t^{-\varepsilon}) \quad \text{as } t \to \infty.$$

Then h *is the principal solution of* (1).

Proof. Let $\gamma \geq 2$. From the assumptions it follows that there exist K_1 such that

$$|G(t)| \leq K_1 t^{-\varepsilon}$$

and hence

$$|G(t)|^{1-\gamma} \geq K_2 t^{\varepsilon(\gamma-1)}$$

for $t \geq T_0$. This shows that

$$|G(t)|^{1-\gamma} \frac{|h'|}{h} \geq K K_2 t^{\beta+\varepsilon(\gamma-1)}$$

and if $\gamma \geq \gamma_0 := \max\{2, 1-(\beta+1)/\varepsilon\}$, then (15) holds and the solution h is principal by Theorem 1. □

Example 1. Consider equation

$$(\Phi_{3/2}(x'))' + \frac{15t^{-3/2}}{(t^9-1)^{1/2}} \Phi_{3/2}(x) = 0, \quad t > 1. \tag{16}$$

The function $h(t) = 1 - 1/t^9$ is a solution of this equation. This solution is a principal solution, as follows easily from Corollary 1 and from the fact that $h(t) \sim 1$, $h'(t) \sim 9t^{-10}$ and $G(t) \sim 3t^{-5}$ near infinity. Note that the principality of this solution has been proved in [1] and [7], but in both cases using more advanced arguments.

4. Nonoscillation criteria

A frequently used approach in the nonoscillation criteria is to establish sufficient conditions which guarantee that the Riccati inequality (4) has a solution in a neighborhood of infinity. This approach has been used in [2] and [6] to show that if an expression involving integral $\int \frac{\mathrm{d}t}{r(t)h^2(t)|h'(t)|^{p-2}}$ and another term arising from the fact that the equation is viewed as a perturbation of another nonoscillatory equation fit into certain bounds (limes inferior is not too small and limes superior is not too large), then the equation is nonoscillatory. See also [8] for summary and refinement of these results. A typical result from [8] is the following.

Theorem B ([8, Theorem 1]). *Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Suppose that the following conditions hold*

$$\begin{cases} \displaystyle\int^{\infty} \frac{\mathrm{d}t}{r(t)h^2(t)|h'(t)|^{p-2}} < \infty, \\ \displaystyle\lim_{t\to\infty} |G(t)| \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} = \infty. \end{cases} \tag{17}$$

If

$$\limsup_{t\to\infty} \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s < \frac{1}{q}\left(-\alpha + \sqrt{2\alpha}\right),$$
$$\liminf_{t\to\infty} \int_t^{\infty} \frac{\mathrm{d}s}{r(s)h^2(s)|h'(s)|^{p-2}} \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s > \frac{1}{q}\left(-\alpha - \sqrt{2\alpha}\right)$$

for some $\alpha > 0$, then equation (1) *is nonoscillatory.*

In view of the results from [7] it seems to be natural to derive a variant of Theorem B and related theorems with $\int \frac{\mathrm{d}t}{r(t)h^2(t)|h'(t)|^{p-2}}$ replaced by the integral from (13). Note that we do not allow $\gamma = 2$ in Theorems 2 and 3 and from this reason these theorems do not include the results from [2, 6, 8] as special cases. Also note that we use opposite estimates than in the previous section and thus the condition $\gamma \geq 2$ from Section 3 is replaced by the condition $\gamma \in (1, 2)$ in the following theorems.

Theorem 2. *Let $\gamma \in (1,2)$ be a real number and $\bar\gamma = \frac{\gamma}{\gamma-1}$ be the conjugate number to γ. Let h be a positive continuously differentiable function such that $h'(t) \neq 0$ in some neighborhood of infnity. Denote*

$$R(t) = r^{\gamma-1}(t)h^{\gamma}(t)|h'(t)|^{(\gamma-q)(p-1)} \tag{18}$$

and suppose that

$$\int^{\infty} R^{-1}(s)\,\mathrm{d}s < \infty \tag{19}$$

and

$$\lim_{t\to\infty} |G(t)| \left[\int_t^{\infty} R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} = \infty. \tag{20}$$

If

$$\limsup_{t\to\infty} \left[\int_t^{\infty} R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s < \infty \tag{21}$$

and

$$\liminf_{t\to\infty} \left[\int_t^{\infty} R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s > -\infty, \tag{22}$$

then (1) *is nonoscillatory.*

Proof. Denote

$$Y(t) := \left[\int_t^{\infty} R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s.$$

Conditions (21), (22) imply that there exist $t_0 \in \mathbb{R}$ and positive constants α, c_0 such that

$$|Y(t)+\alpha|^{\gamma} < \frac{\alpha}{c_0}, \quad \text{for } t \geq t_0. \tag{23}$$

Define the function

$$v(t) = -\alpha \left[\int_t^{\infty} R^{-1}(s)\,\mathrm{d}s\right]^{1-\bar\gamma} - \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s.$$

Then

$$v'(t) = \frac{\alpha(1-\bar\gamma)}{R(t)\left[\int_t^{\infty} R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma}} - h(t)\mathcal{L}[h](t)$$

and

$$\frac{v(t)}{G(t)} = \frac{-\alpha \left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{1-\bar\gamma} - \int^t h(s)\mathcal{L}[h](s)\,\mathrm{d}s}{G(t)}$$
$$= \frac{-\alpha - Y(t)}{G(t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1}}.$$

Conditions of the Theorem imply that $v(t)/G(t) \to 0$ as $t \to \infty$, hence, using inequality (7) with $K = c_0(\bar\gamma - 1)(q-1)$, we obtain that there exists $t_1 \geq t_0$ such that

(24)
$$(p-1)r^{1-q}(t)h^{-q}(t)H(t,v(t)) \leq c_0(\bar\gamma-1)r^{1-q}(t)h^{-q}(t)|G(t)|^q \left|\frac{v(t)}{G(t)}\right|^\gamma$$
$$= c_0(\bar\gamma-1)r(t)|h'(t)|^p \left|\frac{v(t)}{G(t)}\right|^\gamma.$$

holds for $t \geq t_1$.

Consequently, if $w = h^{-p}(v+G)$, we have by identity (5)

$$\begin{aligned}
h^p(t)\mathcal{R}[w](t) &= v'(t) + h(t)\mathcal{L}[h](t) + (p-1)r^{1-q}(t)h^{1-q}(t)H(t,v(t)) \\
&\leq -\frac{\alpha(\bar\gamma-1)}{R(t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma}} \\
&\quad + c_0(\bar\gamma-1)r(t)|h'(t)|^p \left|\frac{-\alpha - Y(t)}{G(t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1}}\right|^\gamma \\
&= \frac{1}{R(t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma}} \\
&\quad \times \left[-\alpha(\bar\gamma-1) + c_0(\bar\gamma-1)\frac{r(t)|h'(t)|^p R(t)}{|G(t)|^\gamma}|\alpha + Y(t)|^\gamma\right] \\
&= \frac{\bar\gamma-1}{R(t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma}}\left[-\alpha + c_0|\alpha+Y(t)|^\gamma\right] \\
&< 0.
\end{aligned}$$

This means that (1) is nonoscillatory by Lemma 1. □

Theorem 3. *Let $\gamma \in (1,2)$ be a real number and $\bar\gamma = \frac{\gamma}{\gamma-1}$ be the conjugate number to γ. Let h be a positive continuously differentiable function such that $h'(t) \neq 0$ in some neighborhood of infinity. Define*

$R(t)$ by (18) and suppose that $\int^\infty h(t)\mathcal{L}[h](t)\,\mathrm{d}t$ is convergent and

$$\lim_{t\to\infty} |G(t)| \left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} = \infty. \tag{25}$$

If

$$\limsup_{t\to\infty} \left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} \int_t^\infty h(s)\mathcal{L}[h](s)\,\mathrm{d}s < \infty$$

and

$$\liminf_{t\to\infty} \left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} \int_t^\infty h(s)\mathcal{L}[h](s)\,\mathrm{d}s > -\infty,$$

then (1) is nonoscillatory.

Proof. We take

$$v(t) = \alpha \left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{1-\bar\gamma} + \int_t^\infty h(s)\mathcal{L}[h](s)\,\mathrm{d}s$$

and similarly as in the proof of Theorem 2, using (5), (7) and (23), we conclude that

$$h^p(t)\mathcal{R}[w](t) \le \frac{\bar\gamma - 1}{R(t)\left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma}} \left[-\alpha + c_0|\alpha + Y(t)|^\gamma\right] < 0,$$

where

$$Y(t) = \left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1} \int_t^\infty h(s)\mathcal{L}[h](s)\,\mathrm{d}s$$

and α, c_0 are positive constants. □

Remark 2. If we compare Theorems 2 and 3 with Theorem B and other related results from [2, 6, 8] we can see that finite nonoscillation constants which appear in [2, 6, 8] are replaced by ∞ and $-\infty$. An explanation for this phenomenon is the fact that an arbitrary constant K can be used in (7) in contrast to the estimate used in [2, 6, 8], where quadratic approximation is used and the constant in this approximation has to be bigger than the second derivative of the function from this approximation.

In the following theorems we view equation (1) as a perturbation of another nonoscillatory equation

$$\widetilde{\mathcal{L}}[x] := \big(r(t)\Phi_p(x')\big)' + \widetilde{c}(t)\Phi_p(x) = 0. \tag{26}$$

Theorem 4. *Let $\gamma \in (1,2)$ be a real number and $\bar\gamma = \frac{\gamma}{\gamma-1}$ be the conjugate number to γ. Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Let R be defined by* (18) *and suppose that both* (19) *and* (20) *hold. If*

$$\limsup_{t\to\infty} R(t)h(t)\widetilde{\mathcal{L}}[h](t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma} < \infty,$$

$$\limsup_{t\to\infty}\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1}\int^t h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s < \infty$$

and

$$\liminf_{t\to\infty}\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1}\int^t h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s > -\infty,$$

then (1) *is nonoscillatory.*

Proof. Denote

$$Y(t) = \left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma-1}\int^t h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s.$$

From the assumptions of the theorem it follows, that there exist positive numbers c_0, T_0 and α (c_0 sufficiently small and T_0, α sufficiently large) such that

$$f(c_0,\alpha,t) := R(t)h(t)\widetilde{\mathcal{L}}[h](t)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{\bar\gamma} + \alpha(1-\bar\gamma) + c_0(\bar\gamma-1)|\alpha+Y(t)|^\gamma < 0$$

for every $t \geq T_0$. Consider functions v and w defined by

$$v(t) = -\alpha\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{1-\bar\gamma} - \int^t h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s \tag{27}$$

and $w(t) = h^{-p}(t)\Big(v(t)+G(t)\Big)$. According to (5) we have

$$\begin{aligned} h^p(t)\mathcal{R}[w](t) &= v'(t) + h(t)\mathcal{L}[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v(t)) \\ &= \alpha(1-\bar\gamma)\left[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right]^{-\bar\gamma} R^{-1} + h(t)\widetilde{\mathcal{L}}[h](t) \\ &\quad + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v(t)). \end{aligned}$$

As in the proof of Theorem 2, there exists $T_1 \geq T_0$ such that (24) holds for every $t \geq T_1$. Sumarizing these computations we get

$$h^p(t)\mathcal{R}[w](t) \leq \alpha(1-\bar\gamma)\Big[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\Big]^{-\bar\gamma} R^{-1}(t) + h(t)\widetilde{\mathcal{L}}[h](t) + c_0(\bar\gamma-1)r(t)|h'(t)|^p\Big|\frac{v(t)}{G(t)}\Big|^\gamma$$

for $t \geq T_1$ and hence

$$h^p(t)\mathcal{R}[w](t) \leq R^{-1}(t)\Big[\int_t^\infty R^{-1}(s)\,\mathrm{d}s\Big]^{-\bar\gamma} f(c_0,\alpha,t) \leq 0$$

holds for $t \leq T_1$. The equation (1) is nonoscillatory by Lemma (1). □

Theorem 5. *Let $\gamma \in (1,2)$ be a real number and $\bar\gamma = \frac{\gamma}{\gamma-1}$ be the conjugate number to γ. Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Let R be defined by (18) and suppose that $\int^\infty h^p(c(t)-\widetilde{c}(t))\,\mathrm{d}t$ is convergent and (25) holds. If*

$$\limsup_{t\to\infty} R(t)h(t)\widetilde{\mathcal{L}}[h](t)\Big[\int^t R^{-1}(s)\,\mathrm{d}s\Big]^{\bar\gamma} < \infty,$$

$$\limsup_{t\to\infty}\Big[\int^t R^{-1}(s)\,\mathrm{d}s\Big]^{\bar\gamma-1}\int_t^\infty h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s < \infty$$

and

$$\liminf_{t\to\infty}\Big[\int^t R^{-1}(s)\,\mathrm{d}s\Big]^{\bar\gamma-1}\int_t^\infty h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s > -\infty$$

then (1) is nonoscillatory.

Proof. Denote

$$Y(t) = \Big[\int^t R^{-1}(s)\,\mathrm{d}s\Big]^{\bar\gamma-1}\int_t^\infty h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s.$$

Analogously to the proof of Theorem 4, there exists $T_0 \in \mathbb{R}$ and positive constants α, c_0 such that

$$f(c_0,\alpha,t) := R(t)h(t)\widetilde{\mathcal{L}}[h](t)\Big[\int^t R^{-1}(s)\,\mathrm{d}s\Big]^{\bar\gamma} + \alpha(1-\bar\gamma) + c_0(\bar\gamma-1)|\alpha+Y(t)|^\gamma < 0.$$

Conditions of the theorem imply that $v(t)/G(t) \to 0$ as $t\to\infty$, where

$$v(t) = \alpha\Big[\int^t R^{-1}(s)\,\mathrm{d}s\Big]^{1-\bar\gamma} + \int_t^\infty h^p(s)\Big(c(s)-\widetilde{c}(s)\Big)\,\mathrm{d}s,$$

hence, defining $w = h^{-p}(v+G)$, using identity (5) and inequality (7), we have

$$\begin{aligned}
h^p(t)\mathcal{R}[w](t) &= \alpha(1-\bar\gamma)\left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{-\bar\gamma} R^{-1}(t) \\
&\quad + h(t)\widetilde{\mathcal{L}}[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v(t)) \\
&\le \alpha(1-\bar\gamma)\left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{-\bar\gamma} R^{-1}(t) \\
&\quad + h(t)\widetilde{\mathcal{L}}[h](t) + c_0(\bar\gamma-1)r(t)|h'(t)|^p\left|\frac{v(t)}{G(t)}\right|^{\gamma} \\
&= R^{-1}(t)\left[\int^t R^{-1}(s)\,\mathrm{d}s\right]^{-\bar\gamma} f(c_0,\alpha,t) \le 0.
\end{aligned}$$

Nonoscillation of (1) follows from Lemma 1. □

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