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# ALMOST PERIODIC SKEW-SYMMETRIC DIFFERENTIAL SYSTEMS

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Abstract. We analyse solutions of almost periodic skew-symmetric homogeneous linear differential systems. We prove that in any neighbourhood of such a system there exists an almost periodic skew-symmetric system which does not possess any non-trivial almost periodic solution.

*Keywords:* almost periodic functions, almost periodic solutions, linear differential equations, skew-symmetric systems

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## 1. Introduction

We study solutions of almost periodic linear differential systems. This field is called the Favard theory what is based on the famous Favard result in [10] (see, e.g., [3, Theorem 1.2] or [28, Theorem 1]). It is a well-known corollary of the Favard (and the Floquet) theory that any bounded solution of a periodic linear differential system is almost periodic (see [12, Corollary 6.5] and [13] for a generalization in the homogeneous case). This result is no longer valid for almost periodic systems. There exist systems whose all solutions are bounded and none of them is almost periodic (see [18, 31]). Homogeneous systems have the zero solution which is almost periodic. But they do not need to have any non-zero almost periodic solution. The existence of a homogeneous system, which has bounded solutions (separated from zero) and, at the same time, all systems from some neighbourhood of it do not possess non-trivial almost periodic solutions, is proved in [33].

In this paper, we consider almost periodic skew-symmetric homogeneous linear differential systems. The basic motivation of our research is paper [38], where skew-Hermitian systems are analysed. The main result of [38] says that, in an arbitrary neighbourhood of a skew-Hermitian system, there exists another skew-Hermitian system which does not possess an almost periodic solution other than the trivial one (not only with a fundamental matrix which is not almost periodic—this problem is discussed in [34]). Our aim is to prove the corresponding result for real skew-symmetric systems. Note that the process from [38] cannot be applied in the real case.

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We use a recurrent method for constructing almost periodic functions. For non-almost periodic solutions of homogeneous linear differential equations, we refer to [27] (and [26]), where a method of constructions of minimal cocycles, which one gets as solutions of recurrent homogeneous linear differential systems, is mentioned. Special constructions of almost periodic homogeneous linear differential systems with given properties can be found in [19, 23, 24] as well. A method to construct fundamental matrices for almost periodic homogeneous linear systems is introduced in [30].

The importance of skew-symmetric systems may be illustrated by the Cameron-Johnson theorem which states that any almost periodic homogeneous linear differential system can be reduced by a Lyapunov transformation to a skew-symmetric system if all solutions of the given system and all of its limit equations are bounded (see [4]). Further, it is known (see [32]) that the skew-symmetric systems, all of whose solutions are almost periodic, form a dense subset in the space of all skew-symmetric systems (special cases are considered in [20, 21] and the corresponding result about unitary difference systems is mentioned in [36]). This fact also motivates the study of skew-symmetric systems without almost periodic solutions.

More precisely, it is proved in [32] that, in any neighbourhood of an almost periodic skew-symmetric system with frequency module $F$, there exists a system with a frequency module contained in the rational hull of $F$ possessing all almost periodic solutions with frequencies belonging to the rational hull of $F$ as well. From [35, Theorem 1] it follows that a neighbourhood of an almost periodic skew-symmetric system with frequency module $F$ may not contain a system with almost periodic solutions and frequency module $F$.

In addition (see [34]), the systems with $k$-dimensional frequency basis, having solutions which are not almost periodic, form a subset of the second category in the space of all systems with $k$-dimensional frequency basis. Thus, it is known (see also [32, Corollary 1]) that the systems with $k$-dimensional frequency basis and with an almost periodic fundamental matrix form a dense subset of the first category in the space of all considered systems with $k$-dimensional frequency basis. For more details concerning the frequency modules and bases of almost periodic linear differential systems and their solutions, we refer to monograph [12, Chapters 4, 6] or to articles [28, 38].

Let us give a short literature overview about almost periodic solutions of almost periodic linear differential equations. Sufficient conditions for the existence of almost periodic solutions are mentioned in [5, 9, 17] (for generalizations and supplements, see [8, 16, 22]). Certain sufficient conditions, under which homogeneous systems that have non-trivial bounded solutions also have non-trivial almost periodic solutions, are given in [29]. Concerning known basic results about skew-symmetric systems and their fundamental matrices, we refer to [2, 11, 25]. For the general theory of almost periodicity in connection with differential equations, see [7]. We add that the elements of the theory of almost periodicity can be found in many classical books, e.g., [1, 6].

## 2. Preliminaries

Let $m \in \mathbb{N} \setminus \{1\}$ be arbitrarily given as the dimension of systems under consideration. Throughout this paper, we will use the following notations: $\mathrm{Mat}\,(\mathbb{R}, m)$ for the set of all $m \times m$ matrices with real elements, $SO(m) \subset \mathrm{Mat}\,(\mathbb{R}, m)$ for the set of all orthogonal matrices with determinant 1, $so(m) \subset \mathrm{Mat}\,(\mathbb{R}, m)$ for the set of all skew-symmetric (i.e., antisymmetric) matrices, $I \in SO(m)$ for the identity matrix, $O \in so(m)$ for the zero matrix. We remark that the Lie algebra associated to the Lie group $SO(m)$ consists of the skew-symmetric $m \times m$ matrices (i.e., this Lie algebra is $so(m)$ and it is sometimes called the special orthogonal Lie algebra).

For the reader's convenience, we recall the definition of almost periodicity and basic properties of almost periodic functions which we will need later. Since we have to consider the almost periodicity of vector valued and, at the same time, matrix valued functions, we formulate the definition and the properties for functions with values in an arbitrary metric space $X$ with a metric $\mu$.

**Definition 1.** *A continuous function $\psi : \mathbb{R} \to X$ is almost periodic if for any $\varepsilon > 0$, there exists a number $l(\varepsilon) > 0$ with the property that any interval of length $l(\varepsilon)$ of the real line contains at least one point $s$ satisfying*

$$\mu\left(\psi(t+s), \psi(t)\right) < \varepsilon, \quad t \in \mathbb{R}.$$

**Theorem 1.** *An almost periodic function with values in $X$ is uniformly continuous on the real line.*

*Proof.* The theorem can be easily proved by modifying the proof of [6, Theorem 6.2]. □

**Theorem 2.** *Let $\psi : \mathbb{R} \to X$ be a continuous function. Then, $\psi$ is almost periodic if and only if from any sequence of the form $\{\psi(t+s_n)\}_{n\in\mathbb{N}}$, where $s_n$ are real numbers, one can extract a subsequence $\{\psi\left(t+r_n\right)\}_{n\in\mathbb{N}}$ satisfying the Cauchy uniform convergence condition on $\mathbb{R}$; i.e., for any $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ with the property that*

$$\mu(\psi(t+r_i), \psi(t+r_j)) < \varepsilon, \quad t \in \mathbb{R},$$

*for all $i, j > n(\varepsilon)$, $i, j \in \mathbb{N}$.*

*Proof.* See, e.g., [38, Theorem 2.4]. □

Let us consider systems of $m$ homogeneous linear differential equations of the form

$$x'(t) = A(t) \cdot x(t), \quad t \in \mathbb{R}, \tag{1}$$

where $A : \mathbb{R} \to so(m)$ is an almost periodic function. Let $\mathcal{S}$ denote the set of all systems (1). We can identify the function $A$ with the system (1) which is determined by $A$. Especially, we will write $A \in \mathcal{S}$. Let $X_S = X_S\,(t)$ denote the principal fundamental matrix of $S \in \mathcal{S}$ satisfying $X_S\,(0) = I$.

In the vector space $\mathbb{R}^m$, we will use the Euclidean norm $\|\cdot\|_2$ (one can also replace it by the absolute norm or the maximum norm). Let $\|\cdot\|$ be the corresponding matrix norm in $\mathrm{Mat}\,(\mathbb{R}, m)$ and let $\varrho$ be the metric given by $\|\cdot\|$. Considering that every almost periodic function is bounded (see directly the definition of almost periodicity), the distance between two systems $A, B \in \mathcal{S}$ is defined by the norm of the matrix valued functions $A, B$, uniformly on $\mathbb{R}$; i.e., we introduce the metric

$$\sigma(A, B) := \sup_{t \in \mathbb{R}} \| A(t) - B(t) \|, \quad A, B \in \mathcal{S}.$$

For $\varepsilon > 0$, the symbol $\mathcal{O}^{\sigma}_{\varepsilon}(A)$ will denote the $\varepsilon$-neighbourhood of a system $A$ in $\mathcal{S}$ and $\mathcal{O}^{\varrho}_{\varepsilon}(M)$ the $\varepsilon$-neighbourhood of a matrix $M$ in a given subset of $\mathrm{Mat}\,(\mathbb{R}, m)$.

Now we can repeat the above mentioned result (see Introduction) in a more explicit form.

**Theorem 3.** *Let $A \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary. There exists $B \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ whose all solutions are almost periodic.*

*Proof.* See [32, Theorem 1, Remark 3]. □

## 3. Results

To prove the announced new result, we need the following lemmas.

**Lemma 1.** *There exist $\xi > 0$ and a neighbourhood $\tilde{\mathcal{O}}\,(O)$ of the zero matrix in $so(m)$ for which the exponential map is a bijection between $\tilde{\mathcal{O}}\,(O)$ and $\mathcal{O}^{\varrho}_{\xi}\,(I) \cap SO(m)$ such that the maps*

$$A \mapsto \exp\,(A)\,, \quad A \in \tilde{\mathcal{O}}\,(O)\,; \qquad A \mapsto \ln\,(A)\,, \quad A \in \mathcal{O}^{\varrho}_{\xi}\,(I) \cap SO(m) \tag{2}$$

*are Lipschitz continuous.*

*Proof.* It is well-known that the exponential map is a bijection between $\tilde{\mathcal{O}}\,(O)$ and $\mathcal{O}^{\varrho}_{\xi}\,(I) \cap SO(m)$ for a sufficiently small $\xi > 0$ and the corresponding neighbourhood $\tilde{\mathcal{O}}\,(O) \subset so(m)$. The fact that the maps in (2) are Lipschitz continuous follows from the inequality

$$\| \exp\,(X + Y) - \exp\,(X) \| \le \| Y \| \cdot \exp\,(\| X \|) \cdot \exp\,(\| Y \|)\,, \quad X, Y \in so(m),$$

and, e.g., from the Richter theorem (see [15, Theorem 11.1])

$$\ln\,(X) = \int_0^1 (X - I)\,[t\,(X - I) + I]^{-1}\,dt, \quad X \in \mathcal{O}^{\varrho}_{\xi}\,(I) \cap SO(m).$$

□

*Remark* 1. Any non-singular matrix has infinitely many logarithms. But symbol $\ln\,(A)$ denotes the principal logarithm, which is the unique logarithm whose spectrum lies in the strip $\{z \in \mathbb{C};\ \mathrm{Im}\, z \in [-\pi, \pi)\}$.

**Lemma 2.** *There exists $p(\vartheta) \in \mathbb{N}$ for all $\vartheta > 0$ with the property that, for any sequence*

$$\{P_0, P_1, \dots, P_n, \dots, P_{2n}\} \subset SO(m), \quad n \geq p(\vartheta),$$

*one can find matrices $Q_2, Q_4, \dots, Q_{2n} \in SO(m)$ for which*

$$Q_{2i} \in \mathcal{O}^{\varrho}_{\vartheta}(P_{2i}), \quad i \in \{1, \dots, n\}, \qquad P_1 \cdot Q_2 \cdot P_3 \cdot Q_4 \cdots P_{2n-1} \cdot Q_{2n} = P_0. \tag{3}$$

*Proof.* First we recall that the group $SO(m)$ is the so-called transformable group (see [37, Remark 2]). This fact implies (see again [37]) the existence of $q(\delta) \in \mathbb{N}$ for all $\delta > 0$ such that, for any sequence $\{P_0, P_1, \dots, P_q, \dots, P_n\} \subset SO(m)$, there exist $T_1, \dots, T_q, \dots, T_n \in SO(m)$ satisfying

$$T_i \in \mathcal{O}^{\varrho}_{\delta}(P_i), \quad i \in \{1, \dots, n\}, \qquad T_1 \cdot T_2 \cdots T_n = P_0.$$

We replace matrices $P_1, \dots, P_{n-1}, P_n$ by $P_1 \cdot P_2, \dots, P_{2n-3} \cdot P_{2n-2}, P_{2n-1} \cdot P_{2n}$ and, using the transformability of $SO(m)$, we obtain matrices $T_i$, $i \in \{1, \dots, n\}$. We put

$$R_1 := (P_1 \cdot P_2)^{-1} \cdot T_1, \dots, R_n := (P_{2n-1} \cdot P_{2n})^{-1} \cdot T_n.$$

Since the multiplication of matrices is Lipschitz continuous on $SO(m)$ as the map $O \mapsto O^T$, there exists $L > 0$ such that

$$R_i \in \mathcal{O}^{\varrho}_{\delta L}(I), \quad i \in \{1, \dots, n\},$$

and, consequently, there exists $K > 0$ for which

$$P_2 \cdot R_1 \in \mathcal{O}^{\varrho}_{\delta K}(P_2), \dots, P_{2n} \cdot R_n \in \mathcal{O}^{\varrho}_{\delta K}(P_{2n}).$$

We see

$$T_1 = P_1 \cdot P_2 \cdot R_1, \dots, T_n = P_{2n-1} \cdot P_{2n} \cdot R_n,$$

i.e., we have (3) for $Q_2 := P_2 \cdot R_1, \dots, Q_{2n} := P_{2n} \cdot R_n$ and $p(\vartheta) := q(\vartheta/K)$. □

We will also use a simple method for constructing almost periodic functions with prescribed values, which is formulated in the next lemma. Note that this lemma is a modification of [38, Theorem 3.1] and that the analogous way, one can generate almost periodic sequences with several given properties, can be found in [39].

**Lemma 3.** *If the sequence of non-negative numbers $a(i)$ for $i \in \mathbb{N}$ has the property that*

$$\sum_{i=1}^{\infty} a(i) < \infty,$$

*then any continuous function $\psi : \mathbb{R} \to so(m)$ for which*

$$\begin{gathered}
\psi(t) = \psi(t-1), \quad t \in (1, 2], \\
\psi(t) = \psi(t+2), \quad t \in (-2, 0], \\
\psi(t) \in \mathcal{O}^{\varrho}_{a(1)}(\psi(t-4)), \quad t \in (2, 6], \\
\psi(t) = \psi(t+8), \quad t \in (-10, -2],
\end{gathered}$$

$$\psi(t) \in \mathcal{O}^{\varrho}_{a(2)}\left(\psi\left(t-2^4\right)\right), \quad t \in \left(2+2^2, 2+2^2+2^4\right],$$
$$\psi(t) = \psi\left(t+2^5\right), \quad t \in \left(-2^5-2^3-2, -2^3-2\right],$$
$$\vdots$$
$$\psi(t) \in \mathcal{O}^{\varrho}_{a(n)}\left(\psi\left(t-2^{2n}\right)\right), \quad t \in \left(2+2^2+\cdots+2^{2n-2}, 2+2^2+\cdots+2^{2n-2}+2^{2n}\right],$$
$$\psi(t) = \psi\left(t+2^{2n+1}\right), \quad t \in \left(-2^{2n+1}-\cdots-2^3-2, -2^{2n-1}-\cdots-2^3-2\right],$$
$$\vdots$$

*is almost periodic.*

*Proof.* Let $\varepsilon > 0$ be arbitrarily given and let $k = k(\varepsilon) \in \mathbb{N}$ satisfy
$$\sum_{i=k}^{\infty} a(i) < \frac{\varepsilon}{2}. \tag{4}$$

From
$$\psi(t) \in \mathcal{O}^{\varrho}_{a(k)}\left(\psi\left(t-2^{2k}\right)\right), \quad t \in \left(2+2^2+\cdots+2^{k-2}, 2+2^2+\cdots+2^{k}\right],$$
$$\psi(t) = \psi\left(t+2^{2k+1}\right), \quad t \in \left(-2^{2k+1}-\cdots-2^3-2, -2^{2k-1}-\cdots-2^3-2\right],$$
$$\psi(t) \in \mathcal{O}^{\varrho}_{a(k+1)}\left(\psi\left(t-2^{2k+2}\right)\right), \quad t \in \left(2+2^2+\cdots+2^{2k}, 2+2^2+\cdots+2^{2k+2}\right],$$
$$\vdots$$

it follows
$$\psi\left(t+2^{2k}\right) \in \mathcal{O}^{\varrho}_{a(k)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\psi\left(t-2^{2k}\right) \in \mathcal{O}^{\varrho}_{a(k)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\psi\left(t-2^{2k+1}\right) \in \mathcal{O}^{\varrho}_{a(k)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\psi\left(t+2^{2k+1}\right) \in \mathcal{O}^{\varrho}_{a(k)+a(k+1)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\psi\left(t+3\cdot 2^{2k}\right) \in \mathcal{O}^{\varrho}_{a(k)+a(k+1)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\psi\left(t+2^{2k+2}\right) \in \mathcal{O}^{\varrho}_{a(k+1)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\psi\left(t+2^{2k}+2^{2k+2}\right) \in \mathcal{O}^{\varrho}_{a(k)+a(k+1)}(\psi(t)),\ t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\vdots$$

Thus (see (4)), it is true
$$\psi\left(t+l\cdot 2^{2k}\right) \in \mathcal{O}^{\varrho}_{\varepsilon/2}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],\ l \in \mathbb{Z}.$$

If we express any $t \in \mathbb{R}$ as $t = t_1 + t_2$, where
$$t_1 \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right], \quad t_2 = j\cdot 2^{2k} \text{ for } j \in \mathbb{Z},$$

then we have

$$\varrho\left(\psi(t),\psi\left(t+l\cdot 2^{2k}\right)\right)\leq\varrho\left(\psi\left(t_1+t_2\right),\psi\left(t_1\right)\right)+\varrho\left(\psi\left(t_1\right),\psi\left(t_1+(j+l)\,2^{2k}\right)\right)$$
$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\quad t\in\mathbb{R},\ l\in\mathbb{Z}.$$

This inequality implies that we can choose $l(\varepsilon):=2^{2k(\varepsilon)}+1$ for any $\varepsilon>0$ (see Definition 1); i.e., the resulting function $\psi$ is almost periodic. $\square$

Now we can prove the result that the systems having no non-zero almost periodic solution form an everywhere dense subset of $\mathcal{S}$.

**Theorem 4.** *Let $A\in\mathcal{S}$ and $\varepsilon>0$ be arbitrary. There exists $B\in\mathcal{O}^{\sigma}_{\varepsilon}(A)$ which does not have an almost periodic solution other than the trivial one.*

*Proof.* Using Theorem 1, the almost periodicity of $A$ implies that there exist $\delta\in(0,1/3)$ and an almost periodic matrix valued function $\tilde{A}:\mathbb{R}\to so(m)$ satisfying $\tilde{A}\in\mathcal{O}^{\sigma}_{\varepsilon/2}(A)$ and $\tilde{A}|_{[k,k+\delta]}\equiv const.$ for any $k\in\mathbb{Z}$. Indeed, it suffices to define $\tilde{A}$ as follows

$$\tilde{A}(t):=A\left(k+\frac{\delta}{2}\right),\quad t\in[k,k+\delta],\ k\in\mathbb{Z},$$
$$\tilde{A}(t):=A(k-\delta)+\frac{t-(k-\delta)}{\delta}\left[A\left(k+\frac{\delta}{2}\right)-A(k-\delta)\right],\quad t\in[k-\delta,k),\ k\in\mathbb{Z},$$
$$\tilde{A}(t):=A\left(k+\frac{\delta}{2}\right)+\frac{t-(k+\delta)}{\delta}\left[A(k+2\delta)-A\left(k+\frac{\delta}{2}\right)\right],\quad t\in(k+\delta,k+2\delta],\ k\in\mathbb{Z},$$
$$\tilde{A}(t):=A(t),\quad t\notin\bigcup_{k\in\mathbb{Z}}[k-\delta,k+2\delta],$$

where $\delta>0$ is sufficiently small. Thus, we will assume without loss of generality that $A\in\mathcal{S}$ is constant on all interval $[k,k+\delta]$, $k\in\mathbb{Z}$.

Every almost periodic function is bounded. Hence, there exists $\eta\in(0,1)$ with the property that

$$\|\,X_S\,(t+s)-X_S\,(t)\,\|<\xi \tag{5}$$

for any $t\in\mathbb{R}$, $s\in[0,\eta]$, and $S\in\mathcal{O}^{\sigma}_{\varepsilon}(A)$, where $\xi>0$ is taken from Lemma 1. We can also assume that $\delta<\eta$. Further (see again Lemma 1), there exists $M\in\mathbb{N}$ satisfying

$$\|\,A-B\,\|<\vartheta\quad\text{if}\quad A,B\in\tilde{\mathcal{O}}\,(O)\,,\ \exp\left(A\right)\in\mathcal{O}^{\varrho}_{\vartheta/M}\left(\exp\left(B\right)\right)\subseteq\mathcal{O}^{\varrho}_{\xi}\left(I\right)\cap SO(m). \tag{6}$$

We choose an increasing sequence of numbers $n(i)\in\mathbb{N}\setminus\{1\}$ for $i\in\mathbb{N}$ arbitrarily so that

$$2^{n(i)-1}\geq p\left(\frac{\varepsilon}{2^iM}\cdot\frac{\delta}{2}\right),\quad i\in\mathbb{N}, \tag{7}$$

where $p(\vartheta)$ is taken from Lemma 2.

Since the sum of skew-symmetric matrices is a skew-symmetric matrix and since the sum of two almost periodic functions is almost periodic as well (see Theorem 2), we have

$A_1 + A_2 \in \mathcal{S}$ for any $A_1, A_2 \in \mathcal{S}$. Thus, it suffices to find $C \in \mathcal{S} \cap \mathcal{O}^{\sigma}_{\varepsilon}(O)$ for which the system $A + C$ does not have any non-zero almost periodic solution. We will construct such a system $C$ (as continuous function) applying Lemma 3 for

$$a\,(n(i)) := \frac{\varepsilon}{2^i}, \quad i \in \mathbb{N}; \qquad a\,(j) := 0, \quad j \notin \{n(i);\, i \in \mathbb{N}\}.$$

Let us denote

$$a_i := 2 + 2^2 + \cdots + 2^{2n(i)-2}, \quad b_i := 2 + 2^2 + \cdots + 2^{2n(i)-2} + 2^{2n(i)},$$
$$d_i^1 := \left(\frac{1}{4} - \frac{1}{2^{2n(i)}}\right)\delta, \quad d_i^2 := \left(\frac{3}{4} + \frac{1}{2^{2n(i)}}\right)\delta, \qquad i \in \mathbb{N}.$$

In the first step of the construction, we put

$$C(t) := O, \quad t \in \left(-2^{2n(1)-1} - \cdots - 2^3 - 2, 2 + 2^2 + \cdots + 2^{2n(1)-2}\right],$$
$$C(t) := O, \quad t \in (a_1, b_1] \setminus \bigcup_{j \in \mathbb{N}} \left(j + d_1^1, j + d_1^2\right],$$
$$C(t) := C_1^{j-a_1+1}, \quad t \in \left(j + d_2^1, j + d_2^2\right] \subset (a_1, b_1],$$

for arbitrary matrices

$$C_1^{j-a_1+1} \in \mathcal{O}^{\varrho}_{\varepsilon/2}(O) \cap so(m), \quad j \in \{a_1, \ldots, b_1 - 1\},$$

and we define $C$ so that it is linear on intervals

$$\left(j + d_1^1, j + d_2^1\right], \quad \left(j + d_2^2, j + d_1^2\right], \qquad j \in \{a_1, \ldots, b_1 - 1\}.$$

In the second step, we put

$$C(t) := C\left(t + 2^{2n(1)+1}\right), \quad t \in \left(-2^{2n(1)+1} - \cdots - 2^3 - 2, -2^{2n(1)-1} - \cdots - 2^3 - 2\right],$$
$$C(t) := C\left(t - 2^{2n(1)+2}\right), \quad t \in \left(2 + 2^2 + \cdots + 2^{2n(1)}, 2 + 2^2 + \cdots + 2^{2n(1)+2}\right],$$
$$\vdots$$
$$C(t) := C\left(t + 2^{2n(2)-1}\right), \quad t \in \left(-2^{2n(2)-1} - \cdots - 2^3 - 2, -2^{2n(2)-3} - \cdots - 2^3 - 2\right],$$
$$C(t) := C\left(t - 2^{2n(2)}\right), \quad t \in (a_2, b_2] \setminus \bigcup_{j \in \mathbb{N}} \left(j + d_2^1, j + d_2^2\right],$$

and we define $C$ as linear on intervals

$$\left(j + d_2^1, j + d_3^1\right], \quad \left(j + d_3^2, j + d_2^2\right], \qquad j \in \{a_2, \ldots, b_2 - 1\}.$$

At the same time, we define

$$C(t) := C_2^{j-a_2+1} \in so(m), \quad t \in \left(j + d_3^1, j + d_3^2\right], \qquad j \in \{a_2, \ldots, b_2 - 1\},$$

arbitrarily so that

$$\left\|\, C(t) - C\left(t - 2^{2n(2)}\right) \right\| < \frac{\varepsilon}{4}, \quad t \in (a_2, b_2]\,.$$

We proceed further in the same way. In the i-th step, we put

$$C(t) := C\left(t + 2^{2n(i-1)+1}\right), \quad t \in \left(-2^{2n(i-1)+1} - \cdots - 2^3 - 2, -2^{2n(i-1)-1} - \cdots - 2^3 - 2\right],$$

$$C(t) := C\left(t - 2^{2n(i-1)+2}\right), \quad t \in \left(2 + 2^2 + \cdots + 2^{2n(i-1)}, 2 + 2^2 + \cdots + 2^{2n(i-1)+2}\right],$$

$$\vdots$$

$$C(t) := C\left(t + 2^{2n(i)-1}\right), \quad t \in \left(-2^{2n(i)-1} - \cdots - 2^3 - 2, -2^{2n(i)-3} - \cdots - 2^3 - 2\right],$$

$$C(t) := C\left(t - 2^{2n(i)}\right), \quad t \in (a_i, b_i] \setminus \bigcup_{j \in \mathbb{N}} \left(j + d_i^1, j + d_i^2\right], \tag{8}$$

and we define $C$ as a linear function on intervals

$$\left(j + d_i^1, j + d_{i+1}^1\right], \quad \left(j + d_{i+1}^2, j + d_i^2\right], \qquad j \in \{a_i, \dots, b_i - 1\},$$

and

$$C(t) := C_i^{j-a_i+1} \in so(m), \quad t \in \left(j + d_{i+1}^1, j + d_{i+1}^2\right], \quad j \in \{a_i, \dots, b_i - 1\},$$

arbitrarily so that

$$\left\| C(t) - C\left(t - 2^{2n(i)}\right) \right\| < \frac{\varepsilon}{2^i}, \quad t \in (a_i, b_i].$$

For

$$\zeta := \max\left\{ \left\| C_1^j \right\| ; \, j \in \left\{1, \dots, 2^{2n(1)}\right\} \right\} < \frac{\varepsilon}{2},$$

we have

$$\| C(t) \| \le \zeta, \quad t \in \left(-2^{2n(1)-1} - \cdots - 2^3 - 2, 2 + 2^2 + \cdots + 2^{2n(1)}\right],$$

$$\| C(t) \| < \zeta + \frac{\varepsilon}{4}, \quad t \in \left(-2^{2n(2)-1} - \cdots - 2^3 - 2, 2 + 2^2 + \cdots + 2^{2n(2)}\right],$$

$$\vdots$$

$$\| C(t) \| < \zeta + \frac{\varepsilon}{4} + \cdots + \frac{\varepsilon}{2^i}, \quad t \in \left(-2^{2n(i)-1} - \cdots - 2^3 - 2, 2 + 2^2 + \cdots + 2^{2n(i)}\right],$$

$$\vdots$$

i.e., there exists $\tilde{\varepsilon} \in (0, \varepsilon)$ with the property that $\| C(t) \| < \tilde{\varepsilon}$, $t \in \mathbb{R}$. Thus, we obtain an almost periodic (continuous) function $C \in \mathcal{S} \cap \mathcal{O}_\varepsilon^\sigma(O)$.

We denote

$$I_i := [a_i, b_i] = \left[2 + 2^2 + \cdots + 2^{2n(i)-2}, 2 + 2^2 + \cdots + 2^{2n(i)-2} + 2^{2n(i)}\right].$$

In the construction, we can choose constant values $C_i^1, \dots, C_i^{2^{2n(i)}}$ on $2^{2n(i)}$ subintervals of $I_i$, where the length of each one of these intervals is

$$d_{i+1}^2 - d_{i+1}^1 \in \left(\frac{\delta}{2}, \delta\right). \tag{9}$$

Each value $C_i^j$ can be chosen arbitrarily from the $(\varepsilon/2^i)$-neighbourhood of a skew-symmetric matrix, which is given by the previous steps of the construction. Further (see (8)), the function $C$ is determined on intervals

$$\left(a_i, a_i + d_i^1\right], \quad \left(a_i + d_i^2, a_i + 1 + d_i^1\right], \quad \dots \quad \left(b_i - 2 + d_i^2, b_i - 1 + d_i^1\right], \quad \left(b_i - 1 + d_i^2, b_i\right]$$

by prescription $C(t) = C\left(t - 2^{2n(i)}\right)$.

We repeat that $C$ is linear on the remaining subintervals of $I_i$. These intervals will be denoted by $J_i^1, \dots, J_i^{2^{2n(i)+1}}$, where

$$\begin{aligned} J_i^{2j-1} &:= \left(a_i + j - 1 + d_i^1, a_i + j - 1 + d_{i+1}^1\right], \quad j \in \left\{1, \dots, 2^{2n(i)}\right\}, \\ J_i^{2j} &:= \left(a_i + j - 1 + d_{i+1}^2, a_i + j - 1 + d_i^2\right], \quad j \in \left\{1, \dots, 2^{2n(i)}\right\}. \end{aligned} \tag{10}$$

Especially, we see that the length of each $J_i^j$ is less than $\delta/2^{2n(i)}$ and that

$$J_i^1, \dots, J_i^{2j} \subset (a_i, a_i + j), \quad J_i^{2j+1}, \dots, J_i^{2^{2n(i)+1}} \subset (a_i + j, b_i), \quad j \in \left\{1, \dots, 2^{2n(i)} - 1\right\},$$

i.e., the total length $l_i^k$ of all subintervals $J_i^j \subset [a_i, a_i + k]$ is

$$l_i^k < \frac{2k\delta}{2^{2n(i)}}, \quad k \in \left\{1, \dots, 2^{2n(i)}\right\}. \tag{11}$$

Let us consider $S = A + C \in \mathcal{O}_\varepsilon^\sigma(A)$. To describe the principal fundamental matrix $X_S$, we define

$$\begin{aligned}
\tilde{X}_S^i(t) &:= X_S(t), \quad t \in \left[a_i, a_i + d_i^1\right], \\
\tilde{X}_S^i(t) &:= \tilde{X}_S^i\left(a_i + d_i^1\right), \quad t \in \left(a_i + d_i^1, a_i + d_{i+1}^1\right], \\
\tilde{X}_S^i(t) &:= \exp\left(\left(A + C_i^1\right)\left(t - a_i - d_{i+1}^1\right)\right) \cdot \tilde{X}_S^i\left(a_i + d_{i+1}^1\right), \quad t \in \left(a_i + d_{i+1}^1, a_i + d_{i+1}^2\right], \\
\tilde{X}_S^i(t) &:= \tilde{X}_S^i\left(a_i + d_{i+1}^2\right), \quad t \in \left(a_i + d_{i+1}^2, a_i + d_i^2\right], \\
\tilde{X}_S^i(t) &:= X_S(t) \cdot \left(X_S\left(a_i + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(a_i + d_i^2\right), \quad t \in \left(a_i + d_i^2, a_i + 1 + d_i^1\right], \\
&\vdots \\
\tilde{X}_S^i(t) &:= X_S(t) \cdot \left(X_S\left(b_i - 2 + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(b_i - 2 + d_i^2\right), \\
&\qquad t \in \left(b_i - 2 + d_i^2, b_i - 1 + d_i^1\right], \\
\tilde{X}_S^i(t) &:= \tilde{X}_S^i\left(b_i - 1 + d_i^1\right), \quad t \in \left(b_i - 1 + d_i^1, b_i - 1 + d_{i+1}^1\right], \\
\tilde{X}_S^i(t) &:= \exp\left(\left(A + C_i^{2^{2n(i)}}\right)\left(t - b_i + 1 - d_{i+1}^1\right)\right) \cdot \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^1\right), \\
&\qquad t \in \left(b_i - 1 + d_{i+1}^1, b_i - 1 + d_{i+1}^2\right], \\
\tilde{X}_S^i(t) &:= \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^2\right), \quad t \in \left(b_i - 1 + d_{i+1}^2, b_i - 1 + d_i^2\right], \\
\tilde{X}_S^i(t) &:= X_S(t) \cdot \left(X_S\left(b_i - 1 + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(b_i - 1 + d_i^2\right), \quad t \in \left(b_i - 1 + d_i^2, b_i\right].
\end{aligned}$$

Since

$$X_S(t_2) - X_S(t_1) = \int_{t_1}^{t_2} S(s)\, X_S(s)\, ds, \quad t_1, t_2 \in \mathbb{R},$$

it is valid that (see also (10))

$$\begin{aligned} \left\| X_S(t) - \tilde{X}_S^i(t) \right\| \leq \sum_{j=1}^{k} & \int_{a_i+j-1+d_i^1}^{a_i+j-1+d_{i+1}^1} \| S(s)\, X_S(s) \|\ ds \\ + \sum_{j=1}^{k} & \int_{a_i+j-1+d_{i+1}^2}^{a_i+j-1+d_i^2} \| S(s)\, X_S(s) \|\ ds \end{aligned} \tag{12}$$

if $t \leq a_i + k$, $k \in \left\{1, \dots, 2^{2n(i)}\right\}$. Considering $S \in \mathcal{O}_\varepsilon^\sigma(A)$ and $X_S(t), \tilde{X}_S^i(t) \in SO(m)$, $t \in \mathbb{R}$, from (11) and (12) it follows that there exists $N \in \mathbb{N}$ satisfying

$$\left\| X_S(t) - \tilde{X}_S^i(t) \right\| < \frac{Nk}{2^{2n(i)-1}} \tag{13}$$

for $t \in [a_i, a_i + k]$, $k \in \left\{1, \dots, 2^{2n(i)}\right\}$.

Let $n_0 \in \mathbb{N}$ be such that

$$N \cdot \frac{4n(i)\, 2^{n(i)}}{2^{2n(i)-1}} < \frac{1}{3}, \quad i \geq n_0\, (i \in \mathbb{N}). \tag{14}$$

We put $X_1 := -I$, $X_2 = -I$, when $m$ is even, and

$$X_1 := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \in SO(m), \quad X_2 := \begin{pmatrix} -1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \in SO(m)$$

for odd $m$. If we express

$$\begin{aligned} \tilde{X}_S^i\left(a_i + d_{i+1}^2\right) &= \exp\left(\left(A + C_i^1\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right) \cdot \tilde{X}_S^i\left(a_i + d_{i+1}^1\right), \\ \tilde{X}_S^i\left(a_i + 1 + d_{i+1}^1\right) &= X_S\left(a_i + 1 + d_i^1\right) \cdot \left(X_S\left(a_i + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(a_i + d_{i+1}^2\right), \\ &\ \ \vdots \\ \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^1\right) &= X_S\left(b_i - 1 + d_i^1\right) \cdot \left(X_S\left(b_i - 2 + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(b_i - 2 + d_{i+1}^2\right), \\ \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^2\right) &= \exp\left(\left(A + C_i^{2^{2n(i)}}\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right) \cdot \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^1\right), \\ \tilde{X}_S^i(b_i) &= X_S(b_i) \cdot \left(X_S\left(b_i - 1 + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^2\right), \end{aligned}$$

then it is seen that we can use Lemma 2 to choose values $C_i^j$ on subintervals

$$\left(a_i + j - 1 + d_{i+1}^1, a_i + j - 1 + d_{i+1}^2\right], \quad j \in \left\{1, \ldots, 2^{2n(i)}\right\},$$

so that we obtain

$$\begin{gathered}
\tilde{X}_S^i\left(a_i + 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 2^{n(i)} + \left(2^{n(i)} - 1\right)\right) = X_1, \\
\tilde{X}_S^i\left(a_i + 3 \cdot 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 1\right)\right) = X_2, \\
\tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 2^{n(i)} + \left(2^{n(i)} - 2^1\right)\right) = X_1, \\
\tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 3 \cdot 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^1\right)\right) = X_2, \\
\vdots \\
\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 2^{n(i)}\right) = I, \\
\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right)\right) = X_1, \\
\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 3 \cdot 2^{n(i)}\right) = I, \\
\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right)\right) = X_2.
\end{gathered}$$

Indeed, it suffices to consider the form of matrices

$$\exp\left(\left(A + C_i^j\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right)$$

for which (see (5), (9))

$$\left\| \exp\left(\left(A + C_i^j\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right) - I \right\| < \xi,$$

inequality (7) with $M \in \mathbb{N}$ satisfying (6) and with

$$d_{i+1}^2 - d_{i+1}^1 > \frac{\delta}{2}, \quad 2^{n(i)} - 1 > 2^{n(i)} - 2^1 > \cdots > 2^{n(i)} - 2^{n(i)-1} = 2^{n(i)-1},$$

and the fact that we can choose all matrix $C_i^j$ from the $(\varepsilon/2^i)$-neighbourhood of a given skew-symmetric matrix arbitrarily. Note that

$$a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right) < a_i + 4n(i) \cdot 2^{n(i)} \tag{15}$$

and $a_i + 4n(i)2^{n(i)} < b_i$ for sufficiently large $i \in \mathbb{N}$, i.e., we can construct the resulting function $C$ with the above mentioned properties on $I_i$ for all $i \geq n_0$ (see also (14)).

Now we use (13) and (14) in connection with (15). For $k \in \left\{1, \ldots, 4n(i)2^{n(i)}\right\}$, where $i \geq n_0$, we have

$$\left\| X_S(t) - \tilde{X}_S^i(t) \right\| < N \cdot \frac{4n(i)\, 2^{n(i)}}{2^{2n(i)-1}} < \frac{1}{3}, \quad t \in [a_i, a_i + k]. \tag{16}$$

Especially, for all $i \geq n_0$ ($i \in \mathbb{N}$), we obtain

$$\left\| X_S\left(s_j^i\right) - \tilde{X}_S^i\left(s_j^i\right) \right\| < \frac{1}{3}, \quad j \in \{1, \ldots, 4n(i)\}, \tag{17}$$

where

$$s_1^i := a_i + 2^{n(i)}, \quad s_2^i := a_i + 2^{n(i)} + \left(2^{n(i)} - 1\right),$$
$$s_3^i := a_i + 3 \cdot 2^{n(i)}, \quad s_4^i := a_i + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 1\right),$$
$$\vdots$$
$$s_{4n(i)-3}^i := a_i + 4\,(n(i) - 1)\,2^{n(i)} + 2^{n(i)},$$
$$s_{4n(i)-2}^i := a_i + 4\,(n(i) - 1)\,2^{n(i)} + 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right),$$
$$s_{4n(i)-1}^i := a_i + 4\,(n(i) - 1)\,2^{n(i)} + 3 \cdot 2^{n(i)},$$
$$s_{4n(i)}^i := a_i + 4\,(n(i) - 1)\,2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right).$$

We recall that we need to prove that any non-trivial solution of $S$ is not almost periodic. By contradiction, suppose that the solution

$$x(t) = X_S(t) \cdot u \tag{18}$$

of the Cauchy problem

$$x'(t) = S(t) \cdot x(t), \quad x(0) = u,$$

where $u \in \mathbb{R}^m$, $||\, u \,||_2 = 1$, is almost periodic. Applying Theorem 2 for $\varepsilon = 1/3$ and $s_i = 2^{n(i)}$, $i \in \mathbb{N}$, we obtain

$$\left\|\, x\left(t + 2^{n(i(1))}\right) - x\left(t + 2^{n(i(2))}\right) \right\|_2 < \frac{1}{3}, \quad t \in \mathbb{R}, \tag{19}$$

for all $i(1), i(2)$ from an infinite set $N_0 \subseteq \mathbb{N}$.

It is immediately seen that

$$\max\left\{||\, X_1 \cdot u - u \,||_2, ||\, X_2 \cdot u - u \,||_2\right\} \geq 1. \tag{20}$$

Thus, from the construction, (17), (20), and from

$$\left\|\, \tilde{X}_S^i(t) \cdot u - \tilde{X}_S^i(s) \cdot u \,\right\|_2 \leq \left\|\, \tilde{X}_S^i(t) \cdot u - X_S(t) \cdot u \,\right\|_2 + \\ \|\, X_S(t) \cdot u - X_S(s) \cdot u \,\|_2 + \left\|\, X_S(s) \cdot u - \tilde{X}_S^i(s) \cdot u \,\right\|_2$$

for

$$t = s_4^i, \quad s = s_3^i; \qquad t = s_2^i, \quad s = s_1^i;$$
$$\vdots$$
$$t = s_{4n(i)}^i, \quad s = s_{4n(i)-1}^i; \qquad t = s_{4n(i)-2}^i, \quad s = s_{4n(i)-3}^i,$$

respectively, it follows

$$1 < \frac{1}{3} + \left\|\, X_S\left(s_{4j}^i\right) \cdot u - X_S\left(s_{4j-1}^i\right) \cdot u \,\right\|_2 + \frac{1}{3}$$

or

$$1 < \frac{1}{3} + \left\|\, X_S\left(s_{4j-2}^i\right) \cdot u - X_S\left(s_{4j-3}^i\right) \cdot u \,\right\|_2 + \frac{1}{3}$$

for $j \in \{1, \dots, n(i)\}$. Hence, we have

$$\max \left\{ \left\| X_S\left(s^i_{4j}\right) \cdot u - X_S\left(s^i_{4j-1}\right) \cdot u \right\|_2, \left\| X_S\left(s^i_{4j-2}\right) \cdot u - X_S\left(s^i_{4j-3}\right) \cdot u \right\|_2 \right\} > \frac{1}{3} \qquad (21)$$

for all $j \in \{1, \dots, n(i)\}$ and $i \geq n_0$. Since

$$\begin{aligned} s^i_2 - s^i_1 &= 2^{n(i)} - 1 = s^i_4 - s^i_3, \\ s^i_6 - s^i_5 &= 2^{n(i)} - 2^1 = s^i_8 - s^i_7, \\ &\vdots \\ s^i_{4n(i)-2} - s^i_{4n(i)-3} &= 2^{n(i)} - 2^{n(i)-1} = s^i_{4n(i)} - s^i_{4n(i)-1}, \end{aligned}$$

inequality (21) implies (see (18))

$$\sup_{t \in \mathbb{R}} \left\| x(t) - x\left(t + 2^{n(i)} - 2^{j-1}\right) \right\|_2 > \frac{1}{3} \qquad (22)$$

for all $i \geq n_0$ and $j \in \{1, \dots, n(i)\}$. Of course, we can rewrite (19) into

$$\sup_{t \in \mathbb{R}} \left\| x(t) - x\left(t + 2^{n(i(2))} - 2^{n(i(1))}\right) \right\|_2 \leq \frac{1}{3}, \quad i(1), i(2) \in N_0.$$

Considering (22), we see that (19) cannot be true for all $i(1), i(2)$ from an infinite set $N_0$. This contradiction proves the theorem. □

The presented process can be applied to prove the existence of systems from $\mathcal{S}$ with several properties. For example, we mention the following result.

**Theorem 5.** *Let $A \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrarily given. There exists $B \in \mathcal{O}^\sigma_\varepsilon(A)$ with the property that*

$$\overline{\{X_B(t);\, t \in \mathbb{R}\}} = SO(m).$$

*Proof.* Let a sequence $\{X_k\}_{k \in \mathbb{N}} \subset SO(m)$ be dense in $SO(m)$. In the proof of Theorem 4, we can replace considered matrices $X_1$, $X_2$ by arbitrary matrices $X_k, X_{k+1}$. Thus, there is shown the existence of a system $S = A + C \in \mathcal{O}^\sigma_\varepsilon(A)$ with the property that (see (16))

$$\left\| X_S\left(s^i_j\right) - X_j \right\| < N \cdot \frac{4n(i)\, 2^{n(i)}}{2^{2n(i)-1}}$$

for some $s^i_j \in \mathbb{R}$ and all $j \in \{1, \dots, 2n(i)\}$, $i \geq n_0$. Now it suffices to consider that

$$\lim_{i \to \infty} N \cdot \frac{4n(i)\, 2^{n(i)}}{2^{2n(i)-1}} = 0.$$

□

At the end, we remark that the question of generalizations of Theorem 4 concerning other homogeneous linear differential systems, which can have only almost periodic solutions, remains open (contrary to the corresponding discrete case, see [14, 37]).

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