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Countably Many Solutions of a Fourth Order Boundary Value Problem

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Abstract

We apply fixed point theorems to obtain sufficient conditions for existence of infinitely many solutions of a nonlinear fourth order boundary value problem

$$u^{(4)}(t) = a(t)f(u(t)), \quad 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $a(t)$ is L^p-integrable and f satisfies certain growth conditions.

Mathematics Subject Classifications: 34B15, 34B16, 34B18.
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1 Introduction

In this paper we are interested in $(2, 2)$ conjugate nonlinear boundary-value problem

$$u^{(4)}(t) = a(t)f(u(t)), \quad 0 < t < 1, \tag{1}$$
$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{2}$$

which describes deformations of elastic beams with fixed end points.

The paper is organized in the following fashion. In the introduction we briefly discuss the background of the problem and give an overview of related results. In section 2 we introduce the assumptions on the inhomogeneous term of (1), discuss the properties of the Green's function of the homogeneous (1), (2), and state the theorems that we use to obtain our main results presented in sections 3 and 4.

Recently Yao [18] applied Krasnosel'skiĭ's fixed point theorem [15] to study the eigenvalue problem

$$\begin{gathered} u^{(4)}(t) = \lambda f(u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered}$$

The author obtained intervals of eigenvalues for which at least one or two solutions are guaranteed. For other related results we refer the reader to [5, 7, 17, 19].

Fixed point theorems have been applied to various boundary value problems to show the existence of multiple positive solutions. An overview of numerous such results can be found in Guo and Lakshmikantham [4] and Agarwal, O'Reagan and Wong [1].

The study of sufficient conditions for the existence of infinitely many positive solutions was originated by Eloe, Henderson and Kosmatov in [3]. The authors of [3] considered $(k, n-k)$ conjugate type BVP

$$\begin{gathered} (-1)^{n-k} u^{(n)}(t) = a(t) f(u(t)), \quad 0 < t < 1, \\ u^{(i)}(0) = 0, \quad i = 0, \dots, k-1, \\ u^{(j)}(1) = 0, \quad j = 0, \dots, n-k-1. \end{gathered}$$

Their approach was based on applications of cone-theoretic theorems due to Krasnosel'skiĭ and Leggett-Williams [16]. For applications of the latter see Davis and Henderson [2] and Henderson and Thompson [6] and the references therein. Later, in [13, 14], the author obtained infinitely many solutions for the second order BVP

$$\begin{gathered} -u''(t) = a(t) f(u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{gathered}$$

where $\alpha, \beta, \gamma, \delta \geq 0$, $\alpha\gamma + \alpha\delta + \beta\gamma > 0$. In addition, we point out that [13, 14] only dealt with a very special choice of a singular (L^1) integrable function $a(t)$. The study of infinitely many solutions was further developed by Kaufmann and Kosmatov [9, 10] to extend the results [13, 14] to the general case of $a(t) \in L^p[0,1]$ for $p \geq 1$. In [9], $a(t)$ was taken to possess countably many singularities (or to be an infinite series of singular functions) and, in [10], $a(t)$ was selected in the form of a finite product of singular functions. It is also relevant to our discussion to mention [8, 11, 12] devoted to BVP's on time scales and three-point BVP's.

In this note, we extend the results of Yao [18] (with $\lambda = 1$). We also generalize and refine the results of [3] (with $n = 4$, $k = 2$) by obtaining sharper sufficient conditions for existence of infinitely many solutions of (1), (2).

2 Auxiliaries and fixed point theorems

The Green's function of

$$u^{(4)} = 0$$

satisfying (2) is

$$G(t,s) = \frac{1}{6}\begin{cases} t^2(1-s)^2((s-t)+2(1-t)s), & 0 \leq t \leq s \leq 1, \\ s^2(1-t)^2((t-s)+2(1-s)t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{3}$$

Definition 2.1 *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subset \mathcal{B}$ be closed and nonempty. Then $\mathcal{K}$ is said to be a cone if*

1. *$\alpha u + \beta v \in \mathcal{K}$ for all $u, v \in \mathcal{K}$ and for all $\alpha, \beta \geq 0$, and*
2. *$u, -u \in \mathcal{K}$ implies $u \equiv 0$.*

We let $\mathcal{B} = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. In the sequel of our note we take $\tau \in [0, \frac{1}{2})$ and define our cone $\mathcal{K}_\tau \subset \mathcal{B}$ by

$$\mathcal{K}_\tau = \{u(t) \in \mathcal{B} \,|\, u(t) \geq 0 \text{ on } [0,1], \min_{t\in[\tau,1-\tau]} u(t) \geq c_\tau \|u\|\}, \tag{4}$$

where $c_\tau = \frac{2}{3}\tau^4$. We define an operator $T \colon \mathcal{B} \to \mathcal{B}$ by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(u(s))ds.$$

The required properties of T are stated in the next lemma.

Lemma 2.2 *The operator T is completely continuous and $T\colon \mathcal{K}_\tau \to \mathcal{K}_\tau$.*

Proof: By Arzela-Ascoli theorem, T is completely continuous.

Now we show that it is cone-preserving. To this end, if $s \in [t,1]$, then

$$\begin{aligned}
\min_{t\in[\tau,1-\tau]} G(t,s) &= \frac{1}{6}(1-s)^2 \min_{t\in[\tau,1-\tau]} t^2((s-t)+2(1-t)s) \\
&\geq \frac{1}{6}\tau^2(1-s)^2 2(1-\tau)\tau \\
&\geq \frac{1}{6}c_\tau 3(1-s)^2 \\
&\geq c_\tau \frac{1}{6}(1-s)^2((s-t')+2(1-t')s) \\
&\geq c_\tau G(t',s)
\end{aligned}$$

for all $t' \in [0,s]$. The case of $s \in [0,t]$ is treated similarly to see that

$$\min_{t\in[\tau,1-\tau]} G(t,s) \geq c_\tau G(t',s)$$

for all $s, t' \in [0,1]$.

With the estimate above we have

$$\begin{aligned}
\min_{t\in[\tau,1-\tau]} Tu(t) &= \min_{t\in[\tau,1-\tau]} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\
&\geq \int_0^1 \min_{t\in[\tau,1-\tau]} G(t,s)a(s)f(u(s))\,ds \\
&\geq c_\tau \int_0^1 G(t',s)a(s)f(u(s))\,ds \\
&= c_\tau Tu(t')
\end{aligned}$$

for all $t' \in [0,1]$. Hence $\min_{t\in[\tau,1-\tau]} Tu(t) \geq c_\tau \|T\|$ and the proof is finished.

□

Fixed points of T are solutions of (1), (2). The existence of a fixed point of T follows from theorems due to Krasnosel'skiĭ and Leggett-Williams. Now we state the former.

Theorem 2.3 *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that either *(i)* $\|Tu\| \leq \|u\|, \; u \in \mathcal{K} \cap \partial\Omega_1, \;$ *and* $\; \|Tu\| \geq \|u\|, \; u \in \mathcal{K} \cap \partial\Omega_2$ *, or*

(ii) $\|Tu\| \geq \|u\|, \; u \in \mathcal{K} \cap \partial\Omega_1, \;$ *and* $\; \|Tu\| \leq \|u\|, \; u \in \mathcal{K} \cap \partial\Omega_2$*.*

Then T *has a fixed point in* $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$*.*

To introduce Leggett-Williams fixed point theorem we need more definitions.

Definition 2.4 *The map* α *is said to be a nonnegative continuous concave functional on a cone* $\mathcal{K}$ *of a (real) Banach space* $\mathcal{B}$ *provided that* $\alpha\colon \mathcal{K} \to [0, \infty)$ *is continuous and*

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in \mathcal{K}$ *and* $0 \leq t \leq 1$*.*

Definition 2.5 *Let* $0 < a < b$ *be given and* α *be a nonnegative continuous concave functional on a cone* $\mathcal{K}$*. Define convex sets*

$$B_r = \{u \in \mathcal{K} \,|\, \|u\| < r\}$$

and

$$P(\alpha, a, b) = \{u \in \mathcal{K} \,|\, a \leq \alpha(u), \; \|u\| \leq b\}.$$

The following fixed point theorem due to Leggett and Williams enables one to obtain triple fixed points of an operator on a cone.

Theorem 2.6 *Let* $T\colon \overline{B}_c \to \overline{B}_c$ *be a completely continuous operator and let* α *be a nonnegative continuous concave functional on a cone* $\mathcal{K}$ *such that* $\alpha(u) \leq \|u\|$ *for all* $u \in \overline{B}_c$*. Suppose there exist* $0 < a < b < d \leq c$ *such that*

(C1) $\{u \in P(\alpha, b, d) | \, \alpha(u) > b\} \neq \emptyset$ *and* $\alpha(Tu) > b$ *for* $u \in P(\alpha, b, d)$*,*

(C2) $\|Tu\| < a$ *for* $\|u\| \leq a$*, and*

(C3) $\alpha(Tu) > b$ *for* $u \in P(\alpha, a, b)$ *with* $\|Tu\| > d$*.*

Then T *has at least three fixed points* u_1*,* u_2*, and* u_3 *such that* $\|u_1\| < a$*,* $b < \alpha(u_2)$*, and* $\|u_3\| > a$ *with* $\alpha(u_3) > b$

To obtain some of the norm inequalities in Theorems 2.3 and 2.6 we employ Hölder's inequality.

Theorem 2.7 (Hölder) *Let $f \in L^p[a,b]$ with $p > 1$, $g \in L^q[a,b]$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1[a,b]$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Let $f \in L^1[a,b]$ and $g \in L^\infty[a,b]$. Then $fg \in L^1[a,b]$ and

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

The following assumptions on the inhomogeneous term of (2) will stand throughout this paper:

(A1) f is nonnegative and continuous;

(A2) $\lim_{t \to t_0} a(t) = \infty$, where $0 < t_0 < 1$;

(A3) $a(t)$ is nonnegative and there exists $m > 0$ such that $a(t) \geq m$ *a.e.* on $[0,1]$;

(A4) $a(t) \in L^p[0,1]$ for some $1 \leq p \leq \infty$.

Any fixed points of T are now positive.

We will need to employ some estimates on (3) that are given below.

One can readily see that

$$\max_{t,s \in [0,1]} G(t,s) = \frac{1}{192}. \tag{5}$$

The function

$$\begin{aligned} \int_\tau^{1-\tau} G(t,s)\,ds &= \frac{1}{6}\int_\tau^t s^2(1-t)^2((t-s)+2(1-s)t)\,ds \\ &+ \frac{1}{6}\int_t^{1-\tau} t^2(1-s)^2((s-t)+2(1-t)s)\,ds \\ &= \frac{1}{6}(\frac{1}{4}t^4 - \frac{1}{2}t^3 + \frac{1}{4}t^2 + t^2\tau^3 - t\tau^3 + \frac{1}{4}\tau^4) \end{aligned}$$

attains its maximum on the interval $[0,1]$ at $t = \frac{1}{2}$ and

$$\max_{t \in [0,1]} \int_\tau^{1-\tau} G(t,s)\,ds = \frac{1}{24}(\tau^4 - \tau^3 + \frac{1}{16}) \tag{6}$$

and its minimum on the interval $[\tau, 1-\tau]$ at $t = \tau, 1-\tau$ so that

$$\min_{t\in[\tau,1-\tau]} \int_{\tau}^{1-\tau} G(t,s)\,ds = \frac{1}{24}\tau^2(1-2\tau)(1-2\tau^2). \tag{7}$$

Using the fact that (7) attains its minimum on the interval $[\tau', \tau''] \subset (0, \frac{1}{2})$ at one of the end-points and defining

$$l(\tau', \tau'') = \frac{1}{24}\min\left\{\tau'^2(1-2\tau')(1-2\tau'^2), \tau''^2(1-2\tau'')(1-2\tau''^2)\right\} \tag{8}$$

we get that

$$\begin{aligned}\min_{t\in[\tau,1-\tau]} \int_{\tau}^{1-\tau} G(t,s)\,ds &= \frac{1}{24}\tau^2(1-2\tau)(1-2\tau^2) \\ &> l(\tau', \tau'')\end{aligned} \tag{9}$$

for all $\tau \in [\tau', \tau'']$.

It follows from (6) that

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_1 = \frac{1}{384}. \tag{10}$$

One can also easily see from (5) that

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_q = \max_{t\in[0,1]} \left(\int_0^1 G^q(t,s)\,ds\right)^{\frac{1}{q}} < \frac{1}{192}. \tag{11}$$

Remark: Other estimates on (3) (used in construction of cones) can be found in [3, 18].

3 Positive solutions and Krasnosel'skiĭ's fixed point theorem

We consider the following three case for $a \in L^p[0,1]$: $p > 1$, $p = 1$, and $p = \infty$. Case $p > 1$ is treated in the following theorem.

Theorem 3.1 *Let $\{\tau_k\}_{k=1}^{\infty}$ be such that $\tau_1 < \frac{1}{2}$ and $\tau_k \downarrow \tau^* > 0$. Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be such that*

$$A_{k+1} < c_k B_k < B_k < C B_k < A_k, \quad k \in \mathbb{N},$$

where

$$C = \max\left\{\frac{384}{m(1+16(\tau_1^4-\tau_1^3))}, 1\right\}.$$

Assume that f satisfies

(H1) $f(z) \le M_1 A_k$ *for all* $z \in [0, A_k], k \in \mathbb{N}$, *where* $M_1 \le 192\backslash\|a\|_p$.

(H2) $f(z) \ge CB_k$ *for all* $z \in [c_{\tau_k} B_k, B_k]$, *where* $c_{\tau_k} = \frac{2}{3}\tau_k^4$.

Then the boundary value problem (1), (2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$. *Furthermore,* $B_k \le \|u_k\| \le A_k$ *for each* $k \in \mathbb{N}$.

Proof: For a fixed k, define $\Omega_{1,k} = \{u \in \mathcal{B} : \|u\| < A_k\}$. The cone $\mathcal{K}_{\tau_k}$ is given by (4) with $\tau = \tau_k$. Then

$$u(s) \le A_k = \|u\|$$

for all $s \in [0,1]$. By (H1),

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\le \max_{t\in[0,1]} \int_0^1 G(t,s)a(s)\,ds\, M_1 A_k. \end{aligned}$$

Since $p > 1$, take $q = \frac{p}{p-1} > 1$. Then, by Theorem 2.7,

$$\|Tu\| \le \max_{t\in[0,1]} \|G(t,\cdot)\|_q \|a\|_p M_1 A_k.$$

From (11) and (H1),

$$\begin{aligned} \|Tu\| &< \frac{1}{192}\|a\|_p M_1 A_k \\ &< A_k. \end{aligned}$$

Since $\|u\| = A_k$ for all $u \in \mathcal{K}_{\tau_k} \cap \partial\Omega_{1,k}$, then

$$\|Tu\| < \|u\|. \tag{12}$$

Remark: Note that since $1 + 16({\tau_1}^4 - {\tau_1}^3) < 1$ and $\|a\|_p \ge m$, we have that $M_1 < C$ (otherwise the theorem is vacuously true).

Now define $\Omega_{2,k} = \{u \in \mathcal{B} : \|u\| < B_k\}$. Let $u \in \mathcal{K}_{\tau_k} \cap \partial\Omega_{2,k}$ and let $s \in [\tau_k, 1-\tau_k]$. Then

$$B_k = \|u\| \geq u(s) \geq \min_{[\tau_k, 1-\tau_k]} u(s) \geq c_{\tau_k}\|u\| = c_k B_k.$$

By (H2),

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)f(u(s))\,ds \\ &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)\,ds\,CB_k. \end{aligned}$$

Now, by (A3) and (6),

$$\begin{aligned} \|Tu\| &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)a(s)\,ds\,CB_k \\ &\geq \max_{t\in[0,1]} \int_{\tau_k}^{1-\tau_k} G(t,s)\,ds\,mCB_k \\ &= \frac{1}{24}(\tau_k^4 - \tau_k^3 + \frac{1}{16})mCB_k \\ &= \frac{\tau_k^4 - \tau_k^3 + \frac{1}{16}}{\tau_1^4 - \tau_1^3 + \frac{1}{16}}B_k \\ &> B_k, \end{aligned}$$

since $\tau_k < \tau_1$. Thus, if $u \in \mathcal{P}_{\tau_k} \cap \partial\Omega_{2,k}$, then

$$\|Tu\| > B_k = \|u\|. \tag{13}$$

Now $0 \in \Omega_{2,k} \subset \overline{\Omega}_{2,k} \subset \Omega_{1,k}$. By (12), (13) it follows from Theorem 2.3 that the operator T has a fixed point $u_k \in \mathcal{P}_{\tau_k} \cap (\overline{\Omega}_{1,k} \setminus \Omega_{2,k})$ such that $B_k \leq \|u_k\| \leq A_k$. Since $k \in \mathbb{N}$ was arbitrary, the proof is complete.

□

The following theorem deals with the case $p = \infty$.

Theorem 3.2 *Let $\{\tau_k\}_{k=1}^{\infty}$ be such that $\tau_1 < \frac{1}{2}$ and $\tau_k \downarrow \tau^* > 0$. Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be such that*

$$A_{k+1} < c_k B_k < B_k < CB_k < A_k, \quad k \in \mathbb{N},$$

where C and c_{τ_k} are as in Theorem 3.1.

Assume that f satisfies (H2) and

(H3) $f(z) \leq M_2 A_k$ for all $z \in [0, A_k], k \in \mathbb{N}$, where $M_2 \leq 384\backslash\|a\|_\infty$.

Then the boundary value problem (1), (2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$. Furthermore, $B_k \leq \|u_k\| \leq A_k$ for each $k \in \mathbb{N}$.

Proof: We now use (10) and repeat the argument above.

□

Our last result corresponds to the case of $p = 1$.

Theorem 3.3 *Let $\{\tau_k\}_{k=1}^{\infty}$ be such that $\tau_1 < \frac{1}{2}$ and $\tau_k \downarrow \tau^* > 0$. Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be such that*

$$A_{k+1} < c_k B_k < B_k < CB_k < A_k, \quad k \in \mathbb{N},$$

where C and c_{τ_k} are as in Theorem 3.1.

Assume that f satisfies (H2) and

(H4) $f(z) \leq M_3 A_k$ for all $z \in [0, A_k], k \in \mathbb{N}$, where $M_3 \leq 192\backslash\|a\|_1$.

Then the boundary value problem (1), (2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$. Furthermore, $B_k \leq \|u_k\| \leq A_k$ for each $k \in \mathbb{N}$.

Proof: For a fixed k, define $\Omega_{1,k} = \{u \in \mathcal{B} : \|u\| < A_k\}$. Then

$$u(s) \leq A_k = \|u\|$$

for all $s \in [0,1]$. By (H4) and (5),

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s)a(s)\,ds\, M_3 A_k \\
&\leq \int_0^1 \max_{t\in[0,1]} G(t,s)a(s)\,ds\, M_3 A_k \\
&\leq \max_{s,t\in[0,1]} G(t,s) \int_0^1 a(s)\,ds\, M_3 A_k \\
&= \frac{1}{192}\|a\|_1 M_3 A_k \\
&\leq A_k
\end{aligned}$$

and thus we obtain (12), which together with (13) completes the proof.

□

4 Positive solutions and Legett-Williams fixed point theorem

In this section we only consider the case of $p > 1$. The existence theorems corresponding to the cases of $p = 1$ and $p = \infty$ are similar to the next theorem and are omitted.

For our cone we now choose

$$\mathcal{K} = \{u \in \mathcal{B} \,|\, u(t) \geq 0\}$$

and our nonnegative continuous concave functionals on $\mathcal{K}$ are defined by

$$\alpha_k(u) = \min_{t\in[\tau_k, 1-\tau_k]} u(t)$$

with $\alpha_k(u) \leq \|u\|$ for each $\tau_k \in (0, \frac{1}{2})$. For the rest of the note c_{τ_k} is denoted by c_k.

Theorem 4.1 *Let $\{\tau_k\}_{k=1}^{\infty}$ be such that $\tau_1 < \frac{1}{2}$ and $\tau_k \downarrow \tau^* > 0$. Let $\{A_k\}_{k=1}^{\infty}$, $\{B_k\}_{k=1}^{\infty}$, and $\{C_k\}_{k=1}^{\infty}$ be such that*

$$C_{k+1} < A_k < B_k < \frac{1}{c_k} B_k < C_k, \quad k \in \mathbb{N},$$

where M_1 is as in Theorem 3.1. Suppose that f satisfies

(H5) $f(z) < M_1 A_k$ *for all* $z \in [0, A_k]$, $k \in \mathbb{N}$,

(H6) $f(z) > LB_k$ *for all* $z \in [B_k, \frac{1}{c_k} B_k]$, $k \in \mathbb{N}$, *where* $L = \frac{1}{ml(\tau^*, \tau_1)}$ *and* $l(\tau^*, \tau_1)$ *is given by (8).*

(H7) $f(z) < M_1 C_k$ *for all* $z \in [0, C_k]$, $k \in \mathbb{N}$.

Then the boundary value problem (1), (2) has three infinite families of solutions $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$, *and* $\{u_{3k}\}_{k=1}^{\infty}$ *satisfying* $\|u_{1k}\| < A_k$, $B_k < \alpha_k(u_{2k})$, *and* $\|u_{3k}\| > A_k$, $B_k > \alpha_k(u_{3k})$ *for each* $k \in \mathbb{N}$.

Proof: As in Definition 2.5, set for each $k \in \mathbb{N}$,

$$B_{A_k} = \{u \in \mathcal{K} \mid \|u\| < A_k\}$$

and

$$B_{C_k} = \{u \in \mathcal{K} \mid \|u\| < C_k\}.$$

We use (H5) and (H7) and repeat the argument leading to (12) to see that $T\colon \overline{B}_{A_k} \to \overline{B}_{A_k}$ and $T\colon \overline{B}_{C_k} \to \overline{B}_{C_k}$. Thus, the condition (C2) of Theorem 2.6 is satisfied.

As in Definition 2.5, set

$$P(\alpha_k, B_k, \frac{1}{c_k} B_k) = \{u \in \mathcal{K} \mid B_k \le \alpha_k(u),\ \|u\| \le \frac{1}{c_k} B_k\}$$

and

$$P(\alpha_k, B_k, C_k) = \{u \in \mathcal{K} \mid B_k \le \alpha_k(u),\ \|u\| \le C_k\}.$$

Choosing $u = \frac{1}{c_k} B_k \in P(\alpha_k, B_k, \frac{1}{c_k} B_k)$, we have $\alpha_k(u) = \frac{1}{c_k} B_k > B_k$, that is, $\{u \in P(\alpha_k, B_k, \frac{1}{c_k} B_k) \mid \alpha_k(u) > B_k\} \ne \emptyset$.

By assumption (A3),

$$\begin{aligned}
\alpha_k(Tu) &= \min_{t \in [\tau_k, 1-\tau_k]} Tu(t) \\
&= \min_{t \in [\tau_k, 1-\tau_k]} \int_0^1 G(t,s) a(s) f(u(s))\, ds \\
&\ge \min_{t \in [\tau_k, 1-\tau_k]} \int_0^1 G(t,s) f(u(s))\, ds\, m \\
&> \min_{t \in [\tau_k, 1-\tau_k]} \int_{\tau_k}^{1-\tau_k} G(t,s) f(u(s))\, ds\, m.
\end{aligned}$$

Now, by (H7), using (7)-(9) we have

$$\begin{aligned}
\alpha_k(Tu) &> \min_{t\in[\tau_k,1-\tau_k]} \int_{\tau_k}^{1-\tau_k} G(t,s)\,ds\,LB_k m \\
&= \frac{1}{24}{\tau_k}^2(1-2\tau_k)(1-2{\tau_k}^2)mLB_k \\
&> B_k,
\end{aligned}$$

since $\tau^* < \tau_k < \tau_1$. Therefore, $\alpha_k(u) > B_k$ for all $u \in P(\alpha_k, B_k, \frac{1}{c_k}B_k)$ and the assumption (C1) of Theorem 2.6 is satisfied.

If, in addition, $u \in P(\alpha_k, B_k, C_k)$ with $\|Tu\| > \frac{1}{c_k}B_k$, then (as in the proof of Lemma 2.2)

$$\begin{aligned}
\alpha_k(Tu) &= \min_{t\in[\tau_k,1-\tau_k]} Tu(t) \\
&= \min_{t\in[\tau_k,1-\tau_k]} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\
&\geq \int_0^1 \min_{t\in[\tau_k,1-\tau_k]} G(t,s)a(s)f(u(s))\,ds \\
&\geq \int_0^1 c_k G(t',s)a(s)f(u(s))\,ds \\
&= c_k Tu(t')
\end{aligned}$$

for all $t' \in [0,1]$, which implies $\alpha_k(Tu) \geq c_k\|Tu\| > B_k$. Thus the assumption (C3) is checked.

Since all hypotheses of Theorem 2.6 are satisfied, the assertion follows. □

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