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Infinitely many solutions for a class of second-order damped vibration systems

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Abstract: In this paper, by using the variational approach, we study the existence of infinitely many solutions for a class of second-order damped vibration systems under super-quadratic and sub-quadratic conditions. Some new results are established and some recent results in the literature are generalized and significantly improved.

Keywords: Damped vibration systems; Infinitely many solutions; Variational methods

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1. Introduction and main results

In this paper, we investigate the existence of infinitely many solutions for the following damped vibration system

$$\begin{cases} \dfrac{d(P(t)\dot{u}(t))}{dt} + (q(t)P(t)+B)\dot{u}(t) + \left(\dfrac{1}{2}q(t)B - A(t)\right)u(t) + \nabla F(t,u(t)) = 0, \text{a.e.} t \in [0,T], \\ u(0)-u(T) = P(0)\dot{u}(0) - P(T)\dot{u}(T) = 0 \end{cases} \tag{1.1}$$

where $T > 0$, $q \in L^1(0,T;\mathbb{R})$ satisfying $\int_0^T q(t)dt = 0$, $P(t)$ and $A(t)$ are symmetric and continuous $N \times N$ matrix-value functions on $[0,T]$, B is a skew-symmetric $N \times N$ constant matrix and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions:

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(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

When $P(t) \equiv I_{N\times N}$, where $I_{N\times N}$ is the $N \times N$ unit matrix , system (1.1) reduces to the following system

$$\begin{cases} \ddot{u}(t) + (q(t)I_{N\times N} + B)\dot{u}(t) + (\frac{1}{2}q(t)B - A(t))u(t) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.2}$$

Recently, system (1.2) has been investigated in [1]. By using Theorem 5.29 in [2], Li-Wu-Wu obtained system (1.2) has a nontrivial solution under (AR)-condition

(AR) there exist constants $\mu > 2$ and $r \ge 0$ such that

$$(\nabla F(t,x), x) \ge \mu F(t,x) > 0, \quad \forall\ |x| \ge r, \ \ a.e. t \in [0,T]$$

and some reasonable conditions (see [1], Theorem 3.3). Moreover, by using symmetric Mountain Pass Theorem in [2] and a critical point theorem in [3], they obtained two existence results of infinitely many solutions under symmetric condition $F(t,-x) = F(t,x)$, (AR)-condition and some reasonable conditions (see Theorem 3.1 and Theorem 3.2 in [1]).

When $q(t) \equiv 0$ and $A(t) \equiv 0$, system (1.1) reduces to the following system

$$\begin{cases} \dfrac{d(P(t)\dot{u}(t))}{dt} + B\dot{u}(t) + \nabla F(t,u(t)) = 0, \text{a.e.} t \in [0,T], \\ u(0) - u(T) = P(0)\dot{u}(0) - P(T)\dot{u}(T) = 0. \end{cases} \tag{1.3}$$

In [10], Han-Wang investigated system (1.3). By using symmetric Mountain Pass Theorem, they obtained system (1.3) has infinitely many solutions under symmetric condition $F(t,-x) = F(t,x)$, (AR)-condition and some reasonable conditions (see Theorem 3.3 in [10]). Moreover, they also investigated the sub-quadratic case. By using a critical point theorem in [8], they obtained system (1.3) has infinitely many solutions under symmetric condition $F(t,-x) = F(t,x)$, the sub-quadratic condition:

(SQ) there exist $0 \le \alpha < 1$ *and* $g(t), h(t) \in L^1(0, T; \mathbb{R}^+)$ *such that*

$$|\nabla F(t,u)| \le g(t)|u|^\alpha + h(t)$$

and some reasonable conditions (see [10], Theorem 3.1). For some related results, one can also see [11]-[18] and the references therein.

In this paper, we will investigate system (1.1) which is the extension of system (1.2) and system (1.3) and under more general super-quadratic conditions than those in [1], we obtain that system (1.1) has infinitely many solutions. Moreover, we also obtain a new result under sub-quadratic conditions. Next, we state our results.

(I) For super-quadratic case

Theorem 1.1. *Assume the following conditions hold:*

(P) there exists a constant $m > \frac{1}{2}$ *such that the matrix* $P(t)$ *satisfies*

$$(P(t)x, x) > m(x, x), \qquad \text{for all } (t,x) \in \mathbb{R} \times \{\mathbb{R}^N / \{0\}\}$$

(H1) $\limsup_{|x| \to 0} \frac{F(t,x)}{|x|^2} \le 0$ *uniformly for a.e.* $t \in [0, T]$*;*

(H2) $\lim_{|x| \to \infty} \frac{F(t,x)}{|x|^2} = +\infty$ *uniformly for a.e.* $t \in [0, T]$;

(H3) there exist constants $L \ge 0$, $\zeta > 0$, $\eta > 0$ *and* $\nu \in [0, 2)$ *such that*

$$\left(2 + \frac{1}{\zeta + \eta |x|^\nu}\right) F(t,x) \le (\nabla F(t,x), x)\,, \quad \text{for } x \in \mathbb{R}^N,\ |x| > L, \quad a.e.\ t \in [0,T];$$

(H4) $F(t, x)$ *is even in* x *and* $F(t, 0) \equiv 0$.

Then system (1.1) has an unbounded sequence of solutions.

When the condition $F(t, 0) \equiv 0$ is deleted, we have the following result:

Theorem 1.2. *Assume that (P) and (H1)-(H3) hold and* $F(t, x)$ *is even in* x*. Then system (1.1) has an infinite sequence of distinct solutions.*

By Theorem 1.1 and Theorem 1.2, we can obtain the following corollaries.

Corollary 1.1. *Assume that (P), (H1), (H4) and (AR)-condtion hold. Then system (1.1) has an unbounded sequence of solutions.*

Corollary 1.2. *Assume that (P), (H1) and (AR)-condtion hold and* $F(t, x)$ *is even in* x*. Then system (1.1) has an infinite sequence of distinct solutions.*

Corollary 1.3. *Assume that (P), (H1), (H2), (H4) and the following condition holds:*

(H3)′ there exist $\vartheta > 2$ and $\mu > \vartheta - 2$ such that

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^{\vartheta}} < \infty \quad \textit{uniformly for a.e. } t \in [0,T],$$
$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^{\mu}} > 0 \quad \textit{uniformly for a.e. } t \in [0,T].$$

Then system (1.1) has an unbounded sequence of solutions.

Corollary 1.4. *Assume that (P), (H1), (H2) and (H3)′ hold and $F(t,x)$ is even in x. Then system (1.1) has an infinite sequence of distinct solutions.*

Remark 1.1. It is remarkable that in [4], the following condition which is similar to (H3) has been presented:

($\hat{S}_2$) *there exist $p > 2$, $c_1, c_2, c_3 > 0$ and $\nu \in (0,2)$ such that, for all $|z| \geq r_1$,*

$$|\nabla H(t,z)||z| \leq c_1(\nabla H(t,z),z), \quad |\nabla H(t,z)| \leq c_2|z|^{p-1},$$
$$H(t,z) \leq \left(\frac{1}{2} - \frac{1}{c_3|z|^{\nu}}\right)(\nabla H(t,z),z),$$

which was used to consider the existence of homoclinic solutions for the first order Hamiltonian system $\dot{z} = J\nabla H(t,z)$. In [5], the author and Tang investigated the existence of periodic and subharmonic solutions for the second order Hamiltonian system

$$\ddot{u}(t) + Au(t) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R} \tag{1.4}$$

and presented that the conditions like

$$|\nabla H(t,z)||z| \leq c_1(\nabla H(t,z),z) \ \text{ and } \ |\nabla H(t,z)| \leq c_2|z|^{p-1}$$

in ($\hat{S}_2$) are not necessary when one considered the existence of periodic solutions for system (1.4). Obviously, when $P(t) \equiv I_{N\times N}$, Corollary 1.1 and Corollary 1.2 reduce to Theorem 3.1-Theorem 3.2 in [1]. There exist examples satisfying our Theorem 1.1 and Theorem 1.2 but not satisfying Theorem 3.1 and Theorem 3.2 in [1]. For example, let

$$F(t,x) \equiv F(x) = |x|^2 \ln(1+|x|^2).$$

(II) For sub-quadratic case

Theorem 1.3. *Assume that (P) and the following conditions hold:*

(F1) $\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = 0$ *uniformly for a.e.* $t \in [0,T]$;

(F2) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = +\infty$ *uniformly for a.e.* $t \in [0,T]$;

(F3) there exists a function $h \in L^1([0,T];\mathbb{R})$ *such that*

$$e^{Q(t)}[2F(t,x) - (\nabla F(t,x), x)] \geq h(t) \quad \text{for } x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T],$$

and

$$\lim_{|x|\to\infty} e^{Q(t)}[2F(t,x) - (\nabla F(t,x), x)] = +\infty, \quad \text{for a.e. } t \in [0,T],$$

where $Q(t) = \int_0^t q(s)ds$;

(F4) $F(t,x)$ *is even in* x *and* $F(t,0) \equiv 0$.

Then system (1.1) has infinitely many nontrivial solutions.

Remark 1.2. Theorem 1.3 is different from Theorem 3.1 and Theorem 3.2 in [10]. There exist examples satisfy Theorem 1.3 but not satisfying Theorem 3.1 and Theorem 3.2 in [10]. For example, let

$$F(t,x) \equiv F(x) = (1+|x|^2)^{1/2} \ln(1+|x|^2) + |x|^{3/2}.$$

2. Preliminaries

In this section, we will present the variational structure of system (1.1), which is the slight modification of those in [1]. Let

$$H_T^1 = \{u : [0,T] \to \mathbb{R}^N |\ u \text{ is absolutely continuous}, u(0) = u(T) \text{ and } \dot{u} \in L^2([0,T])\}.$$

Let

$$Q(t) = \int_0^t q(s)ds.$$

Define

$$\langle u, v\rangle = \int_0^T e^{Q(t)}(u(t), v(t))dt + \int_0^T e^{Q(t)}(P(t)\dot{u}(t), \dot{v}(t))dt$$

and

$$\|u\| = \left[\int_0^T e^{Q(t)}|u(t)|^2 dt + \int_0^T e^{Q(t)}(P(t)\dot{u}(t), \dot{u}(t))dt\right]^{1/2}$$

for each $u, v \in H_T^1$. Then $(H_T^1, \langle \cdot, \cdot \rangle)$ is a Hilbert space. It follows from assumption (A) and Theorem 1.4 in [6] that the functional φ on H_T^1 given by

$$\varphi(u) = \int_0^T e^{Q(t)} \left[\frac{1}{2}(P(t)\dot{u}(t), \dot{u}(t)) + \frac{1}{2}(Bu(t), \dot{u}(t)) + \frac{1}{2}(A(t)u(t), u(t)) - F(t, u(t)) \right] dt$$

is continuously differentiable and

$$\begin{aligned} \langle \varphi'(u), v \rangle \ &= \ \int_0^T e^{Q(t)} [(P(t)\dot{u}(t), \dot{v}(t)) - \frac{1}{2} q(t)(Bu(t), v(t)) - (B\dot{u}(t), v(t)) \\ &\quad + (A(t)u(t), v(t)) - (\nabla F(t, u(t)), v(t))] dt \end{aligned} \tag{2.1}$$

for $u, v \in H_T^1$. It is well known that

$$\|u\|_{H_T^1} = \left[\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right]^{1/2}$$

is also a norm on H_T^1. Obviously, if the condition (P) holds, $\|u\|_{H_T^1}$ and $\|u\|$ are equivalent. Moreover, there exists $C_0 > 0$ such that

$$\|u\|_\infty \leq C_0 \|u\|_{H_T^1}$$

(see Proposition 1.1 in [6]). Hence, there exists $C_* > 0$ such that

$$\|u\|_\infty \leq C_* \|u\|. \tag{2.2}$$

Lemma 2.1. *If $u_0 \in H_T^1$ satisfies $\varphi'(u_0) = 0$, then u_0 is a solution of system (1.1).*

Proof. The proof is a slight modification of Lemma 2.2 in [1]. It follows from $\varphi'(u_0) = 0$ and (2.1) that

$$\int_0^T e^{Q(t)} \left[(P(t)\dot{u}_0, \dot{v}) - \frac{1}{2} q(t)(Bu_0, v) - (B\dot{u}_0, v) + (A(t)u_0, v) - (\nabla F(t, u_0), v) \right] dt = 0$$

for all $v \in H_T^1$, that is

$$\begin{aligned} \int_0^T e^{Q(t)} (P(t)\dot{u}_0, \dot{v}) dt \ &= \ -\int_0^T e^{Q(t)} \left[-\frac{1}{2} q(t)(Bu_0, v) - (B\dot{u}_0, v) \right. \\ &\quad \left. + (A(t)u_0, v) - (\nabla F(t, u_0), v) \right] dt \end{aligned}$$

for all $v \in H_T^1$. By the Fundamental Lemma and Remarks in page 6-7 of [6], we have

$$\begin{aligned} e^{Q(t)} P(t)\dot{u}_0(t) \ &= \ \int_0^t e^{Q(s)} \left[-\frac{1}{2} q(s)Bu_0(s) - B\dot{u}_0(s) + A(t)u_0(s) - \nabla F(t, u_0(s)) \right] ds \\ &\quad + P(0)\dot{u}(0) \end{aligned} \tag{2.3}$$

for a.e. $t \in [0,T]$ and

$$\int_0^T e^{Q(t)}\left[-\frac{1}{2}q(t)Bu_0(t) - B\dot{u}_0(t) + A(t)u_0(t) - \nabla F(t,u_0(t))\right]dt = 0. \tag{2.4}$$

Then by (2.3), we obtain that $e^{Q(t)}P(t)\dot{u}_0$ is completely continuous on $[0,T]$ and

$$\frac{d(P(t)\dot{u}_0(t))}{dt} + (q(t)P(t)+B)\dot{u}_0(t) + \left(\frac{1}{2}q(t)B - A(t)\right)u_0(t) + \nabla F(t,u_0(t)) = 0, \ \text{ a.e. } t \in [0,T].$$

Note that $\int_0^T q(t)dt = 0$. Then by (2.3) and (2.4), it is easy to see that $P(0)\dot{u}(0) = P(T)\dot{u}_0(T)$. Therefore, u_0 is a solution of system (1.1). This completes the proof.

By the Riesz theorem, define the operator $K : H_T^1 \to (H_T^1)^*$ by

$$\langle Ku, v\rangle = \int_0^T e^{Q(t)}(B\dot{u}, v)dt + \int_0^T e^{Q(t)}((I_{N\times N} - A(t))u(t), v(t))dt.$$

for all $u, v \in H_T^1$. Then K is a bounded self-adjoint linear operator (see [1]). By the definition of K, the functional φ can be written as

$$\varphi(u) = \frac{1}{2}((I-K)u,u) - \int_0^T e^{Q(t)}F(t,u)dt.$$

By the classical spectral theory, we have the decomposition: $H_T^1 = H^- \oplus H^0 \oplus H^+$, where $H^0 = \ker(I-K)$ and H^0, H^- are finite dimensional. Moreover, by the spectral theory, there is a $\delta > 0$ such that

$$\langle (I-K)u,u\rangle \le -\delta\|u\|^2, \text{ if } u \in H^- \tag{2.5}$$

$$\langle (I-K)u,u\rangle \ge \delta\|u\|^2, \text{ if } u \in H^+ \tag{2.6}$$

(see [1]).

3. The super-quadratic case

Similar to the proofs in [1], we will also use symmetric mountain pass theorem (see Theorem 9.12 in [2]) to prove Theorem 1.1 and use an abstract critical point theorem due to Bartsch and Ding (see [3]) to prove Theorem 1.2.

Remark 3.1. As shown in [7], a deformation lemma can be proved with replacing the usual (PS)-condition with (C)-condition, and it turns out that symmetric mountain pass theorem in

[2] are true under (C)-condition. We say that φ satisfies (C)-condition, i.e. for every sequence $\{u_n\} \subset H_T^1$, $\{u_n\}$ has a convergent subsequence if $\varphi(u_n)$ is bounded and $(1+\|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.

Let

$$\begin{aligned}\|A\| &= \sup_{t\in[0,T]} \max_{|x|=1, x\in\mathbb{R}^N} |A(t)x| \\ &= \sup_{t\in[0,T]} \max\{\sqrt{\lambda(t)} : \lambda(t) \text{ is the eigenvalue of } A^\tau(t)A(t)\}\end{aligned}$$

and

$$\begin{aligned}\|B\| &= \max_{|x|=1, x\in\mathbb{R}^N} |Bx| \\ &= \max\{\sqrt{\lambda} : \lambda \text{ is the eigenvalue of } B^\tau B\}.\end{aligned}$$

Proof of Theorem 1.1. Step 1. We claim that there exist $\rho > 0$ and $b > 0$ such that

$$\varphi(u) \geq b > 0, \quad \forall\, u \in H^+ \cap \partial B_\rho.$$

In fact, it follows from (H1) that there exist $0 < \varepsilon_0 < \frac{\delta}{2}$ and $r > 0$ such that

$$F(t,x) \leq \varepsilon_0 |x|^2, \quad \text{for all } |x| < r. \tag{3.1}$$

Choosing $\rho = r/C_*$. Then by (2.2), for all $u \in H^+ \cap \partial B_\rho$, we have $\|u\|_\infty \leq r$. Hence, by (P), (2.6) and (3.1), we obtain

$$\varphi(u) \geq \frac{\delta}{2}\|u\|^2 - \varepsilon_0 \int_0^T e^{Q(t)}|u(t)|^2 dt \geq \left(\frac{\delta}{2} - \varepsilon_0\right)\|u\|^2 = \left(\frac{\delta}{2} - \varepsilon_0\right)\rho^2 := b > 0.$$

Step 2. For each finite dimensional space $\tilde{E} \subset E$, we claim that there exists $R > 0$ such that $\varphi(u) \leq 0$ on $\tilde{E}/B_R$. In fact, since $\tilde{E}$ is dimensional, all norms on $\tilde{E}$ are equivalent. Hence, there exist $d_1, d_2 > 0$ such that

$$d_1\|u\|^2 \leq \int_0^T e^{Q(t)}|u(t)|^2 dt \leq d_2\|u\|^2 \tag{3.2}$$

It follows from (H2) that there exist constants

$$\beta > \frac{1}{2d_1} \max\left\{\left(1 + \frac{\|B\|}{2m}\right), \frac{\|B\| + 2\|A\|}{2}\right\}$$

and $M_0 > 0$ such that

$$F(t,x) \geq \beta|x|^2, \quad \forall\ |x| \geq M_0, \ \text{ a.e. } t \in [0,T]. \tag{3.3}$$

It follows from (3.3) and assumption (A) that there exist $D_1 > 0$ and $D_2 > 0$ such that

$$F(t,x) \geq \beta|x|^2 - D_1 - D_2 b(t), \quad \forall\ x \in \mathbb{R}^N, \ \text{ a.e. } t \in [0,T]. \tag{3.4}$$

Then by condition (P), (3.4) and (3.2), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T e^{Q(t)}\left[(P(t)\dot{u}(t),\dot{u}(t)) + (Bu(t),\dot{u}(t)) + (A(t)u(t),u(t))\right]dt \\
&\quad - \int_0^T e^{Q(t)}F(t,u(t))dt \\
&\leq \frac{1}{2}\int_0^T e^{Q(t)}\left[(P(t)\dot{u}(t),\dot{u}(t)) + \frac{\|B\|(|u(t)|^2+|\dot{u}(t)|^2)}{2} + \|A\||u(t)|^2\right]dt \\
&\quad - \int_0^T e^{Q(t)}F(t,u(t))dt \\
&\leq \frac{1}{2}\int_0^T e^{Q(t)}\left[\left(1+\frac{\|B\|}{2m}\right)(P(t)\dot{u}(t),\dot{u}(t)) + \frac{\|B\|+2\|A\|}{2}|u(t)|^2\right]dt \\
&\quad - \int_0^T e^{Q(t)}F(t,u(t))dt \\
&\leq \frac{1}{2}\max\left\{\left(1+\frac{\|B\|}{2m}\right), \frac{\|B\|+2\|A\|}{2}\right\}\|u\|^2 \\
&\quad - \int_0^T e^{Q(t)}[\beta|u(t)|^2 - D_1 - D_2 b(t)]dt \\
&\leq \frac{1}{2}\max\left\{\left(1+\frac{\|B\|}{2m}\right), \frac{\|B\|+2\|A\|}{2}\right\}\|u\|^2 - \beta d_1\|u\|^2 \\
&\quad + D_1\int_0^T e^{Q(t)}dt + D_2\int_0^T e^{Q(t)}b(t)dt.
\end{aligned}$$

Note that

$$\beta > \frac{1}{2d_1}\max\left\{\left(1+\frac{\|B\|}{2m}\right), \frac{\|B\|+2\|A\|}{2}\right\}.$$

So $\varphi(u) \to -\infty$, as $\|u\| \to \infty$.

Step 3. We prove that φ satisfies (C)-condition on H_T^1. Assume that there exists a constant $D_3 > 0$ such that

$$|\varphi(u_n)| \leq D_3, \quad (1+\|u_n\|)\,\|\varphi'(u_n)\| \leq D_3, \ \text{ for all } n \in \mathbb{N}, \tag{3.5}$$

By (H3), we have

$$[(\nabla F(t,x),x) - 2F(t,x)](\zeta + \eta|x|^\nu) \geq F(t,x), \ \ \forall\ x \in \mathbb{R}^N, \ \ |x| > L, \ \text{ a.e. } t \in [0,T]. \tag{3.6}$$

Then by assumption (A) and (3.6), there exists a constant $D_4 > 0$ such that

$$[(\nabla F(t,x),x) - 2F(t,x)](\zeta + \eta|x|^{\nu}) \geq F(t,x) - D_4 b(t), \quad \forall\, x \in \mathbb{R}^N, \quad \text{a.e. } t \in [0,T]. \tag{3.7}$$

It follows from assumption (A), (3.4) and (3.7) that there exist $D_5 > 0, D_6 > 0$ and $D_7 > 0$ such that

$$\begin{aligned} (\nabla F(t,x),x) - 2F(t,x) &\geq \frac{F(t,x) - D_4 b(t)}{\zeta + \eta|x|^{\nu}} \\ &\geq \frac{\beta|x|^2 - D_1 - D_2 b(t) - D_4 b(t)}{\zeta + \eta|x|^{\nu}} \\ &\geq D_5|x|^{2-\nu} - D_6 b(t) - D_7, \qquad \forall\, x \in \mathbb{R}^N. \end{aligned} \tag{3.8}$$

Hence, it follows from (3.8) and antisymmetry of B that

$$\begin{aligned} & 3D_3 \\ \geq\ & 2\varphi(u_n) - \langle \varphi'(u_n), u_n \rangle \\ =\ & \frac{1}{2}\int_0^T e^{Q(t)} q(t)(Bu_n(t), u_n(t))dt + \int_0^T e^{Q(t)}[(\nabla F(t,u_n(t)), u_n(t)) - 2F(t,u_n(t))]dt \\ =\ & \int_0^T e^{Q(t)}[(\nabla F(t,u_n(t)), u_n(t)) - 2F(t,u_n(t))]dt \qquad (3.9) \\ \geq\ & D_5 \int_0^T e^{Q(t)}|u_n(t)|^{2-\nu}dt - D_6 \int_0^T e^{Q(t)} b(t)dt - D_7 \int_0^T e^{Q(t)}dt. \qquad (3.10) \end{aligned}$$

This shows that $\int_0^T e^{Q(t)}|u_n(t)|^{2-\nu}dt$ is bounded. By (3.6) and (3.3), we have

$$\begin{aligned} [(\nabla F(t,x),x) - 2F(t,x)](\zeta + \eta|x|^{\nu}) \geq F(t,x) \geq \beta|x|^2 > 0, \\ \forall\, |x| > L + M_0, \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{3.11}$$

Note that

$$|(A(t)x,x)| \leq \|A\||x|^2, \quad |Bx| \leq \|B\||x|, \quad \forall\, x \in \mathbb{R}^N. \tag{3.12}$$

By (3.5), assumption (A) and (P), (3.12), (3.7), (3.11), (3.9) and (2.2), we have

$$\begin{aligned} & \frac{1}{2}\|u_n\|^2 \\ =\ & \varphi(u_n) - \frac{1}{2}\int_0^T e^{Q(t)}(Bu_n(t), \dot{u}_n(t))dt - \frac{1}{2}\int_0^T e^{Q(t)}(Au_n(t), u_n(t))dt \\ & + \int_0^T e^{Q(t)} F(t,u_n(t))dt + \frac{1}{2}\int_0^T e^{Q(t)}|u_n(t)|^2 dt \end{aligned}$$

$$
\begin{aligned}
&\leq D_3+\frac{1}{4}\int_0^T e^{Q(t)}|\dot{u}_n(t)|^2dt+\frac{\|B\|^2}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt\\
&\quad+\frac{\|A\|}{2}\int_0^T e^{Q(t)}|u_n(t)|^2dt+\frac{1}{2}\int_0^T e^{Q(t)}|u_n(t)|^2dt+D_4\int_0^T e^{Q(t)}b(t)dt\\
&\quad+\int_0^T e^{Q(t)}(\zeta+\eta|u_n(t)|^\nu)[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&\leq D_3+\frac{1}{4m}\int_0^T e^{Q(t)}(P(t)\dot{u}_n(t),\dot{u}_n(t))dt+\frac{2\|A\|+\|B\|^2}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt\\
&\quad+\frac{1}{2}\int_0^T e^{Q(t)}|u_n(t)|^2dt+D_4\int_0^T e^{Q(t)}b(t)dt\\
&\quad+\int_{\{t\in[0,T]:|u_n(t)|\leq L+M_0\}} e^{Q(t)}(\zeta+\eta|u_n(t)|^\nu)[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&\quad+\int_{\{t\in[0,T]:|u_n(t)|> L+M_0\}} e^{Q(t)}(\zeta+\eta|u_n(t)|^\nu)[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&\leq D_3+\frac{1}{4m}\|u_n\|^2+\frac{2\|A\|+\|B\|^2+2}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt+D_8\int_0^T e^{Q(t)}b(t)dt\\
&\quad+(\zeta+\eta\|u_n\|_\infty^\nu)\int_{\{t\in[0,T]:|u_n(t)|> L+M_0\}} e^{Q(t)}[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&= D_3+\frac{1}{4m}\|u_n\|^2+\frac{2\|A\|+\|B\|^2+2}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt+D_8\int_0^T e^{Q(t)}b(t)dt\\
&\quad+(\zeta+\eta\|u_n\|_\infty^\nu)\int_0^T e^{Q(t)}[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&\quad-(\zeta+\eta\|u_n\|_\infty^\nu)\int_{\{t\in[0,T]:|u_n(t)|\leq L+M_0\}} e^{Q(t)}[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&\leq D_3+\frac{1}{4m}\|u_n\|^2+\frac{2\|A\|+\|B\|^2+2}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt+D_8\int_0^T e^{Q(t)}b(t)dt\\
&\quad+(\zeta+\eta\|u_n\|_\infty^\nu)\int_0^T e^{Q(t)}[(\nabla F(t,u_n(t)),u_n(t))-2F(t,u_n(t))]dt\\
&\quad+D_9(\zeta+\eta\|u_n\|_\infty^\nu)\int_0^T e^{Q(t)}b(t)dt\\
&\leq D_3+\frac{1}{4m}\|u_n\|^2+\frac{2\|A\|+\|B\|^2+2}{4}\|u_n\|_\infty^\nu\int_0^T e^{Q(t)}|u_n(t)|^{2-\nu}dt\\
&\quad+3D_3(\zeta+\eta\|u_n\|_\infty^\nu)+D_9(\zeta+\eta\|u_n\|_\infty^\nu)\int_0^T e^{Q(t)}b(t)dt+D_8\int_0^T e^{Q(t)}b(t)dt\\
&\leq D_3+\frac{1}{4m}\|u_n\|^2+\frac{2\|A\|+\|B\|^2+2}{4}C_*^\nu\|u_n\|^\nu\int_0^T e^{Q(t)}|u_n(t)|^{2-\nu}dt\\
&\quad+3D_3(\zeta+\eta C_*^\nu\|u_n\|^\nu)+D_9(\zeta+\eta C_*^\nu\|u_n\|^\nu)\int_0^T e^{Q(t)}b(t)dt\\
&\quad+D_8\int_0^T e^{Q(t)}b(t)dt. \qquad (3.13)
\end{aligned}
$$

Since $\nu<2$ and $m>\frac{1}{2}$, (3.13) and the boundness of $\int_0^T e^{Q(t)}|u_n(t)|^{2-\nu}dt$ imply that $\|u_n\|$ is bounded. Going if necessary to a subsequence, assume that $u_n \rightharpoonup u$ in H_T^1. Then by Proposition 1.2 in [6], we have $\|u_n - u\|_\infty \to 0$ and so $\int_0^T e^{Q(t)}|u_n - u|^2 dt \to 0$ as $n \to \infty$. Similar to the argument of Theorem 3.1 in [1], it is easy to obtain that $\int_0^T e^{Q(t)}(P(t)(\dot{u}_n - \dot{u}), \dot{u}_n - \dot{u})dt \to 0$. Hence, $\|u_n - u\| \to 0$ as $n \to \infty$. Thus we have proved that φ satisfies (C)-condition.

Step 4. We claim that system (1.1) has an unbounded sequence of solutions $\{u_n\}$. In fact, (H4) implies that $\varphi(0) = 0$ and φ is even. Hence, combining step 1-step 3 with symmetric mountain pass theorem (Theorem 9.12 in [2]), we obtain a sequence $\{u_n\}$ such that $\varphi(u_n) \to \infty$. Then, obviously, $\{u_n\}$ is also unbounded.

Proof of Theorem 1.2. Similar to the proofs of Theorem 3.2 in [1], by combining the proofs of Theorem 1.1 and the abstract critical point theorem due to Bartsch and Ding (see [3]), the proof is easy to be completed and so we omit the details.

Proofs of Corollary 1.1 and Corollary 1.2. It is easy to see that (AR)-condition implies that (H2) and (H3). So by Theorem 1.1 and Theorem 1.2, the proofs are easy to be completed.

Proofs of Corollary 1.3 and Corollary 1.4. Similar to the argument of Remark 1.1 in [5], we know that assumption (A), (H2) and (H3)$'$ imply (H3). Then by Theorem 1.1 and Theorem 1.2, the proofs are easy to be completed.

4. The sub-quadratic case

In this section, we will investigate the subquadratic case. The following abstract critical point theorem will be used to prove Theorem 1.3.

Lemma 4.1.(see Lemma 2.4 in [8]) *Let E be an infinite dimensional Banach space and let $f \in C^1(E, \mathbb{R})$ be even, satisfy (PS), and $f(0) = 0$. If $E = E_1 \oplus E_2$, where E_1 is finite dimensional, and f satisfies*

(f_1) f is bounded from above on E_2,

(f_2) for each finite dimensional subspace $\tilde{E} \subset E$, there are positive constants $\rho = \rho(\tilde{E})$ and $\sigma = \sigma(\tilde{E})$ such that $f \geq 0$ on $B_\rho \cap \tilde{E}$ and $f|_{\partial B_\rho \cap \tilde{E}} \geq \sigma$ where $B_\rho = \{x \in E; \|x\| \leq \rho\}$, then f possesses infinitely many nontrivial critical points.

Remark 4.1. By Remark 3.1, Lemma 4.1 also holds when condition (PS) is replaced by

(C)-condition.

Proof of Theorem 1.3. We will consider the functional

$$\begin{aligned}\phi(u) &= -\varphi(u)\\ &= \int_0^T e^{Q(t)}\left[-\frac{1}{2}(P(t)\dot{u}(t),\dot{u}(t))-\frac{1}{2}(Bu(t),\dot{u}(t))-\frac{1}{2}(A(t)u(t),u(t))+F(t,u(t))\right]dt.\end{aligned}$$

Then it is easy to see that the critical point of ϕ is still the solution of system (1.1).

Step 1. We prove that $\phi(=-\varphi)$ satisfies (C)-condition on H_T^1. The proof is motivated by [20], [21] and [9] . For every $\{u_n\}\subset H_T^1$, assume that there exists a constant $C_1>0$ such that

$$|\phi(u_n)|\le C_1,\quad (1+\|u_n\|)\,\|\phi'(u_n)\|\le C_1,\quad \text{for all } n\in\mathbb{N}. \tag{4.1}$$

Then it follows from antisymmetry of B that

$$\begin{aligned}3C_1 &\ge 2\phi(u_n)-(\phi'(u_n),u_n)\\ &= -\frac{1}{2}\int_0^T e^{Q(t)}q(t)(Bu_n(t),u_n(t))dt+\int_0^T e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt\\ &= \int_0^T e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt. \end{aligned}\tag{4.2}$$

Next we prove that $\{u_n\}$ is bounded. Assume that $\|u_n\|\to\infty$ as $n\to\infty$. Let $z_n=\dfrac{u_n}{\|u_n\|}$. Then $\|z_n\|=1$ and so there exists a subsequence, still denoted by $\{z_n\}$, such that $z_n\rightharpoonup z$ on H_T^1. Then by Proposition 1.2 in [6], we get $\|z_n-z\|_\infty\to 0$. Hence, we have $\int_0^T|z_n(t)-z(t)|^2dt\to 0$ and $z_n(t)\to z(t)$ for a.e. $t\in[0,T]$. It follows from (F1) and assumption (A) that there exist constants $0<\varepsilon_1<\delta$ and $C_2>0$ such that

$$F(t,x)\le\varepsilon_1|x|^2+C_2b(t),\quad \text{for all } x\in\mathbb{R}^N \text{ and a.e. } t\in[0,T]. \tag{4.3}$$

Thus by condition (P) and (4.3), we have

$$\begin{aligned}\phi(u_n) &= \int_0^T e^{Q(t)}\left[-\frac{1}{2}(P(t)\dot{u}_n(t),\dot{u}_n(t))-\frac{1}{2}(Bu_n(t),\dot{u}_n(t))\right.\\ &\qquad\left.-\frac{1}{2}(A(t)u_n(t),u_n(t))+F(t,u_n(t))\right]dt\\ &\le \int_0^T e^{Q(t)}\left[-\frac{1}{2}(P(t)\dot{u}_n(t),\dot{u}_n(t))+\frac{\|B\|^2|u_n(t)|^2+|\dot{u}_n(t)|^2}{4}+\frac{\|A\||u_n(t)|^2}{2}\right]dt\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon_1\int_0^T e^{Q(t)}|u_n(t)|^2dt+C_2\int_0^T e^{Q(t)}b(t)dt\\
\le\ & \left(\frac{1}{4m}-\frac{1}{2}\right)\int_0^T e^{Q(t)}\left[(P(t)\dot{u}_n(t),\dot{u}_n(t))\right]dt+\frac{\|B\|^2+2\|A\|}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt\\
&+\varepsilon_1\int_0^T e^{Q(t)}|u_n(t)|^2dt+C_2\int_0^T e^{Q(t)}b(t)dt\\
=\ & \left(\frac{1}{4m}-\frac{1}{2}\right)\|u_n\|^2-\left(\frac{1}{4m}-\frac{1}{2}\right)\int_0^T e^{Q(t)}|u_n(t)|^2dt\\
&+\frac{\|B\|^2+2\|A\|}{4}\int_0^T e^{Q(t)}|u_n(t)|^2dt+\varepsilon_1\int_0^T e^{Q(t)}|u_n(t)|^2dt+C_2\int_0^T e^{Q(t)}b(t)dt\\
=\ & \left(\frac{1}{4m}-\frac{1}{2}\right)\|u_n\|^2+\left[\frac{\|B\|^2+2\|A\|}{4}-\left(\frac{1}{4m}-\frac{1}{2}\right)+\varepsilon_1\right]\int_0^T e^{Q(t)}|u_n(t)|^2dt\\
&+C_2\int_0^T e^{Q(t)}b(t)dt.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\frac{\phi(u_n)}{\|u_n\|^2}\ \le\ & \left(\frac{1}{4m}-\frac{1}{2}\right)+\left[\frac{\|B\|^2+2\|A\|}{4}-\left(\frac{1}{4m}-\frac{1}{2}\right)+\varepsilon_1\right]\int_0^T e^{Q(t)}\frac{|u_n(t)|^2}{\|u_n\|^2}dt\\
&+\frac{C_2\int_0^T e^{Q(t)}b(t)dt}{\|u_n\|^2}.
\end{aligned}
$$

Let $n\to\infty$. Then by (4.1), we get

$$
\frac{1}{2}-\frac{1}{4m}\ \le\ \left[\frac{\|B\|^2+2\|A\|}{4}+\left(\frac{1}{2}-\frac{1}{4m}\right)+\varepsilon_1\right]\int_0^T e^{Q(t)}|z_n(t)|^2dt. \tag{4.4}
$$

Then it follows from $m>\frac{1}{2}$, $\varepsilon_1>0$ and (4.4) that $\int_0^T e^{Q(t)}|z_n(t)|^2dt>0$ and so $z\neq 0$. Let $S=\{t\in[0,T]:\lim_{|x|\to\infty}e^{Q(t)}[2F(t,x)-(\nabla F(t,x),x)]=+\infty\}$ and $S_1=\{t\in S: z(t)\neq 0\}$. Then mes $S>0$ and

$$
\lim_{n\to\infty}|u_n(t)|=+\infty\ \text{ for } t\in S_1. \tag{4.5}
$$

Let $f_n(t)=e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]$. Then (4.5) implies that

$$
\lim_{n\to\infty}f_n(t)=+\infty\ \text{ for } t\in S_1. \tag{4.6}
$$

It follows from (4.6) and Lemma 1 in [19] that there exists a subset S_2 of S_1 with mes $S_2>0$ such that

$$
\lim_{n\to\infty}f_n(t)=+\infty\ \text{ uniformly for } t\in S_2. \tag{4.7}
$$

By (F3), we have

$$
\begin{aligned}
&\int_0^T e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt\\
=\ &\int_{S_2} e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt\\
&+\int_{[0,T]/S_2} e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt\\
\geq\ &\int_{S_2} e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt+\int_{[0,T]/S_2} h(t)dt.
\end{aligned}
$$

Let $n\to\infty$. Then by Fatou's lemma and (4.7), we have

$$
\int_0^T e^{Q(t)}[2F(t,u_n(t))-(\nabla F(t,u_n(t)),u_n(t))]dt\to+\infty
$$

which contradicts (4.2). Hence $\{u_n\}$ is bounded. Similar to the argument of Theorem 1.1, we can obtain that $\{u_n\}$ has a convergent subsequence.

Step 2. We prove that ϕ is bounded from above on H^+. In fact, it follows from (2.6) and (4.3) that for all $u\in H^+$,

$$
\begin{aligned}
\phi(u) &= \int_0^T e^{Q(t)}\left[-\frac{1}{2}(P(t)\dot u(t),\dot u(t))-\frac{1}{2}(Bu(t),\dot u(t))-\frac{1}{2}(A(t)u(t),u(t))+F(t,u(t))\right]dt\\
&= -\frac{1}{2}\langle (I-K)u,u\rangle+\int_0^T e^{Q(t)}F(t,u(t))dt\\
&\leq -\frac{\delta}{2}\|u\|^2+\varepsilon_1\int_0^T e^{Q(t)}|u(t)|^2dt+C_2\int_0^T e^{Q(t)}b(t)dt\\
&\leq (-\frac{\delta}{2}+\varepsilon_1)\|u\|^2+C_2\int_0^T e^{Q(t)}b(t)dt.
\end{aligned}
$$

Note $\varepsilon_1<\frac{\delta}{2}$. So ϕ is bounded from above on H^+.

Step 3. We prove that, for each finite dimensional subspace $\tilde E\subset H_T^1$, there are positive constants $\rho=\rho(\tilde E)$ and $\sigma=\sigma(\tilde E)$ such that $\phi\geq 0$ on $B_\rho\cap\tilde E$ and $\phi|_{\partial B_\rho\cap\tilde E}\geq\sigma$.

In fact, since $\tilde E$ is finite dimensional, all norms on $\tilde E$ are equivalent. Hence there exist $d_3=d_3(\tilde E)>0$ and $d_4=d_4(\tilde E)>0$ such that

$$
d_3\|u\|^2\leq\int_0^T e^{Q(t)}|u(t)|^2dt\leq d_4\|u\|^2.
$$

It follows from (F2) that there exist $C_3>\frac{\|I-K\|}{2d_1}$ and $M_2=M_2(\tilde E)>0$ such that

$$
F(t,x)\geq C_3|x|^2,\ \ \forall\ |x|\leq M_2,\ \text{a.e. } t\in[0,T]. \tag{4.8}
$$

Then by (4.8) and (2.2), for $u \in \tilde{E}$ with $\|u\| \leq \frac{M_2}{C^*}$, we have

$$\begin{aligned}\phi(u) &= -\frac{1}{2}\langle (I-K)u, u\rangle + \int_0^T e^{Q(t)}F(t,u(t))dt \\ &\geq -\frac{1}{2}\|I-K\|\|u\|^2 + C_3\int_0^T e^{Q(t)}|u(t)|^2dt \\ &\geq -\frac{1}{2}\|I-K\|\|u\|^2 + C_3 d_1\|u\|^2 \\ &= (C_3 d_1 - \|I-K\|/2)\|u\|^2.\end{aligned}$$

Let $\rho = \frac{M_2}{C^*}$ and $\sigma = (C_3 d_1 - \|I-K\|/2)\left(\frac{M_2}{C^*}\right)^2$. Then we complete the proof of this step.

Finally, (F4) implies that $\varphi(0) = 0$ and φ is even. Let $E_1 = H^- \oplus H^0$ and $E_2 = H^+$. Then $\dim E_1 < +\infty$. Hence, combining Step 1-Step 3 with Lemma 4.1 and Remark 4.1, we obtain that ϕ has infinitely many nontrivial critical points $\{u_n\}$. Thus we complete the proof.

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