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OSCILLATORY BEHAVIOR OF HIGHER-ORDER NEUTRAL TYPE DYNAMIC EQUATIONS

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Abstract. The oscillation behavior of solutions for higher-order delay dynamic equations of neutral type is investigated by making use of comparison with second-order dynamic equations. The method can be utilized to study other types of higher-order equations on time scales as well.

1. Introduction

In this paper we consider the higher-order neutral dynamic equations of the form

$$[x^\alpha(t) + p(t)x^\alpha(h(t))]^{\Delta^n} + f(t, x(\sigma(g(t)))) = 0 \tag{1.1}$$

and

$$[x^\alpha(t) + p(t)x^\alpha(h(t))]^{\Delta^{2n}} + f(t, x(\sigma(t))) = 0 \tag{1.2}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where $n \geq 2$ is an integer; $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator;

(i) α is the ratio of positive odd integers;
(ii) $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous;
(iii) $f(\cdot, x) : \mathbb{T} \to \mathbb{R}$ is rd-continuous for each fixed $x \in \mathbb{R}$ and $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for each fixed $t \in \mathbb{T}$ such that

$$\frac{f(t,u)}{u^\lambda} \geq q(t) \quad \text{for } u \neq 0 \tag{1.3}$$

with $q : \mathbb{T} \to (0, \infty)$ rd-continuous and λ a ratio of positive odd integers;
(iv) $g, h : \mathbb{T} \to \mathbb{T}$ are rd-continuous such that $g(t) \leq t$, $h(t) \leq t$, g is nondecreasing, h is increasing, and $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = \infty$.

The theory of time scales was introduced by Hilger [1] which unifies continuous and discrete analysis allows one to observe the discrepancies and similarities between discrete and continuous calculus. It also helps avoid proving results separately for both differential equations and difference equations. For a background material on time scale calculus, see [2].

The oscillation problem for dynamic equations on time scales has attracted a lot of attention immediately after the discovery of time scale calculus. Although there are several such works in the literature, the majority is restricted to second-order equations, see [3–20]. An important reason for this is probably due to lack of an inequality included in a Kiguradze's lemma connecting higher-order derivatives and differences to lower-order ones. In this work, we will show how another technique that is introduced by Grace et al. [21] can be used to derive new

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oscillation criteria for (1.1) and (1.2). For some works on higher-order dynamic equations we refer to [22–25]. Further results in both continuous and discrete cases can be found in [26].

By a solution of (1.1) we mean a function $x(t)$ nontrivial for t sufficiently large such that $x^{\alpha}(t) + p(t)x^{\alpha}(h(t))$ is n times differentiable, and (1.1) is fulfilled. Such a solution $x(t)$ of (1.1) is called nonoscillatory if there exists a $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t \geq t_0$; otherwise, it is said to be oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory. For (1.2) we just replace "n times" by "$2n$ times".

We will need the following three lemmas. The last lemma is a time scale version of the well-known Kiguradze's lemma. Indeed, the Lemma has another part in the continuous and discrete cases, see [26, Lemma 1.13.2, Lemma 2.2.2], which is not available on an arbitrary time scale. A special case, however, is given in [27] when $\sigma(t)$ is linear.

Lemma 1.1 ([6]). *Suppose $|x|^{\Delta}$ is of one sign on $[t_0, \infty)_{\mathbb{T}}$ and $\gamma > 0$, $\gamma \neq 1$. Then*

$$\frac{|x|^{\Delta}}{(|x|^{\sigma})^{\gamma}} \leq \frac{(|x|^{1-\gamma})^{\Delta}}{1-\gamma} \leq \frac{|x|^{\Delta}}{|x|^{\gamma}} \quad \text{on } [t_0, \infty)_{\mathbb{T}}. \tag{1.4}$$

Lemma 1.2 ([27]). *Let n be even and consider the equation*

$$x^{\Delta^n}(t) + f(t, x(\phi(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.5}$$

and the inequality

$$x^{\Delta^n}(t) + f(t, x(\phi(t))) \leq 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.6}$$

where $f : [t_0, \infty)_{\mathbb{T}} \times (0, \infty) \to (0, \infty)$ is a function with the property $f(\cdot, w(\cdot)) : [t_0, \infty)_{\mathbb{T}} \to (0, \infty)$ is rd-continuous for any rd-continuous function $w : [t_0, \infty)_{\mathbb{T}} \to (0, \infty)$ and $f(t, \cdot)$ is continuous and nondecreasing for each fixed $t \in [t_0, \infty)_{\mathbb{T}}$, and $\phi : \mathbb{T} \to \mathbb{T}$ is rd-continuous such that $\phi(t) \leq t$ and $\lim_{t\to\infty} \phi(t) = \infty$.

If inequality (1.6) has an eventually positive solution, then equation (1.5) also has an eventually positive solution.

One can easily see that (1.5) and (1.6) can be replaced, respectively, by

$$x^{\Delta^n}(t) + f(t, x(\sigma(\phi(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

and

$$x^{\Delta^n}(t) + f(t, x(\sigma(\phi(t))) \leq 0, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

The proof is similar, hence it is omitted.

Lemma 1.3 ([28]). *Let $x \in C^m_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. If $x^{\Delta^m}(t)$ is of constant sign on $[t_0, \infty)_{\mathbb{T}}$ and not identically zero on $[t_1, \infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$, then there exist a $t_x \geq t_0$ and an integer ℓ, $0 \leq \ell \leq m$ with $m + \ell$ even for $x^{\Delta^m}(t) \geq 0$, or $m + \ell$ odd for $x^{\Delta^m}(t) \leq 0$ such that*

$$\ell > 0 \text{ implies } x^{\Delta^k}(t) > 0 \text{ for } t \geq t_x,\ k \in \{0, 1, \ldots, \ell - 1\} \tag{1.7}$$

and

$$\ell \leq m - 1 \text{ implies } (-1)^{\ell+k} x^{\Delta^k}(t) > 0 \text{ for } t \geq t_x,\ k \in \{\ell, \ell + 1, \ldots, m - 1\}. \tag{1.8}$$

2. The main results

Taylor monomials (see [2, Sect. 1.6]) $h_n, g_n : \mathbb{T}^2 \to \mathbb{R}$, $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$, are defined recursively as

$$\begin{aligned} h_{n+1}(t,s) &= \int_s^t h_n(\tau,s)\Delta\tau, \quad n \in \mathbb{N}_0 \\ h_0(t,s) &= 1 \end{aligned}$$

and

$$\begin{aligned} g_{n+1}(t,s) &= \int_s^t g_n(\sigma(\tau),s)\Delta\tau, \quad n \in \mathbb{N}_0 \\ g_0(t,s) &= 1. \end{aligned}$$

It is easy to observe that $h_1(t,s) = g_1(t,s) = t - s$ and that

$$h_n(t,s) = (-1)^n g_n(s,t), \quad n \in \mathbb{N}_0.$$

2.1. **Oscillation of** (1.1)**.** In this section we give oscillation criteria for higher order neutral type equation (1.1) containing two deviating arguments $g(t)$ and $h(t)$ when $p(t)$ satisfies $-1 \le p(t) \le 0$ and $0 \le p(t) < 1$. The other cases seem to be open.

We will make use of the following functions, where $t_0, \beta \in \mathbb{T}$ with $\beta > t_0$:

$$\begin{aligned} Q_{n-1}(t,t_0,\beta) &:= \left(\frac{\beta - t_0}{\sigma(t) - t_0} h_{n-2}(g(t),\beta)\right)^{\lambda/\alpha} q(t) \\ Q^*_{n-1}(t,t_0,\beta) &:= \left(\frac{\beta - t_0}{\sigma(t) - t_0} h_{n-2}(g(t),\beta)(1 - p(\sigma(g(t))))\right)^{\lambda/\alpha} q(t) \\ Q_\ell(t,t_0,\beta) &:= \int_t^\infty g_{n-\ell-2}(\tau,t)\left(\frac{\beta - t_0}{\sigma(\tau) - t_0} h_{\ell-1}(g(\tau),\beta)\right)^{\lambda/\alpha} q(\tau)\Delta\tau \\ Q^*_\ell(t,t_0,\beta) &:= \int_t^\infty g_{n-\ell-2}(\tau,t)\left(\frac{\beta - t_0}{\sigma(\tau) - t_0} h_{\ell-1}(g(\tau),\beta)(1 - p(\sigma(g(\tau))))\right)^{\lambda/\alpha} q(\tau)\Delta\tau \end{aligned}$$

for $\ell \in \{1, 2, \ldots, n-3\}$. It is assumed that the improper integrals converge.

We begin with the following theorem.

Theorem 2.1. *Let $t_0, \beta \in \mathbb{T}$ with $\beta > t_0$, and assume that*

$$-1 < p \le p(t) \le 0 \tag{2.1}$$

and

$$\eta(t) := (h^{-1} \circ \sigma \circ g)(t) \le t.$$

Then equation (1.1) *is oscillatory if*

(i) *for n even,*

$$y^{\Delta\Delta}(t) + mQ_\ell(t,t_0,\beta) y^{\lambda/\alpha}(\sigma(t)) = 0 \tag{2.2}$$

or

$$y^{\Delta\Delta}(t) + mQ_\ell(t,t_0,\beta) y^{\lambda/\alpha}(\sigma(g(t))) = 0 \tag{2.3}$$

for some $0 < m < 1$ *and for all* $\ell \in \{1, 3, \ldots, n-1\}$ *is oscillatory and*

$$\limsup_{t\to\infty} \int_{\eta(t)}^{t} g_{n-1}^{\lambda/\alpha}(\eta(t), \eta(s)) q(s) \Delta s > \begin{cases} 0 & \textit{when } \lambda < \alpha \\ 1 & \textit{when } \lambda = \alpha; \end{cases} \tag{2.4}$$

(ii) *for* n *odd,* (2.2) *or* (2.3) *for some* $0 < m < 1$ *and for all* $\ell \in \{2, 4, \ldots, n-1\}$ *is oscillatory and*

$$\limsup_{t\to\infty} \int_{\sigma(g(t))}^{\sigma(t)} h_{n-1}^{\lambda/\alpha}(\sigma(g(s)), \sigma(g(t))) q(s) \Delta s > \begin{cases} 0 & \textit{when } \lambda < \alpha \\ 1 & \textit{when } \lambda = \alpha \end{cases} \tag{2.5}$$

and

$$\limsup_{t\to\infty} \int_{\eta(t)}^{t} ((\eta(s) - t_0) g_{n-2}(\eta(t), \eta(s)))^{\lambda/\alpha} q(s) \Delta s > \begin{cases} 0 & \textit{when } \lambda < \alpha \\ 1 & \textit{when } \lambda = \alpha. \end{cases} \tag{2.6}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). We may assume that $x(t)$ is eventually positive, since otherwise the substitution $y := -x$ transforms equation (1.1) into an equation of the same form subject to the assumptions of the theorem. Let $x(t) > 0$, $x(g(t)) > 0$ and $x(h(t)) > 0$ for $t \geq t_0 \in \mathbb{T}$.

Hereafter we set

$$z(t) := x^{\alpha}(t) + p(t) x^{\alpha}(h(t)), \quad t \geq t_0. \tag{2.7}$$

In view of (1.3), from equation (1.1), we have

$$z^{\Delta^n}(t) + q(t) x^{\lambda}(\sigma(g(t))) \leq 0, \quad t \geq t_0, \tag{2.8}$$

and so

$$z^{\Delta^n}(t) < 0, \quad t \geq t_0.$$

Thus, $z^{\Delta^i}(t)$, $0 \leq i \leq n-1$, are monotone. We consider the two possible cases: (i) $z(t) > 0$ for $t \geq t_1$ and (ii) $z(t) < 0$ for $t \geq t_1$ for some $t_1 \geq t_0$.

Suppose that (i) holds. From (2.1) and (2.7), we see that there exists a $t_2 \geq t_1$ such that

$$x(\sigma(g(t))) \geq z^{1/\alpha}(\sigma(g(t))), \quad t \geq t_2,$$

which together with (2.8) gives

$$z^{\Delta^n}(t) + q(t) z^{\lambda/\alpha}(\sigma(g(t))) \leq 0, \quad t \geq t_2. \tag{2.9}$$

By Lemma 1.3, there exist a $t_3 \geq t_2$ and an integer $\ell \in \{0, 1, \ldots, n-1\}$ with $n + \ell$ odd such that (1.7) and (1.8) hold for all $t \geq t_3$.

Let $\ell \in \{1, \ldots, n-1\}$. From

$$z^{\Delta^{\ell-1}}(t) = z^{\Delta^{\ell-1}}(t_3) + \int_{t_3}^{t} z^{\Delta^{\ell}}(s) \Delta s > (t - t_3) z^{\Delta^{\ell}}(t), \quad t \geq t_3,$$

we obtain

$$\left(\frac{z^{\Delta^{\ell-1}}(t)}{t - t_3} \right)^{\Delta} = \frac{z^{\Delta^{\ell}}(t)(t - t_3) - z^{\Delta^{\ell-1}}(t)}{(t - t_3)(\sigma(t) - t_3)} < 0, \quad t > t_3. \tag{2.10}$$

Therefore, the function $z^{\Delta^{\ell-1}}(t)/h_1(t,t_3)$ is decreasing on $(t_3,\infty)_{\mathbb{T}}$. By applying Taylor's formula, for some $t_4 > t_3$, we have

$$\begin{aligned} z(t) &= \sum_{k=0}^{\ell-1} z^{\Delta^k}(t_4)h_k(t,t_4) + \int_{t_4}^{t} h_{\ell-1}(t,\sigma(\tau))z^{\Delta^\ell}(\tau)\Delta\tau \\ &\geq z^{\Delta^{\ell-1}}(t_4)h_{\ell-1}(t,t_4), \quad t \geq t_4. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we obtain

$$z(t) \geq h_{\ell-1}(t,t_4)\frac{t_4-t_3}{t-t_3}z^{\Delta^{\ell-1}}(t), \quad t \geq t_4$$

and hence

$$z(\sigma(g(t))) \geq h_{\ell-1}(\sigma(g(t)),t_4)\frac{t_4-t_3}{\sigma(g(t))-t_3}z^{\Delta^{\ell-1}}(\sigma(g(t))), \quad t \geq t_5 \geq t_4,$$

where we assume $g(t) \geq t_4$ for $t \geq t_5$. It is easy to see that the above inequality leads to

$$z(\sigma(g(t))) \geq h_{\ell-1}(g(t),t_4)\frac{t_4-t_3}{\sigma(t)-t_3}z^{\Delta^{\ell-1}}(\sigma(g(t))) \quad t \geq t_5 \tag{2.12}$$

and

$$z(\sigma(g(t))) \geq h_{\ell-1}(g(t),t_4)\frac{t_4-t_3}{\sigma(t)-t_3}z^{\Delta^{\ell-1}}(\sigma(t)), \quad t \geq t_5. \tag{2.13}$$

Using (2.12) and (2.13) in (2.9), respectively, we obtain

$$-z^{\Delta^n}(t) \geq q(t)h_{\ell-1}^{\lambda/\alpha}(g(t),t_4)\left(\frac{t_4-t_3}{\sigma(t)-t_3}\right)^{\lambda/\alpha}(z^{\Delta^{\ell-1}}(\sigma(g(t))))^{\lambda/\alpha}, \quad t \geq t_5$$

and

$$-z^{\Delta^n}(t) \geq q(t)h_{\ell-1}^{\lambda/\alpha}(g(t),t_4)\left(\frac{t_4-t_3}{\sigma(t)-t_3}\right)^{\lambda/\alpha}(z^{\Delta^{\ell-1}}(\sigma(t)))^{\lambda/\alpha}, \quad t \geq t_5.$$

Since $\lim_{t\to\infty} h_k(t,t_2)/h_k(t,t_1) = 1$ (see [25, Lemma 3.1]), for some $t_6 \geq t_5$ sufficiently large, we have

$$-z^{\Delta^n}(t) \geq mq(t)h_{\ell-1}^{\lambda/\alpha}(g(t),\beta)\left(\frac{\beta-t_0}{\sigma(t)-t_0}\right)^{\lambda/\alpha}(z^{\Delta^{\ell-1}}(\sigma(g(t))))^{\lambda/\alpha}, \quad t \geq t_6 \tag{2.14}$$

and

$$-z^{\Delta^n}(t) \geq mq(t)h_{\ell-1}^{\lambda/\alpha}(g(t),\beta)\left(\frac{\beta-t_0}{\sigma(t)-t_0}\right)^{\lambda/\alpha}(z^{\Delta^{\ell-1}}(\sigma(t)))^{\lambda/\alpha}, \quad t \geq t_6, \tag{2.15}$$

where $0 < m < 1$ is a constant. Setting $\ell = n-1$ in (2.14) and (2.15) leads to

$$-z^{\Delta^n}(t) \geq mq(t)h_{n-2}^{\lambda/\alpha}(g(t),\beta)\left(\frac{\beta-t_0}{\sigma(t)-t_0}\right)^{\lambda/\alpha}(z^{\Delta^{n-2}}(\sigma(g(t))))^{\lambda/\alpha}, \quad t \geq t_6$$

and

$$-z^{\Delta^n}(t) \geq mq(t)h_{n-2}^{\lambda/\alpha}(g(t),\beta)\left(\frac{\beta-t_0}{\sigma(t)-t_0}\right)^{\lambda/\alpha}(z^{\Delta^{n-2}}(\sigma(t)))^{\lambda/\alpha}, \quad t \geq t_6.$$

Thus,

$$y^{\Delta\Delta}(t) + mQ_{n-1}(t, t_0, \beta)y^{\lambda/\alpha}(\sigma(g(t))) \le 0, \quad t \ge t_6$$

and

$$y^{\Delta\Delta}(t) + mQ_{n-1}(t, t_0, \beta)y^{\lambda/\alpha}(\sigma(t)) \le 0, \quad t \ge t_6,$$

respectively, where $y(t) := z^{\Delta^{n-2}}(t)$. Employing Lemma 1.2 and the remark after, we see that

$$y^{\Delta\Delta}(t) + mQ_{n-1}(t, t_0, \beta)y^{\lambda/\alpha}(\sigma(g(t))) = 0$$

and

$$y^{\Delta\Delta}(t) + mQ_{n-1}(t, t_0, \beta)y^{\lambda/\alpha}(\sigma(t)) = 0$$

have eventually positive solutions, which contradicts the hypothesis.

If $\ell \in \{1, 2, \ldots, n-3\}$, then by Taylor's formula, we write

$$\begin{aligned} -z^{\Delta^{\ell+1}}(t) &= \sum_{k=\ell+1}^{n-1} (-1)^{k-\ell} z^{\Delta^k}(s) g_{k-\ell-1}(s, t) + \int_t^s g_{n-\ell-2}(\sigma(\tau), t)(-z^{\Delta^n}(\tau))\Delta\tau \\ &\ge \int_t^s g_{n-\ell-2}(\sigma(\tau), t)(-z^{\Delta^n}(\tau))\Delta\tau, \quad s \ge t \ge t_3, \end{aligned}$$

and hence

$$-z^{\Delta^{\ell+1}}(t) \ge \int_t^\infty g_{n-\ell-2}(\sigma(\tau), t)(-z^{\Delta^n}(\tau))\Delta\tau, \quad t \ge t_3. \tag{2.16}$$

Using (2.14) and (2.15) in (2.16), respectively, and the fact $\sigma(t) \ge t$, we have for $t \ge t_6$,

$$\begin{aligned} -z^{\Delta^{\ell+1}}(t) &\ge m \int_t^\infty g_{n-\ell-2}(\tau, t) h_{\ell-1}^{\lambda/\alpha}(g(\tau), \beta) \left(\frac{\beta - t_0}{\sigma(\tau) - t_0}\right)^{\lambda/\alpha} (z^{\Delta^{\ell-1}}(\sigma(g(\tau))))^{\lambda/\alpha} q(\tau)\Delta\tau \\ &\ge m(z^{\Delta^{\ell-1}}(\sigma(g(t))))^{\lambda/\alpha} \int_t^\infty g_{n-\ell-2}(\tau, t) h_{\ell-1}^{\lambda/\alpha}(g(\tau), \beta) \left(\frac{\beta - t_0}{\sigma(\tau) - t_0}\right)^{\lambda/\alpha} q(\tau)\Delta\tau \end{aligned}$$

and

$$-z^{\Delta^{\ell+1}}(t) \ge m(z^{\Delta^{\ell-1}}(\sigma(t)))^{\lambda/\alpha} \int_t^\infty g_{n-\ell-2}(\tau, t) h_{\ell-1}^{\lambda/\alpha}(g(\tau), \beta) \left(\frac{\beta - t_0}{\sigma(\tau) - t_0}\right)^{\lambda/\alpha} q(\tau)\Delta\tau.$$

Let $w(t) := z^{\Delta^{\ell-1}}(t)$ for $t \ge t_6$. Then $w(t) > 0$ and satisfies

$$w^{\Delta\Delta}(t) + mQ_\ell(t, t_0, \beta)w^{\lambda/\alpha}(\sigma(g(t))) \le 0, \quad t \ge t_6,$$

and

$$w^{\Delta\Delta}(t) + mQ_\ell(t, t_0, \beta)w^{\lambda/\alpha}(\sigma(t)) \le 0, \quad t \ge t_6.$$

Employing Lemma 1.2 and the remark after, we see that

$$w^{\Delta\Delta}(t) + mQ_\ell(t, t_0, \beta)w^{\lambda/\alpha}(\sigma(g(t))) = 0$$

and

$$w^{\Delta\Delta}(t) + mQ_\ell(t, t_0, \beta)w^{\lambda/\alpha}(\sigma(t)) = 0$$

have eventually positive solutions, a contradiction to the hypothesis of the theorem.

Finally, we consider the last case $\ell = 0$, which is possible only if n is odd. By applying Taylor's formula and using (1.8) with $\ell = 0$, we can easily find

$$z(u) \geq h_{n-1}(u,v) z^{\Delta^{n-1}}(v), \quad v \geq u \geq t_3,$$

which implies that for some $t_7 \geq t_3$,

$$z(\sigma(g(s))) \geq h_{n-1}(\sigma(g(s)), \sigma(g(t))) z^{\Delta^{n-1}}(\sigma(g(t))), \quad t \geq s \geq t_7. \tag{2.17}$$

Integrating (2.9) from $\sigma(g(t)) \geq t_7$ to $\sigma(t)$, we get

$$z^{\Delta^{n-1}}(\sigma(g(t))) \geq \int_{\sigma(g(t))}^{\sigma(t)} q(s) z^{\lambda/\alpha}(\sigma(g(s)))\Delta s. \tag{2.18}$$

Using (2.17) in (2.18), we have

$$z^{\Delta^{n-1}}(\sigma(g(t))) \geq \left(z^{\Delta^{n-1}}(\sigma(g(t)))\right)^{\lambda/\alpha} \int_{\sigma(g(t))}^{\sigma(t)} h_{n-1}^{\lambda/\alpha}(\sigma(g(s)), \sigma(g(t))) q(s)\Delta s,$$

or

$$\left(z^{\Delta^{n-1}}(\sigma(g(t)))\right)^{1-\lambda/\alpha} \geq \int_{\sigma(g(t))}^{\sigma(t)} h_{n-1}^{\lambda/\alpha}(\sigma(g(s)), \sigma(g(t))) q(s)\Delta s.$$

Taking the limsup as $t \to \infty$, we obtain a contradiction to condition (2.5).

Now suppose that (ii) holds. Then

$$y(t) := -z(t) = -x^\alpha(t) - p(t)x^\alpha(h(t)) \leq x^\alpha(h(t)), \quad t \geq t_1,$$

and so

$$x(\sigma(g(t))) \geq y^{1/\alpha}((h^{-1} \circ \sigma \circ g)(t)) = y^{1/\alpha}(\eta(t)), \quad t \geq t_2 \geq t_1. \tag{2.19}$$

Clearly, inequality (2.8) implies that

$$y^{\Delta^n}(t) \geq q(t) x^\lambda(\sigma(g(t))), \quad t \geq t_2. \tag{2.20}$$

In view of (2.19) and (2.20), we have

$$y^{\Delta^n}(t) \geq q(t) y^{\lambda/\alpha}(\eta(t)), \quad t \geq t_2. \tag{2.21}$$

Now, as in the proof of [22, Theorem 2], we may show that $x(t)$ and hence $y(t)$ is bounded for $t \geq t_0$. To prove this, assume to the contrary that $x(t)$ is unbounded. Then there exists a sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} x(t_n) = \infty, \quad x(t_n) = \max\{x(t) : t_0 \leq t \leq t_n\}.$$

Since $\lim_{t\to\infty} h(t) = \infty$, for sufficiently large n, we have $h(t_n) > t_0$. From $h(t) \leq t$, we see that

$$\begin{aligned} x(h(t_n)) &\leq \max\{x(t) : t_0 \leq t \leq h(t_n)\} \\ &\leq \max\{x(t) : t_0 \leq t \leq t_n\} = x(t_n). \end{aligned}$$

Therefore, for n sufficiently large, we have

$$y(t_n) \leq -x^\alpha(t_n) - p x^\alpha(h(t_n)) \leq -(1+p)x^\alpha(t_n),$$

and hence

$$\lim_{n\to\infty} y(t_n) = -\infty,$$

which contradicts to $y(t) > 0$ for $t \geq t_1$.

Let n be even. By Lemma 1.3,

$$(-1)^k y^{\Delta^k}(t) > 0, \quad k = 0, 1, \dots n, \ t \geq t_3 \geq t_2.$$

By Taylor's formula, we then have

$$y(u) \geq g_{n-1}(v, u)(-y^{\Delta^{n-1}}(v)), \quad v \geq u \geq t_3,$$

and hence for some $t_4 \geq t_3$,

$$y(\eta(s)) \geq g_{n-1}(\eta(t), \eta(s))(-y^{\Delta^{n-1}}(\eta(t))), \quad t \geq s \geq t_4. \tag{2.22}$$

Now, integrating (2.21) from $\eta(t) \geq t_4$ to t, we obtain

$$-y^{\Delta^{n-1}}(\eta(t)) \geq \int_{\eta(t)}^{t} q(s) y^{\lambda/\alpha}(\eta(s)) \Delta s. \tag{2.23}$$

Using (2.22) in (2.23) gives us

$$-y^{\Delta^{n-1}}(\eta(t)) \geq \left(-y^{\Delta^{n-1}}(\eta(t))\right)^{\lambda/\alpha} \int_{\eta(t)}^{t} g_{n-1}^{\lambda/\alpha}(\eta(t), \eta(s)) q(s) \Delta s,$$

i.e.,

$$\left(-y^{\Delta^{n-1}}(\eta(t))\right)^{1-\lambda/\alpha} \geq \int_{\eta(t)}^{t} g_{n-1}^{\lambda/\alpha}(\eta(t), \eta(s)) q(s) \Delta s.$$

Taking the limsup as $t \to \infty$ in the last inequality, we obtain a contradiction to (2.4).

Now suppose that n is odd. Then we see by Lemma 1.3 that for some $t_5 \geq t_2$,

$$\begin{aligned} &y(t) > 0, \ y^{\Delta}(t) > 0, \quad t \geq t_5, \\ &(-1)^{k-1} y^{\Delta^k}(t) > 0, \quad k = 2, \dots, n, \ t \geq t_5. \end{aligned}$$

We write

$$y(t) = y(t_5) + \int_{t_5}^{t} y^{\Delta}(s) \Delta s \geq (t - t_5) y^{\Delta}(t), \quad t \geq t_5. \tag{2.24}$$

Using (2.24) in (2.21), we have

$$y^{\Delta^n}(t) \geq q(t)(\eta(t) - t_5)^{\lambda/\alpha} (y^{\Delta}(\eta(t)))^{\lambda/\alpha}, \quad t \geq t_6 \geq t_5,$$

and hence

$$w^{\Delta^{n-1}}(t) \geq q(t)(\eta(t) - t_5)^{\lambda/\alpha} w^{\lambda/\alpha}(\eta(t)), \quad t \geq t_6,$$

where $w(t) := y^{\Delta}(t)$. Moreover,

$$(-1)^k w^{\Delta^k}(t) > 0, \quad k = 0, 1, \dots n-1, \ t \geq t_5.$$

As in the above proof for the case n is even, we have

$$w(u) \geq g_{n-2}(v, u)(-w^{\Delta^{n-2}}(v)), \quad v \geq u \geq t_5.$$

The rest of the proof is similar to the case (ii) when n is even. This completes the proof. $\square$

Theorem 2.2. *Let $t_0, \beta \in \mathbb{T}$ with $\beta > t_0$. Assume that*

$$0 \le p(t) < 1 \quad \textit{when } n \textit{ is even}$$

and there exists $p < 1$ such that

$$0 \le p(t) \le p \quad \textit{when } n \textit{ is odd.}$$

(i) *If n is even,*

$$y^{\Delta\Delta}(t) + mQ^*_\ell(t, t_0, \beta) y^{\lambda/\alpha}(\sigma(t)) = 0 \tag{2.25}$$

or

$$y^{\Delta\Delta}(t) + mQ^*_\ell(t, t_0, \beta) y^{\lambda/\alpha}(\sigma(g(t))) = 0 \tag{2.26}$$

for some $0 < m < 1$ and for all $\ell \in \{1, 3, \dots, n-1\}$ is oscillatory, then equation (1.1) is oscillatory.

(ii) *If n is odd, (2.25) or (2.26) for some $0 < m < 1$ and for all $\ell \in \{2, 4, \dots, n-1\}$ is oscillatory and*

$$\int_{t_0}^{\infty} g_{n-1}(\sigma(s), t_0) q(s) \Delta s = \infty, \tag{2.27}$$

then every solution $x(t)$ of equation (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(g(t)) > 0$ and $x(h(t)) > 0$ for $t \ge t_0$. Define the function $z(t)$ by (2.7) and obtain the inequality (2.8). By Lemma 1.3, there exist a $t_1 \ge t_0$ and an integer $\ell \in \{0, 1, \dots, n-1\}$ with $n + \ell$ odd such that (1.7) and (1.8) hold.

We first consider the case $\ell \in \{1, \dots, n-1\}$. Clearly $z^\Delta(t) > 0$ for $t \ge t_1$ and

$$\begin{aligned} x^\alpha(t) &= z(t) - p(t) x^\alpha(h(t)) \\ &= z(t) - p(t)(z(h(t)) - p(h(t)) x^\alpha((h \circ h)(t))) \\ &\ge z(t) - p(t) z(h(t)) \ge (1 - p(t)) z(t), \quad t \ge t_2 \ge t_1. \end{aligned}$$

Thus,

$$x(t) \ge (1 - p(t))^{1/\alpha} z^{1/\alpha}(t), \quad t \ge t_2. \tag{2.28}$$

Using (2.28) in (2.8), we get

$$z^{\Delta^n}(t) + q(t)(1 - p(\sigma(g(t))))^{\lambda/\alpha} z^{\lambda/\alpha}(\sigma(g(t))) \le 0, \quad t \ge t_3 \ge t_2.$$

Proceeding as in case (i) of Theorem 2.1, we obtain a contradiction.

Let $\ell = 0$. As in the proof of [22, Theorem 1], we can show that $\lim_{t\to\infty} x(t) = 0$. Since $z^{1/\alpha}(t) \ge x(t) > 0$ for $t \ge t_0$, it suffices to show that

$$\lim_{t\to\infty} z(t) = 0.$$

From $z(t) > 0, z^\Delta(t) < 0$ for $t \ge t_1$, we see that

$$\lim_{t\to\infty} z(t) := L \ge 0, \quad L < \infty.$$

Assume on the contrary that $L > 0$. Choose $0 < \varepsilon < L(1-p)/p$. Then $L < z(t) < L + \varepsilon$ for $t \ge t_4 \ge t_1$, and

$$x^\alpha(t) \ge z(t) - p z(h(t)) > L - p(L + \varepsilon) > K z(t), \quad t \ge t_5 \ge t_4,$$

where $K := (L - p(L+\varepsilon))/L + \varepsilon > 0$. Thus, we have

$$x(t) > K^{1/\alpha} z^{1/\alpha}(t), \quad t \geq t_5. \tag{2.29}$$

Using (2.29) in inequality (2.8) results in

$$z^{\Delta^n}(t) + K^{\lambda/\alpha} q(t) z^{\lambda/\alpha}(\sigma(g(t))) \leq 0, \quad t \geq t_6 \geq t_5. \tag{2.30}$$

By applying Taylor's formula, it is easy to see that

$$z(t_6) \geq \int_{t_6}^{t} g_{n-1}(\sigma(s), t_6)(-z^{\Delta^n}(s))\Delta s, \quad t \geq t_6. \tag{2.31}$$

From (2.30), (2.31), and $z(t) > L$ for $t \geq t_4$, we obtain

$$z(t_6) \geq (KL)^{\lambda/\alpha} \int_{t_6}^{t} g_{n-1}(\sigma(s), t_6) q(s)\Delta s, \quad t \geq t_6,$$

which however contradicts (2.27). The proof is complete. □

We note that the oscillatory behavior of solutions of second-order dynamic equations of the form (2.2) and (2.25) has been studied extensively in the literature. We refer the reader in particular to [12–18] and the references cited therein.

2.2. **Oscillation of** (1.2)**.** Here we consider even order neutral type equations of the form (1.2) containing a single deviating argument $h(t)$ and the term $p(t)$ with $0 \leq p(t) < 1$. The even order implies that the number l arising from Lemma 1.3 in a way as in the above proofs is positive. It seems interesting to find similar oscillation criteria for odd order equations. The possibility $l = 0$ is crucial there.

For $t \in \mathbb{T}$ and $\ell \in \{1, 3, \ldots, 2n-1\}$, we define

$$\hat{Q}_\ell(t) \;:=\; \int_t^\infty \int_{s_{2n-\ell-1}}^\infty \cdots \int_{s_1}^\infty (1 - p(\sigma(s)))^{\lambda/\alpha} q(s)\Delta s \Delta s_1 \ldots \Delta s_{2n-\ell-1},$$

where it is assumed that the integral is convergent.

The first result is as follows.

Theorem 2.3. *Let $\alpha < \lambda$ and $t_0 \in \mathbb{T}$, and assume that*

$$0 \leq p(t) < 1. \tag{2.32}$$

If for every $\ell \in \{1, 3, \ldots, 2n-1\}$,

$$\int_{t_0}^\infty h_{\ell-1}(s, t_0)\hat{Q}_\ell(s)\Delta s \;=\; \infty, \tag{2.33}$$

then equation (1.2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.2), say, $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_0$. Define $z(t)$ by (2.7) and obtain from (1.2),

$$z^{\Delta^{2n}}(t) + q(t) x^\lambda(\sigma(t)) \leq 0, \quad t \geq t_0. \tag{2.34}$$

Since $z(t) > 0$ and $z^{\Delta^{2n}}(t) < 0$ for $t \geq t_0$, by Lemma 1.3, there exist a $t_1 \geq t_0$ and an integer $\ell \in \{1, 3, \ldots, 2n-1\}$ such that (1.7) and (1.8) hold for all $t \geq t_1$. Using (2.28) in (2.34), we get

$$z^{\Delta^{2n}}(t) + q(t)(1 - p(\sigma(t)))^{\lambda/\alpha} z^{\lambda/\alpha}(\sigma(t)) \leq 0, \quad t \geq t_2 \geq t_1. \tag{2.35}$$

From (1.8), $z^{\Delta^\ell}(t) > 0$ and decreasing on $[t_1, \infty)_{\mathbb{T}}$. Now,

$$z^{\Delta^{\ell-1}}(s) - z^{\Delta^{\ell-1}}(t_1) \;=\; \int_{t_1}^{s} z^{\Delta^\ell}(\tau)\Delta\tau \;\geq\; h_1(s, t_1) z^{\Delta^\ell}(s)$$

gives

$$z^{\Delta^{\ell-1}}(s) \;\geq\; h_1(s, t_1) z^{\Delta^\ell}(s), \quad s \geq t_1. \tag{2.36}$$

Integrating inequality (2.36) $(\ell - 2)$-times, $\ell > 1$, from t_1 to $s \geq t_1$, we have

$$z^{\Delta}(s) \;\geq\; h_{\ell-1}(s, t_1) z^{\Delta^\ell}(s), \quad s \geq t_1, \;\; \ell \geq 1. \tag{2.37}$$

Next, we integrate inequality (2.35) from $s_1 \geq t_1$ to $v \geq s_1$ and let $v \to \infty$ to get

$$z^{\Delta^{2n-1}}(s_1) \;\geq\; \left(\int_{s_1}^{\infty} (1 - p(\sigma(\tau)))^{\lambda/\alpha} q(\tau)\Delta\tau \right) z^{\lambda/\alpha}(\sigma(s_1)).$$

Integrating from $s_2 \geq t_1$ to $v \geq s_2$ and then letting $v \to \infty$ and using (1.8) leads to

$$-z^{\Delta^{2n-2}}(s_2) \;\geq\; \left(\int_{s_2}^{\infty} \int_{s_1}^{\infty} (1 - p(\sigma(\tau)))^{\lambda/\alpha} q(\tau)\Delta\tau\Delta s_1 \right) z^{\lambda/\alpha}(\sigma(s_2)).$$

Continuing in this manner, one can easily find

$$z^{\Delta^\ell}(s) \;\geq\; \left(\int_{s}^{\infty} \int_{s_{2n-\ell-1}}^{\infty} \cdots \int_{s_1}^{\infty} (1 - p(\sigma(\tau)))^{\lambda/\alpha} q(\tau)\Delta\tau\Delta s_1 \ldots \Delta s_{2n-\ell-1} \right) z^{\lambda/\alpha}(\sigma(s)),$$

which we may write as

$$z^{\Delta^\ell}(s) \;\geq\; \hat{Q}_\ell(s) z^{\lambda/\alpha}(\sigma(s)), \quad s \geq t_1. \tag{2.38}$$

From (2.37) and (2.38), we find

$$z^{-\lambda/\alpha}(\sigma(s)) z^{\Delta}(s) \;\geq\; h_{\ell-1}(s, t_1)\hat{Q}_\ell(s), \quad s \geq t_1,$$

and hence

$$\int_{t_1}^{t} z^{-\lambda/\alpha}(\sigma(s)) z^{\Delta}(s)\Delta s \;\geq\; \int_{t_1}^{t} h_{\ell-1}(s, t_1)\hat{Q}_\ell(s)\Delta s.$$

By employing the first inequality in Lemma 1.1,

$$\frac{\alpha}{\alpha - \lambda} \int_{t_1}^{t} (z^{1-\frac{\lambda}{\alpha}}(s))^{\Delta}\Delta s \;\geq\; \int_{t_1}^{t} h_{\ell-1}(s, t_1)\hat{Q}_\ell(s)\Delta s.$$

Thus, we obtain

$$\int_{t_1}^{\infty} h_{\ell-1}(s, t_1)\hat{Q}_\ell(s)\Delta s \;\leq\; \frac{\alpha}{\lambda - \alpha} z^{1-\frac{\lambda}{\alpha}}(t_1),$$

which contradicts (2.33). The proof is complete. □

The calculation of the repeated integrals in (2.33) is in general not easy for an arbitrary time scale. Therefore, we give the following alternative theorems.

Theorem 2.4. *Let $\alpha < \lambda$ and $t_0 \in \mathbb{T}$, and assume that* (2.32) *holds. If*

$$\int_{t_0}^{\infty} h_{\ell-1}(s,t_0)\left(\int_s^{\infty} g_{2n-\ell-1}(\sigma(\tau),s)(1-p(\sigma(\tau)))^{\lambda/\alpha}q(\tau)\Delta\tau\right)\Delta s = \infty,$$

for every $\ell \in \{1,3,\ldots,2n-1\}$, then equation (1.2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.2), say, $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_0$. Define $z(t)$ by (2.7). By Taylor's formula, we see that

$$z^{\Delta^\ell}(s) \geq -\int_s^{\infty} g_{2n-\ell-1}(\sigma(\tau),s)z^{\Delta^{2n}}(\tau)\Delta\tau, \quad s \geq t_1. \tag{2.39}$$

Using inequality (2.35) in (2.39), we get

$$\begin{aligned} z^{\Delta^\ell}(s) &\geq \int_s^{\infty} g_{2n-\ell-1}(\sigma(\tau),s)(1-p(\sigma(\tau)))^{\lambda/\alpha}q(\tau)z^{\lambda/\alpha}(\sigma(\tau))\Delta\tau \\ &\geq \left(\int_s^{\infty} g_{2n-\ell-1}(\sigma(\tau),s)(1-p(\sigma(\tau)))^{\lambda/\alpha}q(\tau)\Delta\tau\right)z^{\lambda/\alpha}(\sigma(s)), \quad s \geq t_2 \geq t_1. \end{aligned} \tag{2.40}$$

Combining (2.37) with (2.40), we find

$$z^{\Delta}(s) \;\geq\; h_{\ell-1}(s,t_1)\left(\int_s^{\infty} g_{2n-\ell-1}(\sigma(\tau),s)(1-p(\sigma(\tau)))^{\lambda/\alpha}q(\tau)\Delta\tau\right)z^{\lambda/\alpha}(\sigma(s)), \quad s \geq t_2.$$

Dividing both sides by $z^{\lambda/\alpha}(\sigma(s))$ and integrating from t_2 to $t \geq t_2$, we have

$$\int_{t_2}^{t} z^{-\lambda/\alpha}(\sigma(s))z^{\Delta}(s)\Delta s \;\geq\; \int_{t_2}^{t} h_{\ell-1}(s,t_1)\left(\int_s^{\infty} g_{2n-\ell-1}(\sigma(\tau),s)(1-p(\sigma(\tau)))^{\lambda/\alpha}q(\tau)\Delta\tau\right)\Delta s.$$

The rest of the proof is similar to that of Theorem 2.3 and hence it is omitted. This completes the proof. ☐

Theorem 2.5. *Let $\alpha > \lambda$ and $t_0 \in \mathbb{T}$, and assume that* (2.32) *holds. If for every $\ell \in \{1,3,\ldots,2n-1\}$,*

$$\int_{t_0}^{\infty} q(s)(1-p(\sigma(s)))^{\lambda/\alpha}\left(\int_{t_0}^{s} h_{\ell-1}(s,\sigma(u))g_{2n-\ell-1}(s,u)\Delta u\right)^{\lambda/\alpha}\Delta s \;=\; \infty, \tag{2.41}$$

then equation (1.2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.2), say, $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_0$. Define the function $z(t)$ by (2.7). As in the proof of Theorem 2.3, we see that (1.7) and (1.8) hold for $t \geq t_1 \geq t_0$. It is not difficult to see that

$$z(t) \;\geq\; \int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))z^{\Delta^\ell}(u)\Delta u$$

and

$$z^{\Delta^\ell}(u) \;\geq\; g_{2n-\ell-1}(t,u)z^{\Delta^{2n-1}}(t), \quad t \geq u \geq t_1.$$

Therefore,

$$z(t) \;\geq\; \left(\int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))g_{2n-\ell-1}(t,u)\Delta u\right) z^{\Delta^{2n-1}}(t), \quad t \geq t_1.$$

Set $w(t) := z^{\Delta^{2n-1}}(t)$. Using this inequality in (2.35), we get for $t \geq t_2 \geq t_1$,

$$\begin{aligned} -w^{\Delta}(t) \;&\geq\; q(t)(1-p(\sigma(t)))^{\lambda/\alpha} z^{\lambda/\alpha}(\sigma(t)) \\ &\geq\; q(t)(1-p(\sigma(t)))^{\lambda/\alpha} z^{\lambda/\alpha}(t) \\ &\geq\; q(t)(1-p(\sigma(t)))^{\lambda/\alpha} \left(\int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))g_{2n-\ell-1}(t,u)\Delta u\right)^{\lambda/\alpha} w^{\lambda/\alpha}(t). \end{aligned}$$

We integrate the last inequality from t_2 to $t \geq t_2$ and apply Lemma 1.1 (the second inequality in (1.4)), to get

$$\frac{\alpha}{\alpha-\lambda}\, w^{1-\frac{\lambda}{\alpha}}(t_2) \;\geq\; \int_{t_2}^{\infty} q(s)(1-p(\sigma(s)))^{\lambda/\alpha} \left(\int_{t_1}^{s} h_{\ell-1}(s,\sigma(u))g_{2n-\ell-1}(s,u)\Delta u\right)^{\lambda/\alpha} \Delta s,$$

a contradiction with condition (2.41). □

To illustrate the last theorem let us consider as a special case the fourth-order equation

$$[x^{\alpha}(t)+p(t)x^{\alpha}(h(t))]^{\Delta^4} + q(t)x^{\lambda}(\sigma(t)) = 0. \tag{2.42}$$

Clearly, conditions in (2.41) read as

$$\int_{t_0}^{\infty} q(s)(1-p(\sigma(s)))^{\lambda/\alpha} \left(\int_{t_0}^{s} g_2(s,u)\Delta u\right)^{\lambda/\alpha} \Delta s \;=\; \infty \tag{2.43}$$

and

$$\int_{t_0}^{\infty} q(s)(1-p(\sigma(s)))^{\lambda/\alpha} \left(\int_{t_0}^{s} h_2(s,\sigma(u))\Delta u\right)^{\lambda/\alpha} \Delta s \;=\; \infty. \tag{2.44}$$

Note that $h_2(t,s) = g_2(s,t)$ but there is no closed form expression for them on an arbitrary time scale. However, if $\sigma(t) = at+b$ with $a \geq 1$ and $b \geq 0$ (see [27]), then

$$h_2(t,s) = \frac{(t-s)(t-\sigma(s))}{1+a},$$

which covers the time scales $\mathbb{R}$, $\mathbb{Z}$, $q^{\mathbb{N}}$, and etc. In this case, the conditions (2.43) and (2.44) can be further simplified for an easier computation.

3. Concluding remarks

We have obtained oscillation criteria for higher-order neutral type equations (1.1) on arbitrary time scales via comparison with second-order dynamic equations with and without delay arguments. Since there are several oscillation criteria for such second-order dynamic equations, one can provide several corresponding results for the higher-order case. This approach has been used quite successfully for differential and difference equations, yet it is at its early stages for dynamic equations on arbitrary time scales. As it is mentioned in Introduction, this is because we do not have the full time scale version of the Kiguradze's lemma, namely the inequality between higher-order derivatives and lower-order ones.

There are several techniques often used in oscillation theory of differential and difference equations separately, but not available for a general time scale. The more tools are made available for time scale calculus the better oscillation criteria can be obtained for higher-order equations. In the present work, we have demonstrated only one method to study such equations. Further work is underway and will be reported in due courses.

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