Bifurcation analysis for a delayed food chain system with two functional responses

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Abstract

A delayed three-species food chain system with two types of functional responses, Holling type and Beddington–DeAngelis type, is investigated. By analyzing the distribution of the roots of the associated characteristic equation, we get the sufficient conditions for the stability of the positive equilibrium and the existence of Hopf bifurcation. In particular, the properties of Hopf bifurcation such as direction and stability are determined by using the normal form theory and center manifold theorem. Finally, numerical simulations are given to substantiate the theoretical results.

Keywords: bifurcation, delay, food chain system, stability, periodic solution.
Subject classification codes: 34C23.

1 Introduction

In population dynamics, two-species predator-prey systems have been studied by many researchers [1, 2, 3, 4, 5, 6]. However, there is often the interaction among multiple species in nature, whose relationships are more complex than those in two species. Therefore, it is more

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realistic to consider the multiple-species predator-prey systems. Recently, Do et al. [7] proposed and studied the following three-species food chain system with Holling type II functional response and Beddington–DeAngelis functional response:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(a - bx(t)) - \frac{c_1 x(t)y(t)}{\alpha_1 + x(t)}, \\ \frac{dy(t)}{dt} = -d_1 y(t) + \frac{c_2 x(t)y(t)}{\alpha_1 + x(t)} - \frac{c_3 y(t)z(t)}{\alpha_2 + y(t) + \beta z(t)}, \\ \frac{dz(t)}{dt} = -d_2 z(t) + \frac{c_4 y(t)z(t)}{\alpha_2 + y(t) + \beta z(t)}, \end{cases} \tag{1}$$

where $x(t)$, $y(t)$ and $z(t)$ denote the population densities of the prey, the mid-predator and the top predator, respectively. All the parameters in system (1) are positive constants. a is the birth rate of the prey. b is the intraspecific competition rate of the prey. c_1 and c_2 are the interspecific interaction coefficients between the prey and the mid-predator. c_3 and c_4 are the interspecific interaction coefficients between the mid-predator and the top predator. d_1 and d_2 are the death rates of the mid-predator and the top predator, respectively. α_1 and α_2 are the half-saturation constants and β scales the impact of the predator interference. In [7], Do et al. proved that system (1) is dissipative and the conditions for the stability and the persistence of system (1) were obtained.

It is well-known that time delays have important effect on predator-prey systems. They could cause a stable equilibrium to become unstable and cause the population to fluctuate. And predator-prey systems with time delay have been investigated widely by many researchers [8, 9, 10, 11, 12, 13]. In [8], Xu investigated the stability and persistence of a predator-prey system with time delay and stage structure for the prey. In [12]. Meng et al. investigated the stability and Hopf bifurcation of a delayed food web consisting of three species. Motivated by the work above, and considering that the consumption of prey by the predator throughout its past history governs the present birth rate of the predator, we incorporate time delay due to gestation of the mid-predator and the top predator into system (1) and get the following delayed predator-prey system:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(a - bx(t)) - \frac{c_1 x(t)y(t)}{\alpha_1 + x(t)}, \\ \frac{dy(t)}{dt} = -d_1 y(t) + \frac{c_2 x(t-\tau)y(t-\tau)}{\alpha_1 + x(t-\tau)} - \frac{c_3 y(t)z(t)}{\alpha_2 + y(t) + \beta z(t)}, \\ \frac{dz(t)}{dt} = -d_2 z(t) + \frac{c_4 y(t-\tau)z(t-\tau)}{\alpha_2 + y(t-\tau) + \beta z(t-\tau)}, \end{cases} \tag{2}$$

where the constant $\tau \geq 0$ is the time delay due to the gestation of the mid-predator and the top predator. In this paper, we shall investigate the effect of the time delay on the dynamics of system (2).

The structure of this paper is arranged as follows. In Section 2, we will consider the local stability of the positive equilibrium and the existence of Hopf bifurcation at the positive equilibrium. In Section 3, we give the formula determining the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions. Finally, we give some simulations to support our theoretical predictions.

2 Local stability and existence of Hopf bifurcation

According to the literature [7], we know that if the following condition holds,

$$(H_1)\ q^2 - 4pr \geq 0, \quad 0 < x_* < \frac{a}{b}, \quad 0 < \frac{\alpha_2 d_2}{c_4 - d_2} < y_*,$$

system (2) has a positive equilibrium $E_*(x_*, y_*, z_*)$, where

$$y_* = \frac{(a - bx_*)(\alpha_1 + x_*)}{c_1}, \quad z_* = \frac{(c_4 - d_2)y_* - d_2\alpha_2}{d_2\beta},$$

and x_* is a positive solution of the quadratic equation

$$px^2 + qx + r = 0,$$

with

$$\begin{aligned}
p =& - bc_2c_4\beta + bc_4d_1\beta + bc_3c_4 - bc_3d_2,\\
q =& ac_2c_4\beta + bc_4d_1\alpha_1\beta - ac_4d_1\beta + ac_3d_2 + bc_3c_4\alpha_1 - ac_3c_4 - bc_3d_2\alpha_1,\\
r =& - ac_4d_1\alpha_1\beta + c_1c_3d_2\alpha_2 - ac_3c_4\alpha_1 + ac_3d_2\alpha_1.
\end{aligned}$$

Let $u_1(t) = x(t) - x_*$, $u_2(t) = y(t) - y_*$, $u_3(t) = z(t) - z_*$. We can rewrite system (2) as the following equivalent system:

$$\begin{cases}
\dot{u}_1(t) = a_{11}u_1(t) + a_{12}u_2(t) + \sum\limits_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} u_1^i(t)u_2^j(t),\\
\dot{u}_2(t) = a_{22}u_2(t) + a_{23}u_3(t) + b_{21}u_1(t-\tau) + b_{22}u_2(t-\tau)\\
\qquad + \sum\limits_{i+j+k+l\geq 2} \frac{1}{i!j!k!l!} f_{ijkl}^{(2)} u_1^i(t-\tau)u_2^j(t-\tau)u_2^k(t)u_3^l(t),\\
\dot{u}_3(t) = a_{33}u_3(t) + b_{32}u_2(t-\tau) + b_{33}u_3(t-\tau)\\
\qquad + \sum\limits_{i+j+k\geq 2} \frac{1}{i!j!k!} f_{ijk}^{(3)} u_2^i(t-\tau)u_3^j(t-\tau)u_3^k(t),
\end{cases} \tag{3}$$

where

$$a_{11} = -bx_* + \frac{c_1x_*y_*}{(\alpha_1 + x_*)^2}, \quad a_{12} = -\frac{c_1x_*}{\alpha_1 + x_*},$$

$$a_{22} = -d_1 - \frac{c_3z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^2},$$

$$a_{23} = -\frac{c_3y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^2}, \quad a_{33} = -d_2,$$

$$b_{21} = \frac{c_2\alpha_1y_*}{(\alpha_1 + x_*)^2}, \quad b_{22} = \frac{c_2x_*}{\alpha_1 + x_*},$$

$$b_{32} = \frac{c_4z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^2}, \quad b_{33} = \frac{c_4y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^2},$$

$$f_{ij}^{(1)} = \frac{\partial^{i+j} f^{(1)}(x_*, y_*, z_*)}{\partial u_1^i(t)\partial u_2^j(t)},$$

$$f_{ijkl}^{(2)} = \frac{\partial^{i+j+k+l} f^{(2)}(x_*, y_*, z_*)}{\partial u_1^i(t-\tau)\partial u_2^j(t-\tau)\partial u_2^k(t)\partial u_3^l(t)},$$

$$f_{ijk}^{(3)} = \frac{\partial^{i+j+k} f^{(3)}(x_*, y_*, z_*)}{\partial u_2^i(t-\tau)\partial u_3^j(t-\tau)\partial u_3^k(t)},$$

$$f^{(1)} = u_1(t)(a - bu_1(t)) - \frac{c_1u_1(t)u_2(t)}{\alpha_1 + u_1(t)},$$

$$f^{(2)} = -d_1 u_2(t) + \frac{c_2 u_1(t-\tau) u_2(t-\tau)}{\alpha_1 + u_1(t-\tau)} - \frac{c_3 u_2(t) u_3(t)}{\alpha_2 + u_2(t) + \beta u_3(t)},$$

$$f^{(3)} = -d_2 u_3(t) + \frac{c_4 u_2(t-\tau) u_3(t-\tau)}{\alpha_2 + u_2(t-\tau) + \beta u_3(t-\tau)}.$$

Then we can obtain the linearized system of system (3)

$$\begin{cases} \dot{u}_1(t) = a_{11} u_1(t) + a_{12} u_2(t), \\ \dot{u}_2(t) = a_{22} u_2(t) + a_{23} u_3(t) + b_{21} u_1(t-\tau) + b_{22} u_2(t-\tau) \\ \dot{u}_3(t) = a_{33} u_3(t) + b_{32} u_2(t-\tau) + b_{33} u_3(t-\tau). \end{cases} \tag{4}$$

The characteristic equation of system (4) is

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau} + (C_1\lambda + C_0)e^{-2\lambda\tau} = 0, \tag{5}$$

where

$$A_2 = -(a_{11} + a_{22} + a_{33}), \quad A_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33},$$
$$A_0 = -a_{11}a_{22}a_{33}, \quad B_2 = -(b_{22} + b_{33}),$$
$$B_1 = (a_{11} + a_{33})b_{22} + (a_{11} + a_{22})b_{33} - a_{12}b_{21} - a_{23}b_{32},$$
$$B_0 = a_{11}a_{23}b_{32} + a_{12}a_{33}b_{21} - a_{11}(a_{22}b_{33} + a_{33}b_{22}),$$
$$C_1 = b_{22}b_{33}, \quad C_0 = a_{12}b_{21}b_{33} - a_{11}b_{22}b_{33}.$$

For $\tau = 0$, Eq.(5) can be reduced to

$$\lambda^3 + (A_2 + B_2)\lambda^2 + (A_1 + B_1 + C_1)\lambda + A_0 + B_0 + C_0 = 0. \tag{6}$$

The Routh–Hurwitz criterion implies that the positive equilibrium $E_*(x_*, y_*, z_*)$ is locally asymptotically stable if the following condition holds:

$$(H_2)\ A_2 + B_2 > 0, \quad (A_2 + B_2)(A_1 + B_1 + C_1) > A_0 + B_0 + C_0.$$

For $\tau > 0$, multiplying $e^{\lambda\tau}$ on both sides of Eq.(5), we can obtain

$$B_2\lambda^2 + B_1\lambda + B_0 + (\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)e^{\lambda\tau} + (C_1\lambda + C_0)e^{-\lambda\tau} = 0. \tag{7}$$

Let $\lambda = i\omega (\omega > 0)$ be a root of Eq.(7). Substituting $\lambda = i\omega$ into Eq.(7) and separating the real and imaginary parts, we can get

$$\begin{cases} (A_0 + C_0 - A_2\omega^2)\cos\tau\omega - (A_1\omega - C_1\omega - \omega^3)\sin\tau\omega = B_2\omega^2 - B_0, \\ (A_0 - C_0 - A_2\omega^2)\sin\tau\omega + (A_1\omega + C_1\omega - \omega^3)\cos\tau\omega = -B_1\omega. \end{cases}$$

It follows that

$$\sin\tau\omega = \frac{m_5\omega^5 + m_3\omega^3 + m_1\omega}{\omega^6 + n_4\omega^4 + n_2\omega^2 + n_0}, \quad \cos\tau\omega = \frac{m_4\omega^4 + m_2\omega^2 + m_0}{\omega^6 + n_4\omega^4 + n_2\omega^2 + n_0},$$

where

$$\begin{aligned}m_0 =&B_0C_0 - A_0B_0, m_1 = A_1B_0 + B_0C_1 - A_0B_1 - B_1C_0,\\ m_2 =&A_0B_2 + A_2B_0 - A_1B_1 + B_1C_1 - B_2C_0,\\ m_3 =&A_2B_1 - A_1B_2 - B_2C_1 - B_0, m_4 = B_1 - A_2B_2, m_5 = B_2,\\ n_0 =&A_0^2 - C_0^2, n_2 = A_1^2 - C_1^2 - 2A_0A_2, n_4 = A_2^2 - 2A_1.\end{aligned}$$

As is know to all, $\sin^2 \tau\omega + \cos^2 \tau\omega = 1$. So we have

$$\omega^{12} + e_5\omega^{10} + e_4\omega^8 + e_3\omega^6 + e_2\omega^4 + e_1\omega^2 + e_0 = 0, \tag{8}$$

with

$$\begin{aligned}e_0 =&n_0^2 - m_0^2, e_1 = 2n_0n_2 - m_1^2 - 2m_0m_2,\\ e_2 =&n_2^2 - m_2^2 - 2n_0n_4 - 2m_0m_4 - 2m_1m_3,\\ e_3 =&2n_0 + 2n_2n_4 - m_3^2 - 2m_1m_5 - 2m_2m_4,\\ e_4 =&n_4^2 - m_4^2 + 2n_2 - 2m_3m_5, e_5 = 2n_4 - m_5^2.\end{aligned}$$

Let $v = \omega^2$, then Eq.(8) becomes

$$v^6 + e_5v^5 + e_4v^4 + e_3v^3 + e_2v^2 + e_1v + e_0 = 0. \tag{9}$$

Next, we give the following assumption.
(H_3) Eq.(9) has at least one positive real root.

Without loss of generality, we assume that Eq.(9) has six real positive roots, which are defined by $v_1, v_2, v_3, \cdots, v_6$, respectively. Then Eq.(8) has six positive roots $\omega_k = \sqrt{v_k}, k = 1, 2, \cdots, 6$. Thus, let

$$\tau_k^{(j)} = \frac{1}{\omega_k} \arccos \frac{m_4\omega_k^4 + m_2\omega_k^2 + m_0}{\omega_k^6 + n_4\omega_k^4 + n_2\omega_k^2 + n_0} + \frac{2j\pi}{\omega_k}, \quad k = 1, 2, \cdots, 6; \quad j = 0, 1, 2 \cdots$$

Then $\pm i\omega_k$ are a pair of purely imaginary roots of Eq.(7) with $\tau = \tau_k^{(j)}$. Define

$$\tau_0 = \tau_k^{(0)} = \min\{\tau_k^{(0)}\}, \quad k = 1, 2, \cdots, 6;\ \omega_0 = \omega_{k_0}.$$

Let $\lambda(\tau) = \xi(\tau) + i\omega(\tau)$ be the root of Eq.(7) near $\tau = \tau_0$ satisfying $\xi(\tau_0) = 0, \omega(\tau_0) = \omega_0$.

Taking the derivative of λ with respect to τ in Eq.(7), we obtain

$$\begin{aligned}(2B_2\lambda + B_1)\frac{d\lambda}{d\tau} &+ (3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau}\frac{d\lambda}{d\tau}\\ &+ (\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)e^{\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right)\\ &+ C_1e^{-\lambda\tau}\frac{d\lambda}{d\tau} + (C_1\lambda + C_0)e^{-\lambda\tau}\left(-\lambda - \tau\frac{d\lambda}{d\tau}\right) = 0.\end{aligned}$$

It follows that

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2B_2\lambda + B_1 + C_1e^{-\lambda\tau} + (3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau}}{(C_1\lambda^2 + C_0\lambda)e^{-\lambda\tau} - (\lambda^4 + A_2\lambda^3 + A_1\lambda^2 + A_0\lambda)e^{\lambda\tau}} - \frac{\tau}{\lambda}.$$

Thus,

$$\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{P_R Q_R + P_I Q_I}{Q_R^2 + Q_I^2},$$

where

$$\begin{aligned}
P_R =&(A_1 + C_1 - 3\omega_0^2)\cos\tau_0\omega_0 - 2A_2\omega_0 \sin\tau_0\omega_0 + B_1,\\
P_I =&(A_1 - C_1 - 3\omega_0^2)\sin\tau_0\omega_0 + 2A_2\omega_0 \cos\tau_0\omega_0 + 2B_2\omega_0,\\
Q_R =&(C_0\omega_0 + A_0\omega_0 - A_2\omega_0^3)\sin\tau_0\omega_0 - (C_1\omega_0^2 - A_1\omega_0^2 + \omega_0^4)\cos\tau_0\omega_0,\\
Q_I =&(C_0\omega_0 - A_0\omega_0 + A_2\omega_0^3)\cos\tau_0\omega_0 + (C_1\omega_0^2 + A_1\omega_0^2 - \omega_0^4)\sin\tau_0\omega_0.
\end{aligned}$$

Thus, if the condition (H_4) $P_R Q_R + P_I Q_I \neq 0$ holds, then $\text{Re}[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau_0} \neq 0$. Namely, if the condition (H_4) holds, then the transversality condition is satisfied. Through the analysis above, we have the following results.

Theorem 1 *For system (2), if the conditions $(H_1)--(H_4)$ hold, then the positive equilibrium $E_*(x_*, y_*, z_*)$ is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. And system (2) undergoes a Hopf bifurcation at the positive equilibrium $E_*(x_*, y_*, z_*)$ when $\tau = \tau_0$.*

3 Properties of bifurcating periodic solutions

In Section 2, we have obtained the conditions for the existence of Hopf bifurcation when $\tau = \tau_0$. In this section, we shall investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using normal form theory and center manifold theorem in [15].

Let $\tau = \tau_0 + \mu, \mu \in R$, then $\mu = 0$ is the Hopf bifurcation value of system (2). Rescaling the time delay $t \to (t/\tau)$, then system (2) can be transformed into an FDE in $C = C([-1, 0], R^3)$ as:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{10}$$

where

$$L_\mu\phi = (\tau_0 + \mu)\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}\phi(0) + (\tau_0 + \mu)\begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & 0 \\ 0 & b_{32} & b_{33} \end{pmatrix}\phi(-1),$$

and

$$F(\mu, \phi) = (\tau_0 + \mu)(F_1, F_2, F_3)^T,$$

with

$$\begin{aligned}
F_1 =& g_1\phi_1^2(0) + g_2\phi_1(0)\phi_2(0) + g_3\phi_1^3(0) + g_4\phi_1^2(0)\phi_2(0) + \cdots,\\
F_2 =& h_1\phi_2^2(0) + h_2\phi_3^2(0) + h_3\phi_2(0)\phi_3(0) + h_4\phi_1^2(-1)\\
&+h_5\phi_1(-1)\phi_2(-1) + h_6\phi_2^3(0) + h_7\phi_3^3(0) + h_8\phi_2^2(0)\phi_3(0)\\
&+h_9\phi_2(0)\phi_3^2(0) + h_{10}\phi_1^3(-1) + h_{11}\phi_1^2(-1)\phi_2(-1) + \cdots,\\
F_3 =& k_1\phi_2^2(-1) + k_2\phi_3^2(-1) + k_3\phi_2(-1)\phi_3(-1) + k_4\phi_2^3(-1)\\
&+k_5\phi_3^3(-1) + k_6\phi_2^2(-1)\phi_3(-1) + k_7\phi_2(-1)\phi_3^2(-1) + \cdots,
\end{aligned}$$

with

$$g_1 = -b + \frac{c_1\alpha_1 y_*}{(\alpha_1 + x_*)^3}, \quad g_2 = -\frac{c_1\alpha_1}{(\alpha_1 + x_*)^2},$$

$$g_3 = -\frac{c_1\alpha_1 y_*}{(\alpha_1 + x_*)^4}, \quad g_4 = \frac{c_1\alpha_1}{(\alpha_1 + x_*)^3},$$

$$h_1 = \frac{c_3 z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^3}, \quad h_2 = \frac{\beta c_3 y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^3},$$

$$h_3 = \frac{c_3 y_*(\alpha_2 + y_* - \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^3} - \frac{c_3(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^2},$$

$$h_4 = -\frac{c_2\alpha_1 x_* y_*}{(\alpha_1 + x_*)^3}, \quad h_5 = \frac{c_2\alpha_1}{(\alpha_1 + x_*)^2},$$

$$h_6 = -\frac{c_3 z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^4}, \quad h_7 = -\frac{\beta c_3 y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^4},$$

$$h_8 = \frac{c_3(\alpha_2 + 2\beta z_*)}{(\alpha_2 + y_* + \beta z_*)^3} - \frac{3\beta c_3 z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^4},$$

$$h_9 = \frac{\beta c_3(\alpha_2 + 2y_*)}{(\alpha_2 + y_* + \beta z_*)^3} - \frac{3\beta c_3 y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^4},$$

$$h_{10} = \frac{c_2\alpha_1 y_*}{(\alpha_1 + x_*)^4}, \quad h_{11} = -\frac{c_2\alpha_1}{(\alpha_1 + x_*)^3},$$

$$k_1 = -\frac{c_4 z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^3}, \quad k_2 = -\frac{\beta c_4 y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^3},$$

$$k_3 = \frac{c_4(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^2} - \frac{c_4 y_*(\alpha_2 + y_* - \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^3},$$

$$k_4 = \frac{c_4 z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^4}, \quad k_5 = \frac{\beta c_4 y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^4},$$

$$k_6 = \frac{3\beta c_4 z_*(\alpha_2 + \beta z_*)}{(\alpha_2 + y_* + \beta z_*)^4} - \frac{c_4(\alpha_2 + 2\beta z_*)}{(\alpha_2 + y_* + \beta z_*)^3},$$

$$k_7 = \frac{3\beta c_4 y_*(\alpha_2 + y_*)}{(\alpha_2 + y_* + \beta z_*)^4} - \frac{\beta c_4(\alpha_2 + 2y_*)}{(\alpha_2 + y_* + \beta z_*)^3}.$$

By the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta, \mu), \theta \in [-1, 0]$ whose components are of bounded variation, such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \phi \in C([-1, 0], R^3).$$

In fact, we choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}\phi(0) + (\tau_0 + \mu)\begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & 0 \\ 0 & b_{32} & b_{33} \end{pmatrix}\phi(-1).$$

For $\phi \in C([-1, 0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Then system (10) can be transformed into the following operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{11}$$

The adjoint operator A^* of A is defined by

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases}$$

associated with a bilinear form

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{12}$$

where $\eta(\theta) = \eta(\theta, 0)$.

By the discussion above, we know that $\pm i\tau_0\omega_0$ are eigenvalues of $A(0)$ and $A^*(0)$. Let $q(\theta) = (1, q_2, q_3)^T e^{i\tau_0\omega_0\theta}$ be the eigenvector of $A(0)$ corresponding to $i\tau_0\omega_0$ and $q^*(s) = D(1, q_2^*, q_3^*)^T e^{i\tau_0\omega_0 s}$ be the eigenvector of A^* corresponding to $-i\tau_0\omega_0$. Then we have

$$A(0)q(\theta) = i\tau_0\omega_0 q(\theta), \quad A^*(0)q^*(\theta) = -i\tau_0\omega_0 q^*(\theta).$$

By a simple computation, we can get

$$q_2 = \frac{i\omega_0 - a_{11}}{a_{12}}, \quad q_3 = \frac{(i\omega_0 - a_{11})b_{32}e^{-i\tau_0\omega_0}}{(i\omega_0 - a_{33} - b_{33}e^{-i\tau_0\omega_0})a_{12}},$$

$$q_2^* = -\frac{i\omega_0 + a_{11}}{b_{21}e^{i\tau_0\omega_0}}, \quad q_3^* = \frac{(i\omega_0 + a_{11})a_{33}e^{-i\tau_0\omega_0}}{(i\omega_0 + a_{33} + b_{33}e^{i\tau_0\omega_0})b_{21}}.$$

From (12), we obtain

$$\bar{D} = [1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* + (\bar{q}_2^*(b_{21} + b_{22}q_2) + \bar{q}_3^*(b_{32}q_2 + b_{33}q_3))\tau_0 e^{-i\tau_0\omega_0}]^{-1},$$

such that $\langle q^*, q \rangle = 1$, $\langle q^*, \bar{q} \rangle = 0$.

Following the algorithm given in [15] and using the similar computation process in [16], we can get the coefficients used to determine the qualities of the bifurcating periodic solutions:

$$\begin{aligned} g_{20} =& 2\tau_0\bar{D}[g_1 + g_2 q^{(2)}(0) + \bar{q}_2^*(h_1(q^{(2)}(0))^2 + h_2(q^{(3)}(0))^2 + h_3 q^{(2)}(0)q^{(3)}(0) \\ &+ h_4(q^{(1)}(-1))^2 + h_5 q^{(1)}(-1)q^{(2)}(-1)) + \bar{q}_3^*(k_1(q^{(2)}(-1))^2 + k_2(q^{(3)}(-1))^2 \\ &+ k_3 q^{(2)}(-1)q^{(3)}(-1))], \\ g_{11} =& \tau_0\bar{D}[2g_1 + g_2(q^{(2)}(0) + \bar{q}^{(2)}(0)) + \bar{q}_2^*(2h_1 q^{(2)}(0)\bar{q}^{(2)}(0) + 2h_2 q^{(3)}(0)\bar{q}^{(3)}(0) \\ &+ h_3(q^{(2)}(0)\bar{q}^{(3)}(0) + \bar{q}^{(2)}(0)q^{(3)}(0))) + 2h_4 q^{(1)}(-1)\bar{q}^{(1)}(-1) \\ &+ h_5(q^{(1)}(-1)\bar{q}^{(2)}(-1) + \bar{q}^{(1)}(-1)q^{(2)}(-1))) + \bar{q}_3^*(2k_1 q^{(2)}(-1)\bar{q}^{(2)}(-1) \\ &+ 2k_2 q^{(3)}(-1)\bar{q}^{(3)}(-1) + k_3(\bar{q}^{(2)}(-1)q^{(3)}(-1) + q^{(2)}(-1)\bar{q}^{(3)}(-1)))], \end{aligned}$$

$$
\begin{aligned}
g_{02} =& 2\tau_0 \bar{D}[g_1 + g_2\bar{q}^{(2)}(0) + \bar{q}_2^*(h_1(\bar{q}^{(2)}(0))^2 + h_2(\bar{q}^{(3)}(0))^2 + h_3\bar{q}^{(2)}(0)\bar{q}^{(3)}(0)) \\
&+ h_4(\bar{q}^{(1)}(-1))^2 + h_5\bar{q}^{(1)}(-1)\bar{q}^{(2)}(-1) + \bar{q}_3^*(k_1(\bar{q}^{(2)}(-1))^2 \\
&+ k_2(\bar{q}^{(3)}(-1))^2 + k_3\bar{q}^{(2)}(-1)\bar{q}^{(3)}(-1))], \\
g_{21} =& 2\tau_0 \bar{D}[g_1(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + g_2(W_{11}^{(1)}(0)q^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\bar{q}^{(2)}(0) \\
&+ W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)) + 3g_3 + g_4(\bar{q}^{(2)}(0) + 2q^{(2)}(0)) \\
&+ \bar{q}_2^*(h_1(2W_{11}^{(2)}(0)q^{(2)}(0) + W_{20}^{(2)}(0)\bar{q}^{(2)}(0)) + h_2(2W_{11}^{(3)}(0)q^{(3)}(0) \\
&+ W_{20}^{(3)}(0)\bar{q}^{(3)}(0)) + h_3(W_{11}^{(2)}(0)q^{(3)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\bar{q}^{(3)}(0) \\
&+ W_{11}^{(3)}(0)q^{(2)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\bar{q}^{(2)}(0)) + h_4(2W_{11}^{(1)}(-1)q^{(1)}(-1) \\
&+ W_{20}^{(1)}(-1)\bar{q}^{(1)}(-1)) + h_5(W_{11}^{(1)}(-1)q^{(2)}(-1) + \frac{1}{2}W_{20}^{(1)}(-1)\bar{q}^{(2)}(-1) \\
&+ W_{11}^{(2)}(-1)q^{(1)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1)\bar{q}^{(1)}(-1)) + 3h_6(q^{(2)}(0))^2\bar{q}^{(2)}(0) \\
&+ 3h_7(q^{(3)}(0))^2\bar{q}^{(3)}(0) + h_8((q^{(2)}(0))^2\bar{q}^{(3)}(0) + 2q^{(2)}(0)\bar{q}^{(2)}(0)q^{(3)}(0)) \\
&+ h_9((q^{(3)}(0))^2\bar{q}^{(2)}(0) + 2q^{(3)}(0)\bar{q}^{(3)}(0)q^{(2)}(0)) \\
&+ 3h_{10}(q^{(1)}(-1))^2\bar{q}^{(1)}(-1) + h_{11}(q^{(1)}(-1))^2\bar{q}^{(2)}(-1) \\
&+ 2q^{(1)}(-1)\bar{q}^{(1)}(-1)q^{(2)}(-1))) + \bar{q}_3^*(k_1(2W_{11}^{(2)}(-1)q^{(2)}(-1) \\
&+ W_{20}^{(2)}(-1)\bar{q}^{(2)}(-1)) + k_2(2W_{11}^{(3)}(-1)q^{(3)}(-1) + W_{20}^{(3)}(-1)\bar{q}^{(3)}(-1)) \\
&+ k_3(W_{11}^{(2)}(-1)q^{(3)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1)\bar{q}^{(3)}(-1) + W_{11}^{(3)}(-1)q^{(2)}(-1) \\
&+ \frac{1}{2}W_{20}^{(3)}(-1)\bar{q}^{(2)}(-1)) + 3k_4(q^{(2)}(-1))^2\bar{q}^{(2)}(-1) \\
&+ 3k_5(q^{(3)}(-1))^2\bar{q}^{(3)}(-1) \\
&+ k_6((q^{(2)}(-1))^2\bar{q}^{(3)}(-1) + 2q^{(2)}(-1)\bar{q}^{(2)}(-1)q^{(3)}(-1)) \\
&+ k_7((q^{(3)}(-1))^2\bar{q}^{(2)}(-1) + 2q^{(3)}(-1)\bar{q}^{(3)}(-1)q^{(2)}(-1)))],
\end{aligned}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_{20}e^{2i\tau_0\omega_0\theta}, \\
W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_{11},
\end{aligned}
$$

where E_{20} and E_{11} can be computed by the following equations, respectively

$$
\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} & 0 \\ -b_{21}e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_{22} - b_{22}e^{-2i\tau_0\omega_0} & -a_{23} \\ 0 & -b_{32}e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_{33} - b_{33}e^{-2i\tau_0\omega_0} \end{pmatrix} E_{20} = 2\begin{pmatrix} E_{20}^{(1)} \\ E_{20}^{(2)} \\ E_{20}^{(3)} \end{pmatrix}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & 0 \\ b_{21} & a_{22} + b_{22} & a_{23} \\ 0 & b_{32} & a_{33} + b_{33} \end{pmatrix} E_{11} = -\begin{pmatrix} E_{11}^{(1)} \\ E_{11}^{(2)} \\ E_{11}^{(3)} \end{pmatrix}
$$

with

$$
\begin{aligned}
E_{20}^{(1)} =& g_1 + g_2 q^{(2)}(0), \\
E_{20}^{(2)} =& h_1(q^{(2)}(0))^2 + h_2(q^{(3)}(0))^2 \\
& + h_3 q^{(2)}(0) q^{(3)}(0) + h_4 (q^{(1)}(-1))^2 \\
& + h_5 q^{(1)}(-1) q^{(2)}(-1), \\
E_{20}^{(3)} =& k_1(q^{(2)}(-1))^2 + k_2(q^{(3)}(-1))^2 \\
& + k_3 q^{(2)}(-1) q^{(3)}(-1), \\
E_{11}^{(1)} =& 2g_1 + g_2(q^{(2)}(0) + \bar{q}^{(2)}(0)), \\
E_{11}^{(2)} =& 2h_1 q^{(2)}(0)\bar{q}^{(2)}(0) + 2h_2 q^{(3)}(0)\bar{q}^{(3)}(0) \\
& + h_3(q^{(2)}(0)\bar{q}^{(3)}(0) + \bar{q}^{(2)}(0) q^{(3)}(0))) \\
& + 2h_4 q^{(1)}(-1)\bar{q}^{(1)}(-1) \\
& + h_5(q^{(1)}(-1)\bar{q}^{(2)}(-1) + \bar{q}^{(1)}(-1) q^{(2)}(-1)), \\
E_{11}^{(3)} =& 2k_1 q^{(2)}(-1)\bar{q}^{(2)}(-1) + 2k_2 q^{(3)}(-1)\bar{q}^{(3)}(-1) \\
& + k_3(\bar{q}^{(2)}(-1) q^{(3)}(-1) + q^{(2)}(-1)\bar{q}^{(3)}(-1)).
\end{aligned}
$$

Then, we can get the following coefficients:

$$
\begin{aligned}
C_1(0) =& \frac{i}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 =& -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}}, \\
\beta_2 =& 2\mathrm{Re}\{C_1(0)\}, \\
T_2 =& -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2 \mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}.
\end{aligned}
\tag{13}
$$

Based on the discussion above, we can obtain the following results.

Theorem 2 *The direction of the Hopf bifurcation is determined by the sign of μ_2: if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical). The stability of bifurcating periodic solutions is determined by the sign of β_2: if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable).The period of the bifurcating periodic solutions is determined by the sign of T_2: if $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solutions increases (decreases).*

4 Numerical simulation

In this section, we give a numerical example to support the theoretical analysis. We consider the following system:

$$
\begin{cases}
\frac{dx(t)}{dt} = x(t)(1.5 - x(t)) - \frac{0.6x(t)y(t)}{1+x(t)}, \\
\frac{dy(t)}{dt} = -0.1y(t) + \frac{0.6x(t-\tau)y(t-\tau)}{1+x(t-\tau)} - \frac{0.7y(t)z(t)}{1+y(t)+z(t)}, \\
\frac{dz(t)}{dt} = -0.45z(t) + \frac{0.7y(t-\tau)z(t-\tau)}{1+y(t-\tau)+z(t-\tau)},
\end{cases}
\tag{14}
$$

which has a unique positive equilibrium $E_*(0.4097, 2.5617, 0.4232)$. Then, we can get $A_2+B_2 =$

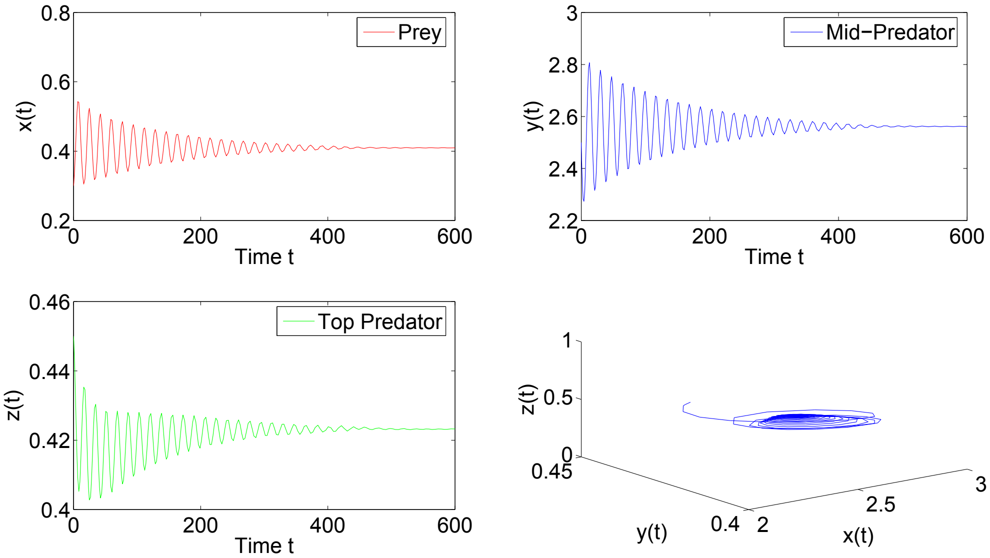

Figure 1: E_* is locally asymptotically stable for $\tau = 0.225 < \tau_0 = 0.2766$ with initial value 0.3, 2.5, 0.45.

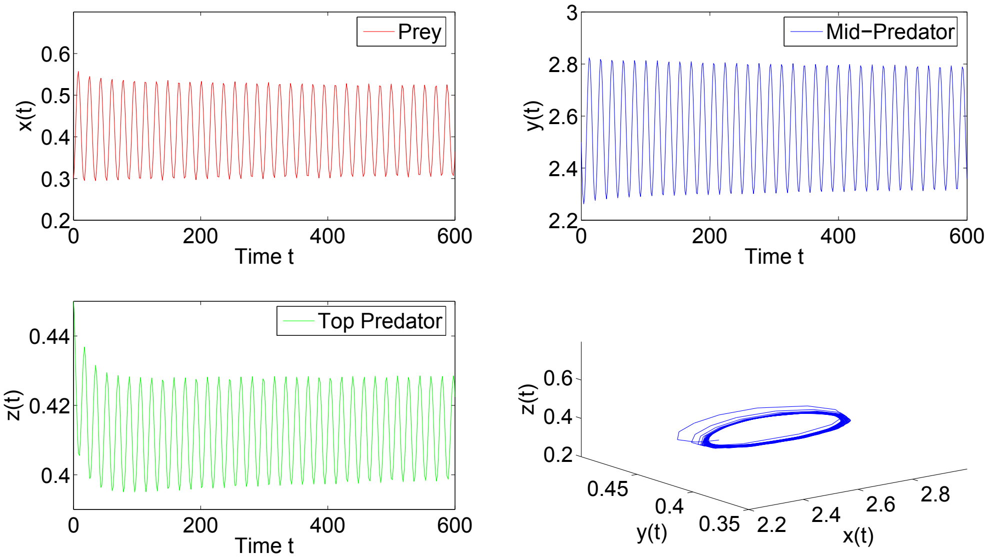

Figure 2: E_* is unstable for $\tau = 0.355 > \tau_0 = 0.2766$ with initial value 0.3, 2.5, 0.45.

0.0928 > 0, $(A_2 + B_2)(A_1 + B_1 + C_1) = 0.0133 > A_0 + B_0 + C_0 = 0.0073$. Further, we obtain $\omega_0 = 0.1921$, $\tau_0 = 0.2766$ and $P_RQ_R + P_IQ_I = 1.5071e - 004$, $\lambda'(\tau_0) = -0.0431 + 0.1673i$. That is, the conditions $(H_1) - (H_4)$ hold. Thus, from Theorem 1, the positive equilibrium $E_*(0.4097, 2.5617, 0.4232)$ is locally asymptotically stable when $0 \le \tau < \tau_0$, as is illustrated by Fig.1. When the time delay τ passes through the critical value τ_0, the positive equilibrium $E_*(0.4097, 2.5617, 0.4232)$ loses its stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from the positive equilibrium $E_*(0.4097, 2.5617, 0.4232)$. This property can be seen from Fig.2. In addition, from (13), we have $C_1(0) = -0.1317 + 0.2352i$, $\mu_2 = -3.0557 < 0$, $\beta_2 = -0.2634 < 0$, $T_2 = 5.1947 > 0$. Therefore, from Theorem 2, we know that the Hopf bifurcation is subcritical, the bifurcating periodic solutions are stable and the period of the bifurcating periodic solutions increases.

5 Conclusion

In the present paper, a three-species food chain system with time delay and the hybrid type of functional responses, Holling type and Beddington–DeAngelis type is studied. Compared with the system considered in [7], we mainly investigate the effect of the time delay due to gestation of the mid-predator and the top predator on the system. The sufficient conditions for the local stability of the positive equilibrium and the existence of periodic solutions via Hopf bifurcation at the positive equilibrium of system (2) are obtained. It is proved that when the conditions are satisfied, then there exists a critical value τ_0 of the time delay below which system (2) is stable and above which the system is unstable. Especially, system (2) undergoes a Hopf bifurcation at the positive equilibrium when $\tau = \tau_0$. In reality, the occurrence of Hopf bifurcation means that the existence of the species in system (2) changes from the positive equilibrium to a limit cycle. For the further investigation, formulae are derived to determine the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions by using the normal form theory and center manifold theorem. If the bifurcating periodic solutions are stable, then we can conclude that the species in system (2) may coexist in an oscillatory mode. A numerical example verifying the theoretical results is also included. And from the numerical example, we can see that the species in system (2) may coexist in an oscillatory mode under some certain conditions. Do et al. [7] obtained that the species in system (2) without delay could coexist. However, we get that the species could also coexist with the time delay due to gestation of the mid-predator and the top predator. This is very valuable from the view of ecology.

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