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Positive Solutions to an N^{th} Order Right Focal Boundary Value Problem

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Abstract

The existence of a positive solution is obtained for the n^{th} order right focal boundary value problem $y^{(n)} = f(x,y)$, $0 < x \leq 1$, $y^{(i)}(0) = y^{(n-2)}(p) = y^{(n-1)}(1) = 0, i = 0, \cdots, n-3$, where $\frac{1}{2} < p < 1$ is fixed and where $f(x,y)$ is singular at $x = 0, y = 0$, and possibly at $y = \infty$. The method applies a fixed-point theorem for mappings that are decreasing with respect to a cone.

Key words: Fixed point theorem, boundary value problem.
AMS Subject Classification: 34B15.

1 Introduction

In this paper, we establish the existence of a positive solution for the n^{th} order right focal boundary value problem,

$$y^{(n)} = f(x,y), \text{ for } x \in (0,1], \tag{1}$$

$$y^{(i)}(0) = y^{(n-2)}(p) = y^{(n-1)}(1) = 0, \quad i = 0, \cdots, n-3, \tag{2}$$

where $\frac{1}{2} < p < 1$ is fixed and $f(x,y)$ is singular at $x = 0$, $y = 0$, and may be singular at $y = \infty$.

We assume the following conditions hold for f:

(H1) $f(x, y) : (0, 1] \times (0, \infty) \longrightarrow (0, \infty)$ is continuous, and $f(x, y)$ is decreasing in y for every x.

(H2) $\lim_{y\to 0^+} f(x, y) = +\infty$ and $\lim_{y\to\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1]$.

We reduce the problem to a third order integro-differential problem. We establish decreasing operators for which we find fixed points that are solutions to this third order problem. Then, we use Gatica, Oliker, and Waltman methods to find a positive solution to the integro-differential third order problem. We integrate the positive solution $n-3$ times to obtain the positive solution to the n^{th} order right focal boundary value problem. The role of $\frac{1}{2} < p < 1$ is fundamental for the positivity of the Green's function which in turn is fundamental for the positivity of desired solutions. The existence of positive solutions to a similar third order right focal boundary value problem was established in [22].

Singular boundary value problems for ordinary differential equations have arisen in numerous applications, especially when only positive solutions are useful. For example, when $n = 2$, Taliaferro [28] has given a nice treatment of the general problem, Callegari and Nachman [9] have studied existence questions of this type in boundary layer theory, and Lunning and Perry [21] have established constructive results for generalized Emden-Fowler boundary value problems. Also, Bandle, Sperb, and Stakgold [3] and Bobisud, *et al.* [6], [7], [8], have obtained results for singular boundary value problems that arise in reaction-diffusion theory, while Callegari and Nachman [10] have considered such boundary conditions in non-Newtonian fluid theory as well as in the study of pseudoplastic fluids. Nachman and Callegari point to applications in glacial advance and transport of coal slurries down conveyor belts. See [10] for references. Other applications for these boundary value problems appear in problems such as in draining flows [1], [5] and semi-positone and positone problems [2].

In addition, much attention has been devoted to theoretical questions for singular boundary value problems. In some studies on singular boundary value problems, the underlying technique has been to obtain *a priori* estimates on solutions to an associated two-parameter family of problems, and then use these bounds along with topological transversality theorems to obtain solutions of the original problem; for example, see Granas, Guenther, and Lee [15] and Dunninger and Kurtz [11]. This method has been fairly exploited in a number of recent papers by O'Regan, [24], [25], [26]. Baxley [4] also used to some degree this latter technique in his work on singular boundary value problems for membrane response of a spherical cap. Wei [30] gave necessary and sufficient conditions for the existence of positive solutions for the singular Emden-Fowler equation satisfying Sturm-Liouville boundary conditions employing upper and lower solutions methods. Guoliang [16] also gave necessary and sufficient conditions for a higher order singular boundary value problem, using superlinear and sublinear conditions to show the existence of a positive solution.

2 Definitions and Properties of Cones

In this section, we begin by giving some definitions and some properties of cones in a Banach space.

Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. A nonempty set $\mathcal{K} \subset \mathcal{B}$ is called a *cone* if the following conditions are satisfied:
(a) the set $\mathcal{K}$ is closed;
(b) if $u, v \in \mathcal{K}$ then $\alpha u + \beta v \in \mathcal{K}$, for all $\alpha, \beta \geq 0$;
(c) $u, -u \in K$ imply $u = 0$.
Given a cone, $\mathcal{K}$, a *partial order*, $\leq$, is induced on $\mathcal{B}$ by $x \leq y$, for $x, y \in \mathcal{B}$ iff $y - x \in \mathcal{K}$. (For clarity we sometimes write $x \leq y$ (w.r.t. $\mathcal{K}$)). If $x, y \in \mathcal{B}$ with $x \leq y$, let $< x, y >$ denote the *closed order interval between* x *and* y given by, $< x, y >= \{z \in \mathcal{K} | x \leq z \leq y\}$. A cone $\mathcal{K}$ is *normal* in $\mathcal{B}$ provided there exists $\delta > 0$ such that $\| e_1 + e_2 \| \geq \delta$, for all $e_1, e_2 \in \mathcal{K}$ with $\| e_1 \| = \| e_2 \| = 1$.

Remark: If $\mathcal{K}$ is a normal cone in $\mathcal{B}$, then closed order intervals are norm bounded.

3 Gatica, Oliker, and Waltman Fixed Point Theorem

Now we state the fixed point theorem due to Gatica, Oliker, and Waltman on which most of the results of this paper depend.

Theorem 3.1 *Let $\mathcal{B}$ be a Banach space, $\mathcal{K}$ a normal cone in $\mathcal{B}$, $\mathcal{C}$ a subset of $\mathcal{K}$ such that if x, y are elements of $\mathcal{C}$, $x \leq y$, then $< x, y >$ is contained in $\mathcal{C}$, and let $T{:}\mathcal{C} \to \mathcal{K}$ be a continuous decreasing mapping which is compact on any closed order interval contained in $\mathcal{C}$. Suppose there exists $x_0 \in \mathcal{C}$ such that $T^2(x_0)$ is defined (where $T^2(x_0) = T(Tx_0)$), and furthermore, Tx_0 and T^2x_0 are order comparable to x_0. Then T has a fixed point in $\mathcal{C}$ provided that either,*
(I) $Tx_0 \leq x_0$ and $T^2x_0 \leq x_0$, or $Tx_0 \geq x_0$ and $T^2x_0 \geq x_0$, or
(II) The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined, and there exists $y_0 \in \mathcal{C}$ such that $y_0 \leq T^n x_0$, for every n.

We consider the following Banach space, $\mathcal{B}$, with associated norm, $\|\cdot\|$:

$$\mathcal{B} = \{u : [0,1] \to R \mid \text{u is continuous}\},$$

$$\|u\| = \sup_{x\in[0,1]} |u(x)|.$$

We also define a cone, $\mathcal{K}$, in $\mathcal{B}$ by,

$$\mathcal{K} = \{u \in \mathcal{B} | u(x) \geq 0, g(x)u(p) \leq u(x) \leq u(p) \text{ and } u(x) \text{ is concave on } [0,1]\},$$

where

$$g(x) = \frac{x(2p-x)}{p^2}, \text{ for } 0 \leq x \leq 1.$$

4 The Integral Operator

In this section, we will define a decreasing operator $\mathcal{T}$ that will allow us to use the stated fixed point theorem.

First, we define $k(x)$ by,

$$k(x) = \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} g(s)ds,$$

Given $g(x)$ and $k(x)$ above, we define $g_\theta(x)$ and $k_\theta(x)$, for $\theta > 0$, by

$$g_\theta(x) = \theta \cdot g(x),$$

and

$$k_\theta(x) = \theta \cdot k(x),$$

and we will assume hereafter

(H3) $\int_0^1 f(x, k_\theta(x))dx < \infty$, for each $\theta > 0$.

We note that the function $f(x,y) = \frac{1}{\sqrt[4]{xy}}$ also satisfies (H3).
In particular, for each $\theta > 0$,
$\int_0^1 f(x, g_\theta(x))dx = \sqrt[4]{(\frac{p^2}{\theta})}[\frac{2}{\sqrt[4]{2p}} + 4\frac{(2p-1)^{\frac{3}{4}} - (2p)^{\frac{3}{4}}}{3(\sqrt{2p})} < \infty.$

If y is a solution of (1)-(2), then

$$u(x) = y^{(n-3)}(x),$$

is positive and concave. Hence, if in addition $u \in \mathcal{K}$, then $||u|| = u(p)$.

Also, we get,

$$u''' = f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u(s)ds\Big), \tag{3}$$

$$u(0) = u'(p) = u''(1) = 0. \tag{4}$$

Since $g(x)$ is concave with $g(0) = 0$ and $||g(x)|| = g(p)$, then we observe, that for each positive solution, $u(x)$, of (3)-(4), there is some $\theta > 0$, such that $g_\theta(x) \le u(x)$, for $0 \le x \le 1$.

Next, we let $\mathcal{D} \subseteq \mathcal{K}$ be defined by

$$\mathcal{D} = \{u \in \mathcal{K} | \text{ there exists } \theta(u) > 0 \text{ so that } g_\theta(x) \le u(x), 0 \le x \le 1\}.$$

We note that for each $u \in \mathcal{K}$,

$$\|u\| = \sup_{x \in [0,1]} |u(x)| = u(p).$$

Next, we define an integral operator $\mathcal{T} : \mathcal{D} \to \mathcal{K}$ by

$$(\mathcal{T}u)(x) = \int_0^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u(s)ds\Big)dt,$$

where $G(x,t)$ is the Green's function for $y''' = 0$ satisfying (4), and given by

$$G(x,t) = \begin{cases} \frac{x(2t-x)}{2}, x \le t \le p \ , \\ \frac{t^2}{2}, t \le x, t \le p \ , \\ \frac{x(2p-x)}{2}, x \le t, t \ge p \ , \\ \frac{x(2p-x)}{2} + \frac{(x-t)^2}{2}, x \ge t \ge p \ ; \end{cases}$$

see [14] .

First, we show $\mathcal{T}$ is a decreasing operator. Let $u \in \mathcal{D}$ be given. Then there exists $\theta > 0$ such that $g_\theta(x) \le u(x)$. Then, by condition (H1), $f(x, u(x)) \le f(x, g_\theta(x))$. Now, let $u(x) \le v(x)$ for $u(x), v(x) \in \mathcal{D}$. Then,

$$\int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u(s)ds \le \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} v(s)ds.$$

Then by condition (H1),

$$f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} v(s)ds\Big) \le f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u(s)ds\Big).$$

And since $G(x,t) > 0$, we have by (H1) and (H3),

$$\begin{aligned}\int_0^1 G(x,t) f\Big(t, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} v(s)ds\Big)dt &< \int_0^1 G(x,t) f\Big(t, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u(s)ds\Big)dt \\ &\leq \int_0^1 G(x,t) f\Big(t, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} g_\theta(s)ds\Big)dt \\ &= \int_0^1 G(x,t) f(x, k_\theta(x)) \\ &< \infty,\end{aligned}$$

where $g_\theta(x) \leq u(x)$.

Therefore, $\mathcal{T}$ is well-defined on $\mathcal{D}$ and $\mathcal{T}$ is a decreasing operator.

Remark: We claim that $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. To see this, suppose $u \in \mathcal{D}$ and let

$$w(x) = (\mathcal{T}u)(x) = \int_0^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u(s)ds\Big)dt \geq 0.$$

Thus, for $0 \leq x \leq 1$, $w(x) \geq 0$. Also by properties of G,

$$w'''(x) = f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u(s)ds\Big) > 0, \text{ for } 0 < x \leq 1,$$

and $w(x)$ satisfies (5.4). As we argued previously, $\|w\| = w(p)$.
Since we have that $w''(1) = 0$ and $w'''(x) > 0$, then w is concave.
Also, with $w(p) = ||w(x)||$, then $w(x) \geq w(p)g(x) = g_{w(p)}(x)$. Therefore, $w \in \mathcal{D}$, and $\mathcal{T} : \mathcal{D} \to \mathcal{D}$.
Remark: It is well-known that $\mathcal{T}u = u$ iff u is a solution of (3)-(4). Hence, we seek solutions of (3)-(4) that belong to $\mathcal{D}$.

5 *A Priori* Bounds on Norms of Solutions

In this section, we will show that solutions of (3)-(4) have positive *a priori* upper and lower bounds on their norms. The proofs will be done by contradiction.

Lemma 5.1 *If f satisfies (H1)-(H3), then there exists $S > 0$ such that $||u|| \leq S$ for any solution u of (3)-(4) in $\mathcal{D}$.*

Proof: We assume that the conclusion of the lemma is false. Then there exists a sequence, $\{u_m\}_{m=1}^\infty$, of solutions of (3)-(4) in $\mathcal{D}$ such that $u_m(x) > 0$, for $x \in (0,1]$, and

$$\|u_m\| \leq \|u_{m+1}\| \text{ and } \lim_{m\to\infty} \|u_m\| = \infty.$$

For a solution u of (3)-(4), we have

$$u''' = f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u(s)ds\Big) > 0, \text{ for } 0 < x \le 1,$$

$$\text{or } u'' < 0, \text{ for } 0 < x \le 1.$$

This says that u is concave. In particular, the graphs of the sequence of solutions, u_m, are concave. Furthermore, for each m, the boundary conditions (4) and the concavity of u_m give us,

$$u_m(x) \ge u_m(p)g(x) = \|u_m\|g(x) = g_{\|u_m\|}(x) \text{ for all } x,$$

and so for every $0 < c < 1$,

$$\lim_{m\to\infty} u_m(x) = \infty \text{ uniformly on } [c, 1].$$

Now, let us define

$$M := \max\{G(x,t) : (x,t) \in [0,1] \times [0,1]\}.$$

Then, from condition (H2), there exists m_0 such that, for all $m \ge m_0$ and $x \in [p, 1]$,

$$f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u_m(s)ds\Big) \le \frac{1}{M(1-p)}.$$

Let

$$\theta = \|u_{m_0}\| = u_{m_0}(p).$$

Then, for all $m \ge m_0$,

$$u_m(x) \ge g_\theta(x) = \|u_{m_0}\|g(x), \text{ for } 0 \le x \le 1.$$

So, for $m \ge m_0$, and for $0 \le x \le 1$, we have

$$\begin{aligned}
u_m(x) &= (\mathcal{T}u_m)(x) \\
&= \int_0^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big)dt \\
&= \int_0^p G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big)dt \\
&\quad + \int_p^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big)dt \\
&\le \int_0^p G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} g_\theta(s)ds\Big)dt + \int_p^1 M \frac{1}{M(1-p)} dt \\
&\le \int_0^p G(x,t) f(t, k_\theta(t))dt + 1 \\
&\le \int_0^1 G(x,t) f(t, k_\theta(t))dt + 1 \\
&\le M \int_0^1 f(t, k_\theta(t))dt + 1.
\end{aligned}$$

This is a contradiction to $\lim_{m\to\infty}\|u_m\| = \infty$. Hence, there exists an $S > 0$ such that $\|u\| < S$ for any solution $u \in \mathcal{D}$ of (3)-(4).
□

Now we deal with positive *a priori* lower bounds on the solution norms.

Lemma 5.2 *If f satisfies (H1)-(H3), then there exists $R > 0$ such that $||u|| \geq R$ for any solution u of (5.3)-(5.4) in $\mathcal{D}$.*

Proof: We assume the conclusion of the lemma is false. Then, there exists a sequence $\{u_m\}_{m=1}^{\infty}$ of solutions of (3)-(4) in $\mathcal{D}$ such that $u_m(x) > 0$, for $x \in (0, 1]$, and

$$\|u_m\| \geq \|u_{m+1}\|$$

and

$$\lim_{m\to\infty}\|u_m\| = 0.$$

Now we define

$$\bar{m} := \min\{G(x, t) : (x, t) \in [p, 1] \times [p, 1]\} > 0.$$

From condition (H2), $\lim_{y\to 0^+} f(x, y) = \infty$ uniformly on compact subsets of $(0, 1]$.
Thus, there exists $\delta > 0$ such that, for $x \in [p, 1]$ and $0 < y < \delta$,

$$f(x, y) > \frac{1}{\bar{m}(1-p)}.$$

In addition, there exists m_0 such that, for all $m \geq m_0$ and $x \in (0, 1]$

$$0 < u_m(x) < \frac{\delta}{2},$$

$$0 < \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}u_m(s)ds < \frac{\delta}{2}.$$

So, for $x \in [p, 1]$ and $m \geq m_0$,

$$\begin{aligned} u_m(x) &= (\mathcal{T}u_m)(x) \\ &= \int_0^1 G(x,t) f\Big(t, \int_0^t \frac{(x-s)^{n-4}}{(n-4)!}u_m(s)ds\Big)dt \\ &\geq \int_p^1 G(x,t) f\Big(t, \int_0^t \frac{(x-s)^{n-4}}{(n-4)!}u_m(s)ds\Big)dt \end{aligned}$$

$$\begin{aligned}
&\geq \bar{m}\int_p^1 f\Big(t,\int_0^t \frac{(x-s)^{n-4}}{(n-4)!}u_m(s)ds\Big)dt\\
&> \bar{m}\int_p^1 f\Big(t,\frac{\delta}{2}\Big)dt\\
&> \bar{m}\int_p^1 \frac{1}{\bar{m}(1-p)}dt\\
&= 1.
\end{aligned}$$

Which is a contradiction to $\lim_{m\to\infty}\|u_m(x)\| = 0$ uniformly on $[0,1]$. Thus, there exists $R > 0$ such that $R \leq \|u\|$ for any solution u in $\mathcal{D}$ of (5.3)-(5.4). □

In summary, there exist $0 < R < S$ such that, for $u \in \mathcal{D}$, a solution of (3)-(4), Lemma 5.1 and Lemma 5.2 give us

$$R \leq \|u\| \leq S.$$

The next section gives the main result, an existence theorem, for this problem.

6 Existence Result

In this section, we will construct a sequence of operators, $\{\mathcal{T}_m\}_{m=1}^{\infty}$, each of which is defined on all of $\mathcal{K}$. We will then show, by applications of Theorem 3.1, that each $\mathcal{T}_m$ has a fixed point, ϕ_m, for every m, in $\mathcal{K}$. Then, we will show that some subsequence of the $\{\phi_m\}_{m=1}^{\infty}$ converges to a fixed point of $\mathcal{T}$.

Theorem 6.1 *If f satisfies (H1)-(H3), then (3)-(4) has at least one positive solution u in $\mathcal{D}$, such that $y(x) = \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}u(s)ds$ is a positive solution of (1)-(2).*

Proof: For all m, let $u_m(x) := \mathcal{T}(m)$, where m is the constant function of that value on $[0,1]$. In particular,

$$\begin{aligned}
u_m(x) &= \int_0^1 G(x,t)f\Big(t,\int_0^t \frac{(t-s)^{n-4}}{(n-4)!}mds\Big)dt\\
&= \int_0^1 G(x,t)f\Big(t,\frac{m(-s)^{n-3}}{(n-3)!}\Big)dt, \text{ for } 0 \leq x \leq 1.
\end{aligned}$$

But f is decreasing in its second component, giving us,

$$0 < u_{m+1}(x) \leq u_m(x), \text{ for } 0 \leq x \leq 1.$$

By condition (H2), $\lim_{m\to\infty} u_m(x) = 0$, uniformly on $[0,1]$.

Now, we define $f_m(x,y) : (0,1] \times [0,\infty) \to (0,\infty)$ by

$$f_m(x,y) = f\Big(x, \max\Big\{y, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u_m(s)ds\Big\}\Big).$$

Then f_m is continuous and f_m does not possess the singularities as found in f at $y=0$. Moreover, for $(x,y) \in (0,1] \times (0,\infty)$ we have that,

$$f_m(x,y) \le f(x,y)$$

and, moreover,

$$f_m(x,y) = f\Big(x, \max\Big\{y, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u_m(s)ds\Big\}\Big) \le f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!} u_m(s)ds\Big).$$

Next, we define a sequence of operators, $\mathcal{T}_m : \mathcal{K} \to \mathcal{K}$, for $\phi \in \mathcal{K}$ and $x \in [0,1]$, by

$$\mathcal{T}_m\phi(x) := \int_0^1 G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi(s)ds\Big)dt.$$

It is standard that each $\mathcal{T}_m$ is a compact mapping on $\mathcal{K}$. Moreover,

$$\begin{aligned}
\mathcal{T}_m(0) &= \int_0^1 G(x,t) f_m(t,0)dt \\
&= \int_0^1 G(x,t) f\Big(t, max\Big\{0, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big\}\Big)dt \\
&= \int_0^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big)dt \\
&> 0.
\end{aligned}$$

Also,

$$\mathcal{T}_m^2(0) = \mathcal{T}_m\Big(\int_0^1 G(x,t) f_m(t,0)dt\Big) \ge 0.$$

Then, by theorem (3.1) with $x_0 = 0$, $\mathcal{T}_m$ has a fixed point in $\mathcal{K}$ for every m. Thus, for every m, there exists a $\phi_m \in \mathcal{K}$ so that

$$\mathcal{T}_m\phi_m(x) = \phi_m(x), \ 0 \le x \le 1.$$

Hence, for $m \ge 1$, ϕ_m satisfies the boundary conditions (4) of the problem. Also,

$$\begin{aligned}
\mathcal{T}_m\phi_m(x) &= \int_0^1 G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&= \int_0^1 G(x,t) f\Big(t, max\Big\{\int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big\}\Big)dt \\
&\le \int_0^t G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!} u_m(s)ds\Big)dt \\
&= \mathcal{T}u_m(x).
\end{aligned}$$

That is, $\phi_m(x) = \mathcal{T}_m\phi_m(x) \le \mathcal{T}u_m(x)$, for $0 \le x \le 1$, and for every m.

Proceeding as in lemmas 5.1 and 5.2, there exists $S > 0$ and $R > 0$ such that

$$R < \|\phi_m\| < S$$

for every m.

Now, let $\theta = R$. Since $\phi_m \in \mathcal{K}$, then for $x \in [0,1]$ and every m,

$$\phi_m(x) \ge \phi_m(p)g(x) = \|\phi_m\| \cdot g(x) > R \cdot g(x) = \theta \cdot g(x) = g_\theta(x).$$

Thus, with $\theta = R$, $g_\theta(x) \le \phi_m(x)$ for $x \in [0,1]$, for every m. Thus, $\{\phi_m\}_{m=1}^\infty$ is contained in the closed order interval $< g_\theta, S >$. Therefore, the sequence $\{\phi_m\}_{m=1}^\infty$ is contained in $\mathcal{D}$. Since $\mathcal{T}$ is a compact mapping, we may assume $\lim_{m\to\infty} \mathcal{T}\phi_m$ exist; say the limit is ϕ^*.

To conclude the proof of this theorem, we still need to show that

$$\lim_{m\to\infty} \Big(\mathcal{T}\phi_m(x) - \phi_m(x)\Big) = 0$$

uniformly on $[0,1]$. This will give us that $\phi^* \in < g_\theta, S >$. Still with $\theta = R$, then $k_\theta(x) = \int_0^1 \frac{(x-s)^{n-4}}{(n-4)!} g_\theta(s)ds \le \int_0^1 \frac{(x-s)^{n-4}}{(n-4)!}\phi_m(s)ds$ for every m and $0 \le x \le 1$. Let $\epsilon > 0$ be given and choose δ, $0 < \delta < 1$, such that

$$\int_0^\delta f(t, k_\theta(t))dt < \frac{\epsilon}{2M},$$

where again $M := \max\{G(x,t) : (x,t) \in [0,1] \times [0,1]\}$. Then, there exists m_0 such that, for $m \ge m_0$ and for $x \in [\delta, 1]$,

$$\int_0^1 \frac{(x-s)^{n-4}}{(n-4)!} u_m(x) \le k_\theta(x) \le \int_0^1 \frac{(x-s)^{n-4}}{(n-4)!}\phi_m(x).$$

So, for $x \in [\delta, 1]$,

$$\begin{aligned} f_m\Big(\quad x \quad , \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big) &= \\ f\Big(x, \max\Big\{ \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}\phi_m(s)ds, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}u_m(s)ds\Big\}\Big) &= \\ f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big). \end{aligned}$$

Then, for $0 \le x \le 1$,

$$\begin{aligned}
\mathcal{T}\phi_m(x) - \phi_m(x) &= \mathcal{T}\phi_m(x) - \mathcal{T}_m\phi_m(x) \\
&= \int_0^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad - \int_0^1 G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&= \int_0^\delta G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad + \int_\delta^1 G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad - \int_0^\delta G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad - \int_\delta^1 G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&= \int_0^\delta G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad - \int_0^\delta G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt.
\end{aligned}$$

Thus, for $0 \le x \le 1$, we have,

$$\begin{aligned}
\Big|\mathcal{T}\phi_m(x) - \phi_m(x)\Big| &= \Big|\int_0^\delta G(x,t) f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad - \int_0^\delta G(x,t) f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt\Big| \\
&\le M\Big[\Big|\int_0^\delta f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt\Big| \\
&\quad + \Big|\int_0^\delta f_m\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)d\Big|\Big] \\
&= M\Big[\int_0^\delta f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad + \int_0^\delta f\Big(t, \max\Big\{\int_0^t \frac{(t-s)^{n-4}}{(n-4)!}u_m(s)ds, \\
&\qquad \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big\}\Big)dt\Big] \\
&= M\Big[\int_0^\delta f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\quad + \int_0^\delta f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt\Big] \\
&= 2M\int_0^\delta f\Big(t, \int_0^t \frac{(t-s)^{n-4}}{(n-4)!}\phi_m(s)ds\Big)dt \\
&\le 2M\int_0^\delta f\Big(t, \int_0^1 \frac{(x-s)^{n-4}}{(n-4)!}g_\theta(s)ds\Big)ds \\
&= 2M\int_0^\delta f(t, k_\theta(t))dt \\
&= 2M\frac{\epsilon}{2M} = \epsilon.
\end{aligned}$$

Thus, for $m \ge m_0$,

$$\|\mathcal{T}\phi_m - \phi_m\| < \epsilon.$$

In particular, $\lim_{m\to\infty}\Big(\mathcal{T}\phi_m(x) - \phi_m(x)\Big) = 0$ uniformly on $[0,1]$, and for $0 \le x \le 1$

$$\begin{aligned}
\mathcal{T}\phi^*(x) &= \mathcal{T}\Big(\lim_{m\to\infty}\mathcal{T}\phi_m(x)\Big) \\
&= \mathcal{T}\Big(\lim_{m\to\infty}\phi_m(x)\Big) \\
&= \lim_{m\to\infty}\Big(\mathcal{T}\phi_m(x)\Big) \\
&= \phi^*(x).
\end{aligned}$$

Thus,

$$\mathcal{T}\phi^* = \phi^*,$$

and ϕ^* is a desired solution of (3)-(4).

Now, if $\phi^*(x)$ is the solution of (3)-(4), let $y(x) = \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}\phi^*(s)ds$. Then we have

$$y(0) = \int_0^0 \frac{(x-s)^{n-4}}{(n-4)!}\phi^*(s)ds = 0,$$

and by the Fundamental Theorem of Calculus,

$$y^{(n-3)}(x) = \phi^*(x).$$

Thus,

$$y^{(n-3)}(0) = \phi^*(0) = 0.$$

Also,

$$y^{(n-2)}(x) = (\phi^*)'(x),$$

thus,

$$y^{(n-2)}(p) = (\phi^*)'(p) = 0.$$

And,

$$y^{(n-1)}(x) = (\phi^*)''(x)$$
$$y^{(n-1)}(1) = (\phi^*)''(1) = 0.$$

Moreover,

$$y^{(n)}(x) = (\phi^*)'''(x) = (\mathcal{T}\phi^*)'''(x) = f\Big(x, \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}\phi^*(s)ds\Big) = f(x,y).$$

Thus, $y(x) = \int_0^x \frac{(x-s)^{n-4}}{(n-4)!}\phi^*(s)ds > 0$, $0 \le x \le 1$ solves (1)-(2).

This completes the proof. $\square$

Remark: The results of this paper extend to Boundary Value Problems for $y^{(n)} = f(x, y, y', \cdot, y^{(n-3)})$ under the same boundary conditions.

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