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ON THE EXISTENCE OF SOLUTIONS TO SOME NONLINEAR INTEGRODIFFERENTIAL EQUATIONS WITH DELAYS

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Abstract. Existence of solutions to some nonlinear integral equations with variable delays are obtained by the use of a fixed point theorem due to Dhage. As applications of the main results, existence results to some initial value problems concerning differential equations of higher order as well as integro-differential equations are derived. The case of Lipschitz-type conditions is also considered. Our results improve and generalize, in several ways, existence results already appeared in the literature.

1. INTRODUCTION AND PRELIMINARIES

In the recent paper [5], Dhage and Karande considered the initial value problem

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = \int_0^t g(t,x(s))ds, \quad t \in J \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $J = [0, T]$ for some positive number T, $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, and $g : J \times \mathbb{R} \to \mathbb{R}$, and established existence results under mixed generalized Lipschitz and Caratheodory conditions. A similar problem concerning existence of solutions to the initial value problem (i.v.p., for short)

$$\begin{cases} \frac{d^2}{dt^2}\left[\frac{x(t)}{f(t,x(t))}\right] = \int_0^t g(t,x(s))ds, \quad t \in J \\ x(0) = x_0, \quad x'(0) = x_1 \end{cases} \tag{1.2}$$

has been investigated by the authors in [7]. Both problems are studied by means of fixed point theory (see [8]). For some existence results concenring initial value problems for differential or integrodifferential equations where the derivative of the function $\frac{x(t)}{f(t,x(t))}$ is involved, the reader is refered to the papers [2-7].

Motivated by the work in [5] and [7], the purpose of this note is to generalize and extend the results presented in these papers to an integral equatiion that includes (1.1) and (1.2) (as special cases) meanwhile relaxing the assumptions posed on the functions f and g in [5] and [7]. More precisely, we consider the following integral equation

$$x(t) = f\left(t, x(t), x(\vartheta(t))\right)\left[Q(t) + \int_0^t H(t,s)g(s,x(s),x(\eta(s)))ds\right], \quad t \in J, \tag{1.3}$$

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where f *and* g *are real-valued functions defined on* $J \times \mathbb{R}^2$, $H(t,s)$ *is a continuous function on* $J \times J$, ϑ, $\eta \in C(J,J)$ *with* $\theta(t), \eta(t) \in [0,t]$, *for all* $t \in J$, *and* x_0 *is a real number.* For our convenience, we set

$$f_0 = \sup_{t\in J} |f(t,0,0)| \qquad \text{and} \quad q_0 = \sup_{t\in J} |Q(t)| .$$

By a *solution* of the integral equation (1.3) we mean a function $x : J \to \mathbb{R}$ such that the function $H(t,s)g(s,x(s),x(\eta(s)))$ is integrable on J with respect to s for any $t \in J$ and satisfies (1.3) for all $t \in J$. We note that, as the assumptions on the delay θ imply that $\theta(0) = 0$, it immediately follows that for any solution x of the integral equation (1.3) it will hold

$$x_0 = f\left(0, x_0, x_0\right) Q(0), \tag{1.4}$$

where $x_0 = x(0)$. Let $BM(J,\mathbb{R})$ and $C(J,\mathbb{R})$ denote the space of real valued bounded measurable functions on J and the space of continuous real valued functions defined on J, respectively. Clearly, $C(J,\mathbb{R})$ equipped with the norm

$$\|x-y\| = \sup_{t\in J} |x(t)-y(t)| , \quad x,y \in C(J,\mathbb{R}),$$

becomes a Banach space while $(C(J,\mathbb{R}), \|\cdot\|)$ with the usual multiplication is a Banach algebra. Whenever there is no case of misunderstanding we'll use the same symbol in denoting the usual max-norm of $\mathbb{R}^n$, i.e.,

$$\|(x_1,...,x_n)-(y_1,...,y_n)\| = \max\left\{|x_1-y_1|, ..., |x_n-y_n|\right\}.$$

We seek solutions of the integral equation (1.3) that belong to the space $C(J,\mathbb{R})$.

It is not difficult to see that the integral equation (1.3) includes not only the problems (1.1) and (1.2) as special cases but, also, some i.v.p.'s concerning more general equations. We deal with this matter in Section 3 where we apply our main result to problems (1.2), (1.3) and to some more general problems, and in Section 4 where some discussion on this subject is cited.

In order to state our results, we need some definitions.

Definition. *A function* $f : J \times \mathbb{R}^n \to \mathbb{R}$ *is called* $k-$*Lipschitz if there exists a function* $k \in B(J,\mathbb{R})$ *such that* $k(t) > 0$ *a.e. in* I *and*

$$|f(t,\overline{x}) - f(t,\overline{y})| \le k(t)\,\|\overline{x}-\overline{y}\| , \quad t \in J$$

for all $\overline{x}, \overline{y} \in \mathbb{R}^n$.

Definition: *Let* $(X, \|.\|)$ *be a normed space. An operator* $T : X \to X$ *is called*

(i) *totally bounded if* T *maps bounded subsets of* X *into relatively compact subsets of* X.

(ii) *completely continuous if* T *is totally bounded and continuous.*

Definition. *Let* $(X, \|\cdot\|)$ *be a normed space. A mapping* $T : X \to X$ *is called*

(i) *contraction on* X *if there exists a real constant* $\alpha \in (0,1)$ *such that*

$$\|T(x)-T(y)\| \le \alpha \|x-y\| \quad \text{for any } x,y \in X.$$

(ii) *nonlinear contraction with contraction function* ϕ *if there exists a continuous function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\|Tx - Ty\| \leq \phi(\|x - y\|) \quad \text{for all } x, y \in X \quad \text{and} \quad \phi(r) < r \text{ for all } r > 0.$$

The function ϕ *is called a* $D-$*function for* f *on* X *with contraction function* ϕ.

(iii) $D-$*Lipschitzian if there exists a continuous and nondecreasing function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\|Tx - Ty\| \leq \phi(\|x - y\|) \quad \text{for all } x, y \in X \quad \text{and } \phi(0) = 0.$$

The function ϕ *is called a* $D-$*function of* f *on* X.

Clearly any Lipschitzian mapping is $D-$Lipschitzian and any nonlinear contraction is $D-$Lipschitzian but the converses may not hold (See [1]).

Our results are based on the following theorem by Dhage [2].

Theorem D. *Let* S *be a closed, convex and bounded subset of a Banach algebra* X *be and let* $A : X \to X$ and $B : S \to X$ *be two operators such that*

(a) A *is* $D-$*Lipschitzian with a* $D-$*function* ϕ,

(b) B *is completely continuous, and*

(c) $x = AxBy \Longrightarrow x \in S$, *for all* $y \in S$.

Then the operator equation $AxBx = x$ *has a solution whenever* $M\phi(r) < r$, $r > 0$, *where* $M = \|B(S)\| := \sup\{\|B(x)\| : x \in S\}$.

2. MAIN RESULTS

Before we prove the main result of the paper, we state assumptions (h_1) and (h_2) posed on the functions f and g respectively. These assumptions describe, in a way, the "allowable growth" of the functions f and g that guarantee existence of solutions to equation (1.3). Note that the bound function on f is not assumed to be Lipschitz while the bound functions on g need not posess any kind of monotonicity.

(h_1) *The function* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \phi(\max\{|x_1 - y_1|, |x_2 - y_2|\}) \quad \text{for } (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, t \in J,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and nondecreasing with* $\phi(0) = 0$.

(h_2) *There exist a continuous function* $\Omega : [0, \infty) \times \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\} \to (0, \infty)$, *and a function* $\gamma \in L^1(J, \mathbb{R}^+)$ *such that* $\gamma(t) > 0$ *a.e. on* J *satisfying*

$$|g(t, x, y)| \leq \gamma(t)\Omega(x, y), \quad \text{a.e. on } J \text{ for all } x, y \in [0, \infty) \times \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\}.$$

We note here that, for our convenience, the notation

$$\omega(r) = \sup_{0 \leq y \leq x \leq r} \Omega(r, r),$$

will be used in the rest of the paper without any further mention.

Theorem 1. *Assume that* (h_1) *and* (h_2) *hold. If there exists an* $r > 0$ *such that*

$$\text{(C)} \qquad [\phi(r) + f_0] \sup_{t \in J} \left[|Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds \right] < r,$$

then the integral equation (1.3) *has a solution on* J.

Proof. Let $X = C(J, \mathbb{R})$ and recall that X equipped with the usual sup-norm is a Banach algebra. We define the mapping $A : X \to X$ by

$$Ax(t) = f(t, x(t), x(\vartheta(t))), \quad t \in J. \tag{2.1}$$

Then A is $D-$Lipschitz on X with a $D-$function ϕ. Indeed, in view of (h_1) we have for any $x, y \in X$ and $t \in J$

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(t), x(\vartheta(t))) - f(t, y(t), y(\vartheta(t)))| \\ &\leq \phi(\max\{|x(t) - y(t)|, |x(\vartheta(t)) - y(\vartheta(t))|\}) \\ &= \phi(\|x - y\|) \end{aligned}$$

which immediatley implies that

$$\|Ax - Ay\| \leq \phi(\|x - y\|) \quad \text{for } x, y \in X.$$

Set $S_r = \{x \in X : \|x\| \leq r\}$, where r is a positive real number r satisfying condition (C) Clearly, S_r is a closed, convex and bounded subset of X. We define a mapping $B : S_r \to X$ by

$$Bx(t) = Q(t) + \int_0^t H(t,s) g(s, x(s), x(\eta(s))) ds, \quad t \in J. \tag{2.2}$$

We will show that the operators B and A satisfy (b) and (c) of Theorem D.

The continuity of the operator B is an immediate consequence of the continuity of the functions g, η, ϑ as well as of the continuity of the integral operator on J. We claim that B is completely continuous.

First, let us show that $|B|$ is bounded on S_r by a constant depending on r. Indeed, in view of (h_2), for any x with $\|x\| \leq r$ we have for $t \in J$

$$\begin{aligned} |Bx(t)| &= \left| Q(t) + \int_0^t H(t,s) g(s, x(s), x(\eta(s))) ds \right| \\ &\leq |Q(t)| + \int_0^t |H(t,s)| \, |g(s, x(s), x(\eta(s)))| \, ds \\ &\leq |Q(t)| + \int_0^t |H(t,s)| \gamma(s) \Omega(x(s), x(\eta(s))) ds \\ &\leq |Q(t)| + \int_0^t |H(t,s)| \gamma(s) \sup_{0 \leq y \leq x \leq r} \Omega(x,y) ds \\ &= |Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds \end{aligned}$$

and so, it holds

$$|Bx(t)| \leq |Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds, \quad t \in J.$$

Taking the supremum over t, from the last inequality it follows that

$$\|Bx\| \le R, \quad x \in S_r, \tag{2.3}$$

where we have set $R = \sup_{t\in J}\left[|Q(t)| + \omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right]$. Since the constant R is independent of x, it follows that the operator B is uniformly bounded in S_r.

Now we show that $B(S_r)$ is an equicontinuous subset of X. Let $x \in S_r \subset X$ and $t, \tau \in J$. Without loss of generality we may assume that $t \le \tau$. We have

$$\begin{aligned}
&|Bx(t) - Bx(\tau)| \\
= & \left|Q(t) - Q(\tau) + \int_0^t H(t,s)g(s,x(s),x(\eta(s)))ds - \int_0^\tau H(\tau,s)g(s,x(s),x(\eta(s)))ds\right| \\
\le & |Q(t) - Q(\tau)| + \left|\int_0^t H(t,s)g(s,x(s),x(\eta(s)))ds - \int_0^t H(\tau,s)g(s,x(s),x(\eta(s)))ds\right| \\
& + \left|\int_0^t H(\tau,s)g(s,x(s),x(\eta(s)))ds - \int_0^\tau H(\tau,s)g(s,x(s),x(\eta(s)))ds\right| \\
\le & |Q(t) - Q(\tau)| + \left|\int_0^t \left[H(t,s) - H(\tau,s)\right] g(s,x(s),x(\eta(s)))ds\right| \\
& + \left|\int_\tau^t H(\tau,s)g(s,x(s),x(\eta(s)))ds\right| \\
\le & |Q(t) - Q(\tau)| + \int_0^t |H(t,s) - H(\tau,s)|\,|g(s,x(s),x(\eta(s)))|\,ds \\
& + \int_\tau^t |H(\tau,s)|\,|g(s,x(s),x(\eta(s)))|\,ds \\
\le & |Q(t) - Q(\tau)| + \int_0^t |H(t,s) - H(\tau,s)|\,\gamma(s)\Omega(x(s),x(\eta(s)))ds \\
& + \int_\tau^t |H(\tau,s)|\,\gamma(s)\Omega(x(s),x(\eta(s)))ds \\
\le & |Q(t) - Q(\tau)| + \sup_{0\le s\le t}|H(t,s) - H(\tau,s)| \int_0^t \gamma(s) \sup_{0\le y\le x\le r}\Omega(x,y)ds \\
& + \sup_{(u,s)\in[\tau,t]\times J}|H(u,s)| \int_\tau^t \gamma(s) \sup_{0\le y\le x\le r}\Omega(x,y)ds
\end{aligned}$$

and

$$\begin{aligned}
|Bx(t) - Bx(\tau)| \le |Q(t) - Q(\tau)| &+ \sup_{0\le s\le t}|H(t,s) - H(\tau,s)|\,\omega(r)\left|\int_0^t \gamma(s)ds\right| \\
&+ \sup_{(u,s)\in[\tau,t]\times[\tau,t]}|H(u,s)|\,\omega(r)\int_0^T \gamma(s)ds
\end{aligned}$$

As H is assumed to be continuous on $J \times J$ it follows that

$$\lim_{t\to\tau}\left[\sup_{(u,s)\in[\tau,t]\times J}|H(u,s)|\right] = 0 \quad \text{and} \lim_{t\to\tau}\left[\sup_{0\le s\le t}|H(t,s) - H(\tau,s)|\right] = 0$$

From this fact and the continuity of Q we obtain

$$|Bx(t) - Bx(\tau)| \to 0 \quad \text{as } t \to \tau.$$

Hence $B(S_r)$ is an equicontinuous subset of X, which, in view of the Ascoli-Arzelá theorem implies that $B(X)$ is relatively compact. Consequently, B is a completely continuous operator.

Now we show that if y is an arbitrary element in S_r and x is an element in X for which $x = AxBy$, then $x \in S_r$, i.e.

$$y \in S_r \text{ and } x \in X \text{ with } x = AxBy \Longrightarrow \|x\| \leq r.$$

To this end, let y be an arbitrary function in S_r. Then, for any $x \in X$ with $x = AxBy$ we have for $t \in J$

$$\begin{aligned}
|x(t)| &= |AxBy| \\
&= |f(t, x(t), x(\vartheta(t)))| \left| Q(t) + \int_0^t H(t,s) g(s, y(s), y(\eta(s))) ds \right| \\
&\leq |f(t, x(t), x(\vartheta(t))) - f(t,0,0) + f(t,0,0)| \\
&\quad \times \left(|Q(t)| + \left| \int_0^t H(t,s) g(s, y(s), y(\eta(s))) ds \right| \right) \\
&\leq [\phi(\|x\|) + f_0] \\
&\quad \times \left(|Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds \right) \\
&\leq [\phi(r) + f_0] \sup_{t \in J} \left[|Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds \right]
\end{aligned}$$

which, in view of (C), implies

$$|x(t)| < r \quad \text{for all } t \in J.$$

As the last inequality holds for any $t \in J$, it follows that $\|x\| \leq r$, hence $x \in S_r$. This clearly implies that the operators A and B satisfy (c) of Theorem D.

It remains to show that $M\phi(r) < r$, where $M = \|B(S_r)\| = \sup\{\|B(x)\| : x \in S_r\}$. Indeed, by (2.3), (C) and the nonnegativity of f_0, we have

$$\begin{aligned}
M\phi(r) &\leq R[\phi(r) + f_0] \\
&= [\phi(r) + f_0] \sup_{t \in J} \left[|Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds \right] \\
&< r.
\end{aligned}$$

The proof of our theorem is now completed.■

Before we proceed to our next result, we cite two remarks concerning condition (C).

Remark 1. In the case that $f_0 \neq 0$, i.e., if $f(t,0,0)$ *is not identically zero on* J, then condition (C) may be relaxed by substituting "$<$" by "$\leq$", i.e., by

$$(\widetilde{\mathrm{C}}) \qquad [\phi(r) + f_0] \sup_{t \in J} \left[|Q(t)| + \omega(r) \int_0^t |H(t,s)| \gamma(s) ds \right] \leq r,$$

Note that the strict inequality in (C) is needed so that the condition "$M\phi(r) < r$" in Theorem D is satisfied. Thus, if $f_0 \neq 0$, then (C) may replaced by $(\widetilde{\mathrm{C}})$.

Remark 2. As it concerns the function H, we notice the following:

(i) In case that the function $|H|$ is monotone in its first argument and $|Q|$ has the same type of monotonicity as $|H|$, then (C) may be simplified. For example, if H is nondecreasing in its first argument, i.e., if $|H(t_1, s)| \leq |H(t_2, s)|$ for (t_1, s), (t_2, s) in $\{(t, s) : 0 \leq s \leq t,\ t \in J\}$ with $t_1 \leq t_2$ and $|Q|$ is nondecreasing on J, then (C) becomes

$$(\mathrm{C}^*) \qquad [\phi(r) + f_0]\left[|Q(T)| + \omega(r)\int_0^T |H(T, s)|\,\gamma(s)ds\right] < r,$$

However, if this is not the case, then the difference between the real numbers $\sup_{t\in J}\left[|Q(t)| + \omega(r)\int_0^t |H(t, s)|\,\gamma(s)ds\right]$ and $|Q(T)| + \omega(r)\int_0^T |H(T, s)|\,\gamma(s)ds$ may become too large to be ignored (see, also, the first application in the next section).

(ii) One can easily see that, in fact, there is no need to assume that H is defined on the whole rectangle $J \times J$ but only on $J \times \{(t, s) : 0 \leq s \leq t,\ t \in J\}$.

Now we state three propositions concerning the way that condition (C) may be modified according to the behavior of the functions ϕ and ω at infinity. First we deal with the case that ϕ and ω are unbounded.

Proposition 1. *Assume that* (h_1) *and* (h_2) *hold. Moreover, assume that both functions* ϕ *and* ω *are unbounded. If*

$$(\mathrm{C}_1) \qquad \left[\sup_{t\in J}\int_0^t |H(t, s)|\,\gamma(s)ds\right]\left[\liminf_{u\to\infty}\frac{\phi(u)\omega(u)}{u}\right] < 1,$$

then the integral equation (1.3) *has a solution on* J.

Proof. Assume that none of the functions ϕ and ω is bounded. It suffices to show that (C_1) implies (C). Consider an arbitrary positive number ε. Due to (C_1) we may find a sufficently large r such that

$$(2.4) \qquad \left[\sup_{t\in J}\int_0^t |H(t, s)|\,\gamma(s)ds\right]\phi(r)\omega(r) < r,$$

and

$$f_0 < \varepsilon\phi(r), \qquad q_0 < \varepsilon\omega(r)\sup_{t\in J}\int_0^t |H(t, s)|\,\gamma(s)ds.$$

Then we have

$$\begin{aligned}
& [\phi(r) + f_0]\sup_{t\in J}\left[|Q(t)| + \omega(r)\int_0^t |H(t, s)|\,\gamma(s)ds\right] \\
\leq \quad & [\phi(r) + f_0]\left[q_0 + \omega(r)\sup_{t\in J}\int_0^t |H(t, s)|\,\gamma(s)ds\right] \\
< \quad & [\phi(r) + \varepsilon\phi(r)]\left[\varepsilon\omega(r)\sup_{t\in J}\int_0^t |H(t, s)|\,\gamma(s)ds + \omega(r)\sup_{t\in J}\int_0^t |H(t, s)|\,\gamma(s)ds\right]
\end{aligned}$$

and so

$$[\phi(r)+f_0]\sup_{t\in J}\left[|Q(t)|+\omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right]$$
$$< (1+\varepsilon)^2\,\phi(r)\omega(r)\left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]$$

In view of (2.4), from the last inequality we have

$$[\phi(r)+f_0]\sup_{t\in J}\left[|Q(t)|+\omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right] < (1+\varepsilon)^2 r,$$

which, as ε is arbitrary, implies that (C) holds true.

Our assertion is proved.■

Let us note, here, that in case that the function $\int_0^t |H(t,s)|\,\gamma(s)ds$ is bounded on $[0,\infty)$ (e.g., if it is nonincreasing on $[0,\infty)$), then (C_1) implies that (1.3) has a solution on $[0,T]$ for any $T>0$. We may, also, notice that, if the (nondecreasing) functions ϕ and ω are both unbounded and such that $\liminf_{u\to\infty}\frac{\phi(u)\omega(u)}{u}=0$, then condition ($C_1$) is satisfied for any choice of the initial interval $[0,T]$. The next corollary is an immediate consequence of Proposition 1. We note that, in the sequel, we use the notation $h(u)\sim u^p$ to denote that there exists some $k\in\mathbb{R}$ such that $\lim_{t\to\infty}\frac{h(u)}{u^p}=k$, and the notation $h(\infty)$ to denote the limit $\lim_{u\to\infty}h(u)$.

Corollary 1. *Assume that* (h_1) *and* (h_2) *hold.*

(i) *If both functions ϕ and ω are unbounded and*

$$\liminf_{u\to\infty}\frac{\phi(u)\omega(u)}{u}=0,$$

then the integral equation (1.3) *has a solution on $J=[0,T]$ for any $T>0$.*

(ii) *If*

$$\phi(u)\sim u^p \quad \text{and} \quad \omega(u)\sim u^q \;\; \text{for some } p,q\in(0,1) \;\; \text{with } p+q<1,$$

then the integral equation (1.3) *has a solution on $J=[0,T]$ for any $T>0$.*

Obviously, if the function $\frac{\phi(u)\omega(u)}{u}$, $u>0$ is nondecreasing then $\liminf_{u\to\infty}\frac{\phi(u)\omega(u)}{u}$ may be replaced by $\lim_{u\to\infty}\frac{\phi(u)\omega(u)}{u}$. It is pointed out that it may happen that both functions ϕ and ω be strictly increasing but the function $\frac{\phi(u)\omega(u)}{u}$ not be monotone, still satisfying $\liminf_{u\to\infty}\frac{\phi(u)\omega(u)}{u}=0$.

Proposition 2, below, deals with the case in which at least one of the functions ϕ and ω is bounded. Recalling that the functions ϕ and ω are nondecreasing, as it concerns the boundedness of ϕ and ω there are only three cases to be considered.

Proposition 2. (I) *Assume that $\phi(\infty)=M_1$ and ω is unbounded. If*

$$(C_2) \qquad \left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]\left[\liminf_{u\to\infty}\frac{\omega(u)}{u}\right] < \frac{1}{M_1+f_0},$$

then the integral equation (1.3) *has a solution on* J.

(II) *Assume that* $\omega(\infty) = M_2$ *and* ϕ *is unbounded. If*

$$(C_3) \qquad \left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]\left[\liminf_{u\to\infty}\frac{\phi(u)}{u}\right] < \frac{1}{q + M_2\sup\limits_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds},$$

then the integral equation (1.3) *has a solution on* J.

(III) *If both* ω *and* ϕ *are bounded, then the integral equation* (1.3) *has a solution on* $J = [0,T]$ *for any* $T > 0$.

The proof of Proposition 2 may be easily obtained following the same arguments as those in the proof of Corollary 1 and so it will be omitted.

Proposition 3 refers to the case that at least one of the functions ϕ or ω behaves at ∞ like t^p for some $p \in (0,1)$.

Proposition 3. *Assume that* (h_1) *and* (h_2) *hold. Moreover, asssume that*

$$\phi(u) \sim u^p \qquad or \qquad \omega(u) \sim u^p \ \ for\ some\ p \in (0,1).$$

If (C_1) *holds then the integral equation* (1.3) *has a solution on* J.

Proof. It suffices to prove that, in view of our assumption on the behavior on ϕ or ω, (C_1) implies (C). Let θ be a positive number such that

$$\left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]\liminf_{u\to\infty}\frac{\phi(u)\omega(u)}{u} < \theta < 1$$

and assume that $\phi(u) \sim u^p$ at $+\infty$. By $\phi(u) \sim u^p$ it follows that for any $q \in (0,p)$ it holds $u^q < \phi(u)$ for sufficiently large t hence, by our assumption there exists an arbitrary large u such that

$$u^q\omega(u) \le \phi(u)\omega(u) < u\frac{1}{\sup\limits_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds}$$

and so

$$\omega(u) < \frac{1}{\sup\limits_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds}u^{1-q}$$

or

$$\frac{\omega(u)}{u} < \frac{1}{\sup\limits_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds}u^{-q},$$

from which it follows that

$$\frac{\omega(r)}{r}f_0\left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right] \to 0 \text{ as } r \to \infty.$$

Therefore, in view of $\phi(u) \sim u^p$, we may consider a sufficiently large $r > 0$ such that

$$\frac{\phi(r) + f_0}{r}q_0 < \frac{1-\theta}{3} \quad \text{and} \quad \frac{\omega(r)}{r}f_0\left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right] < \frac{1-\theta}{3}$$

and so

$$
\begin{aligned}
& [\phi(r)+f_0]\sup_{t\in J}\left[|Q(t)|+\omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right] \\
\leq\ & [\phi(r)+f_0]\sup_{t\in J}\left[q_0+\omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right] \\
=\ & [\phi(r)+f_0]\,q_0+\phi(r)\omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds+\omega(r)f_0\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds \\
\leq\ & \frac{1-\theta}{3}r+\theta r+\frac{1-\theta}{3}r \\
=\ & \frac{2+\theta}{3}r \\
<\ & r
\end{aligned}
$$

i.e., (C) is satisfied. The proof for the case $\omega(u)\sim u^p$ for some $p\in(0,1)$ is similar.■

Now we explore condition (h$_2$) a little more. In a way, condition $(\widetilde{C}_2)$ in Theorem 2, below, gives an example of a "suitable" function Ω such that (h$_2$) holds.

Theorem 2. A*ssume that* f *satisfies* (h$_1$) *and*
$(\widetilde{\text{h}}_2)$ *There exists a function* $\gamma\in L^1(J,\mathbb{R}^+)$ *such that for some* $r>0$ *it holds*

$$
(\widetilde{C}_2)\qquad \sup_{0\leq y\leq x\leq r}|g(t,x,y)|<\gamma(t)\left[\frac{r}{\phi(r)+f_0}-q_0\right]\frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds}
$$

for all $t\in J$.

Then the integral equation (1.3) *has a solution on* J.
Note. If $f_0\neq 0$, then "$<$" in $(\widetilde{C}_2)$ may be replaced by "$\leq$".

Proof. We define the operators A and B and consider the real number r and the set S_r as in the proof of Theorem 1. It suffices to prove that the operators A and B satisfy (b) and (c) of Theorem D.

First, let us note that in view of $(\widetilde{\mathrm{h}}_2)$, for any x with $\|x\| \le r$ we have for $t \in J$

$$\begin{aligned}
|Bx(t)| &= \left| Q(t) + \int_0^t H(t,s) g(s, x(s), x(\eta(s))) ds \right| \\
&\le |Q(t)| + \int_0^t |H(t,s)| \, |g(s, x(s), x(\eta(s)))| \, ds \\
&\le |Q(t)| + \int_0^t |H(t,s)| \sup_{0 \le y \le x \le r} |g(s,x,y)| \, ds \\
&< |Q(t)| + \int_0^t |H(t,s)| \, \gamma(s) \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,u)| \, \gamma(u) du} ds \\
&\le |Q(t)| + \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,u)| \, \gamma(u) du} \int_0^t |H(t,s)| \, \gamma(s) ds \\
&\le |Q(t)| + \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \\
&\le \frac{r}{\phi(r) + f_0}
\end{aligned}$$

and so, it holds

$$|Bx(t)| < \frac{r}{\phi(r) + f_0}, \quad t \in J.$$

Since $\frac{r}{\phi(r)+f_0}$ is independent of x, it follows that the operator B is uniformly bounded in S_r.

As in Theorem 1, we can prove that $B(S_r)$ is an equicontinuous subset of X. This, in view of the Ascoli-Arzelá theorem implies that $B(X)$ is relatively compact and so B is a completely continuous operator.

Now we show that if y is an arbitrary element in S_r and x is an element in X for which $x = AxBy$, then $x \in S_r$. For an arbitrary function y in S_r and for any

$x \in X$ with $x = AxBy$ we have for $t \in J$

$$
\begin{aligned}
|x(t)| &= |AxBy| \\
&= |f(t, x(t), x(\vartheta(t)))| \left| Q(t) + \int_0^t H(t,s) g(s, y(s), y(\eta(s))) ds \right| \\
&\leq |f(t, x(t), x(\vartheta(t))) - f(t,0,0) + f(t,0,0)| \\
&\quad \times \left(|Q(t)| + \left| \int_0^t H(t,s) g(s, y(s), y(\eta(s))) ds \right| \right) \\
&\leq [\phi(\|x\|) + f_0] \\
&\quad \times \left(|Q(t)| + \int_0^t |H(t,s)| \gamma(s) \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,u)| \gamma(u) du} ds \right) \\
&\leq [\phi(\|x\|) + f_0] \\
&\quad \times \left(|Q(t)| + \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,u)| \gamma(u) du} \int_0^t |H(t,s)| \gamma(s) ds \right) \\
&\leq [\phi(\|x\|) + f_0] \\
&\quad \times \left(q_0 + \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,u)| \gamma(u) du} \sup_{t \in J} \int_0^t |H(t,s)| \gamma(s) ds \right) \\
&= [\phi(\|x\|) + f_0] \left(q_0 + \frac{r}{\phi(r) + f_0} - q_0 \right) \\
&= [\phi(r) + f_0] \frac{r}{\phi(r) + f_0} \\
&= r
\end{aligned}
$$

i.e.,

$$|x(t)| \leq r \quad \text{for all } t \in J.$$

As the last inequality holds for any $t \in J$, it follows that $\|x\| \leq r$, hence $x \in S_r$. This clearly implies that the operators A and B satisfy (c) of Theorem D.

It remains to show that $M\phi(r) < r$, where $M = \|B(S_r)\| = \sup\{\|B(x)\| : x \in S_r\}$. In view of the definition of B by (2.2), we have for $t \in J$

$$
\begin{aligned}
|Bx(t)| &= \left| Q(t) + \int_0^t H(t,s) g(s, x(s), x(\eta(s))) ds \right| \\
&\leq |Q(t)| + \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,s)| \gamma(s) ds} \int_0^t |H(t,s)| \gamma(s) ds \\
&\leq q_0 + \left[\frac{r}{\phi(r) + f_0} - q_0 \right] \frac{1}{\sup\limits_{t \in J} \int_0^t |H(t,s)| \gamma(s) ds} \sup_{t \in J} \int_0^t |H(t,s)| \gamma(s) ds \\
&= \frac{r}{\phi(r) + f_0},
\end{aligned}
$$

which implies that

$$\|B(x)\| \leq \frac{r}{\phi(r)+f_0} \quad \text{for any } x \in S_r,$$

and so,

$$M = \sup\{\|B(x)\| : x \in S_r\} \leq \frac{r}{\phi(r)+f_0}.$$

Thus,

$$M\phi(r) \leq \frac{r}{\phi(r)+f_0}\phi(r) < r.$$

The proof of our theorem is now completed.■

It is not difficult to see that, in comparison with Theorem 1, what Theorem 2 really states is that condition (h_1) may be replaced by (a more easily verified condition such as) (C_2) thus allowing us to ask only for the existence of one function (namely the function γ) rather than two functions needed in (h_2) (namely the functions H and γ).

Now let us suppose that there exists an $r > 0$ such that it holds

$$(\mathrm{C}_4) \qquad \sup_{0\leq y\leq x\leq r} |g(t_0,x,y)| \leq \left[\frac{r}{\phi(r)+f_0} - q_0\right] \frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,ds},$$

for some $t_0 \in J$.

Let

$$\gamma(t) = \sup_{s\in[0,t]} \left\{ \sup_{0\leq y\leq x\leq r} |g(s,x,y)| \right\}, \quad t \in J.$$

It follows that γ is nondecreasing and continuous on J, hence for $t_0 \in J$ such that (C_4) holds, we have

$$\begin{aligned}
\sup_{0\leq y\leq x\leq r} |g(t_0,x,y)| &\leq \left[\frac{r}{\phi(r)+f_0} - q_0\right] \frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,ds} \\
&\leq \frac{\gamma(t_0)}{\gamma(t)} \left[\frac{r}{\phi(r)+f_0} - q_0\right] \frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,ds} \\
&\leq \gamma(t_0) \left[\frac{r}{\phi(r)+f_0} - q_0\right] \frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(t)ds}
\end{aligned}$$

Consequently, if for some $r > 0$ it holds

$$(\mathrm{C}_5) \qquad \sup_{t\in J}\left\{ \sup_{0\leq y\leq x\leq r} |g(t,x,y)| \right\} \leq \left[\frac{r}{\phi(r)+f_0} - q_0\right] \frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,ds}$$

then (h_2) is always satisfied. We have thus proved the next result.

Theorem 3. A*ssume that* (h_1) *holds. If there exists some* $r > 0$ *such that* (C_5) *is satisfied, then* the *integral equation* (1.3) *has a solution on* J.

Finally we observe that, as the assumption of Theorem D asks that $M\phi(r) < r$, it follows that the most intense "allowable" growth of f is that of ϕ tending to be "linear from below" with $\lim_{r\to\infty} \frac{r}{\phi(r)} = m$ and so $\lim_{r\to\infty} \frac{r}{\phi(r)+f_0} = m$. Clearly, $m > q_0$, is a necessity. It turns out that, either the function g is bounded by $\frac{m-q_0}{\sup_{t\in J}\int_0^t |H(t,s)|ds}$ or an appropriate r has to be sought in the interval $[0, R]$ where $\sup_{0\le y\le x\le R} |g(t,x,y)| < \frac{m-q_0}{\sup_{t\in J}\int_0^t |H(t,s)|ds}$.

As a consequence of the above remarks, Theorem 3 can be restated as Theorem 3* below.

Theorem 3*. A*ssume that* (h$_1$) *holds and the function* $G : [o,\infty) \to [o,\infty)$ *with*

$$G(r) = \sup_{t\in J}\left\{\sup_{0\le y\le x\le r} |g(t,x,y)|\right\}, \quad r \ge 0$$

satisfies

$$\liminf_{r\to\infty}\left[\frac{G(r)}{\frac{r}{\phi(r)+f_0} - q_0}\right] < \frac{1}{\sup_{t\in J}\int_0^t |H(t,s)|\,ds},$$

Then the integral equation (1.3) *has a solution on* J.

3. APPLICATIONS

In this section we apply the main results of the paper to generalize and extend some known existence results for some initial value problems concerning differential as well as integro-differential equations, still relaxing, in some cases, the assumptions placed on f and g. First we deduce results concerning the case where the function f is Lipschitz and then we show in some detail how Theorems 1 and 2 may be applied to a second-order initial value problem concerning differential equations with delays. Finally, we present the results obtained by the application of Theorems 1 and 2 to initial value problems concerning higher order differential or integro-differential equations.

3.1. **The case of a Lipschitz function.** Let us now assume that the function f is Lipschitz with a Lipschitz function k, i.e., assume that

(h$_1^L$) *There exists a (bounded) function* $k : J \to \mathbb{R}^+$ *such that*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le k(t)\,\|(x_1,y_1) - (x_2,y_2)\|$$

for $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$, $t \in J$.

Clearly, the function f is $D-$Lipschitzian with a $D-$function $\phi(r) = Kr$ where $K = \sup_{t\in J}\|k(t)\|$. Theorem 4, below, is an immediate consequence of Theorem 1.

Theorem 4. (*Lipschitz*) *Assume that* (h_1^L) and (h_2) *hold. If there exists an* $r > 0$ *such that*

$$(\mathrm{C}_L) \qquad \left[\|k\| + \frac{f_0}{r}\right] \sup_{t\in J} \left[|Q(t)| + \omega(r) \int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1,$$

then the integral equation (1.3) *has a solution on* J.

Remark 3. Note that (C_L) implies that $\|k\|\,\omega(r)\sup_{t\in J}\left[\int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1$. Hence, in order that ($\mathrm{C}_L$) is satisfied it is necessary that there exists an $r > 0$ such that

$$\omega(r) < \omega_0 = \frac{1}{\|k\| \sup_{t\in J}\left[\int_0^t |H(t,s)|\,\gamma(s)ds\right]}.$$

As ω is nondecreasing, we consider the following two cases.

(i) The function ω is bounded on $[0,\infty)$ by the real number ω_0.

In this case, condition (C_L) may be replaced by

$$(\mathrm{C}'_L) \qquad \|k\| \sup_{t\in J} \left[|Q(t)| + \omega_0 \int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1$$

(ii) The function ω exceeds ω_0 on $[0,\infty)$. Then any appropriate r such that (C_L) holds has to belong in the interval $\left[0, \omega^{-1}(\omega_0)\right]$. Note that if $\omega_0 = \omega(\infty)$, then the left part of (C'_L) becomes

$$\sup_{t\in J}\left[\|k\|\,|Q(t)| + \frac{\int_0^t |H(t,s)|\,\gamma(s)ds}{\sup_{t\in J}\left[\int_0^t |H(t,s)|\,\gamma(s)ds\right]}\right]$$

which is greater than one.

As a consequence of Remark 3 (i), we have the folowing corollary.

Corollary 2. *Assume that* (h_1^L) and (h_2) hold. *If*

$$\lim_{u\to\infty} \omega(u) < \frac{1}{\|k\| \left[\sup_{t\in J} \int_0^t |H(t,s)|\,\gamma(s)ds\right]}$$

then the integral equation (1.3) *has a solution on* J.

Remark 4. Checking whether the requirement $M\phi(r) < r$ (in Theorem D) is satisfied, we see that it suffices to be verified that

$$M\phi(r) = \left\{\sup_{t\in J}\left[|Q(t)| + \omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right]\right\} \|k\|\, r < r$$

i.e., that

$$\|k\| \sup_{t\in J}\left[|Q(t)| + \omega(r)\int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1.$$

The last inequality is a consequence of (C_L). Note that in case that $f(t,0,0)$ is not identically zero on J, then (C_L) may be replaced by (C'_L). Also, observe that the assumption on the function ϕ is to be $D-$Lipschitz, hence it is not required that $\phi(r) < r$. This may allow us to consider Lipschitz functions k with $\|k\| > 1$. For example, if the function ω is bounded by ω_0 and $s = \sup_{t\in J}\left[|Q(t)| + \omega_0 \int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1$, then $\|k\|$ may well belong to $\left(0, \frac{1}{s}\right)$. In particular, condition (C'_L) does not not necessarily require that $\|k\|$ is less than one, i.e., f may not be a contraction. Also, in Corollary 2, $\|k\|$ is allowed to exceed unity provided that $\lim_{u\to\infty}\omega(u)\left[\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1$.

Towards a different direction, we may note that if ω is bounded and the function $\int_0^T |H(T,s)|\,\gamma(s)ds$ tends to zero as $T\to\infty$, then the interval J may be extended arbitrarily.

Corollary 3. *Assume that* (h_1^L) and (h_2) hold. *If*

$$\lim_{T\to\infty}\int_0^T |H(T,s)|\,\gamma(s)ds = 0 \quad \text{and} \quad \lim_{u\to\infty}\omega(u) < \infty$$

then the integral equation (1.3) *has a solution on* $[0,T]$ *for any* $T > 0$.

Example 1. Some suitable functions H and γ so that the first limit in the condition of Corollary 3 is zero may be

$$H(t,s) = \frac{1}{t^3+1}s^{1/2}, \quad 0 \le s \le t \le T$$

and

$$\gamma(s) = s^{1/2}, \quad s \in J.$$

Clearly,

$$\int_0^T |H(T,s)|\,\gamma(s)ds = \int_0^T \frac{1}{T^3+1}sds = \frac{1}{2}\frac{T^2}{T^3+1} \to 0 \quad \text{as } T\to\infty.$$

As an immediate consequence of Theorem 4 we obtain the following corollary.

Corollary 4. *Assume that* (h_1^L) *and* (h_2) *hold.*

(i) *If there exists an* $r > 0$ *such that*

$$(\mathrm{C}_L^0) \qquad [\|k\|\,r + f_0]\left[q_0 + \omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right] < r,$$

then the integral equation (1.3) *has a solution on* J.

(ii) *Suppose that the functions* $|Q|$ *and* $\int_0^t |H(t,s)|\,\gamma(s)ds$ *are nondecreasing on* J. *If there exists an* $r > 0$ *such that*

$$[\|k\|\,r + f_0]\left[q_0 + \omega(r)\int_0^T |H(T,s)|\,\gamma(s)ds\right] < r,$$

then the integral equation (1.3) *has a solution on* J.

Finally, by condition (C_L^0) in Corollary 4 we may, equivalently, take

$$f_0\left[q+\omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right] \le r - \left[q+\omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]\|k\|\,r$$

which leads to

$$\frac{f_0\left[q+\omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]}{1-\|k\|\left[q+\omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right]} < r,$$

provided that

$$\|k\|\left[q+\omega(r)\sup_{t\in J}\int_0^t |H(t,s)|\,\gamma(s)ds\right] < 1.$$

Conditions involving the last two inequalities appear in several papers presenting existence results to initial value problems similar to the one considered here (see, for example, assumption (5.4) in Theorem (5.3) in [5]).

Next we apply the main results of this paper to two initial value problems concerning a second-order differential equation (P_2) and a higher-order differential equation (P_n). Though (P_2) may be regarded as a special case of (P_n), we choose to state the results concerning both, (P_2) and (P_n). Results for the first-order differential initial value problem (P_1) may be obtained from the general case of (P_n).

3.2. **A second-order differential i.v.p..** Consider the initial value problem (P_2)

$$(\mathrm{P}_2)\qquad \begin{cases} \frac{d^2}{dt^2}\left[\frac{x(t)}{f(t,x(t),x(\vartheta(t)))}\right] = g(t,x(t),x(\eta(t))), & t\in J\\ x(0)=x_0, \quad x'(0)=x_1 \end{cases}$$

where f *and* g *are real-valued functions defined on* $J\times\mathbb{R}^2$, *with* $f(t,x,y)\neq 0$ *on* $J\times\mathbb{R}^2$, ϑ, $\eta\in C(J,J)$ *with* $\theta(t)\le t$, $\eta(t)\le t$ *for all* $t\in J$, *and* x_0 *and* x_1 *are real numbers.* Let $AC^2(J,\mathbb{R})$ denote the space of continuous functions whose second derivative exists and is absolutely continuous on J.

Following [6], we say that a function $x\in AC^2(J,\mathbb{R})$ is a *solution* of the (P_2) if the mapping $t\to\left(\frac{x}{f(t,x)}\right)$ is differentiable and the derivative $\left(\frac{x}{f(t,x)}\right)'$ is absolutely continuous on J for all $x\in AC^2(J,\mathbb{R})$ and x satisfies the equation in (P_2) for all $t\in J$ and fulfills the initial value conditions in (P_2).

The next lemma verifies that the initial value problem (P_2) is equivalent to an integrodifferential equation. As it is quite elementary, its proof is omitted stating it only for the sake of completeness.

Lemma 2.1. *Let $g(s, x(s), x(\eta(s))$ be a function in $L^1(J, \mathbb{R})$ for any $x \in AC^2(J, \mathbb{R})$. Then a function x is a solution of the initial value problem* (P_2) *if and only if x satisfies the integral equation*

$$x(t) = [f(t, x(t), x(\vartheta(t)))] \left[c_0 + c_1 t + \int_0^t (t-s) g(s, x(s), x(\eta(s))) ds \right], \quad t \in J. \tag{3.1}$$

where

$$c_0 = \frac{x_0}{f(0, x_0, x_0)} \tag{3.2}$$

and

$$\begin{aligned} c_1 &= \frac{d}{dt} \left[\frac{x(t)}{f(t, x(t), x(\vartheta(t)))} \right]_{t=0} \\ &= \frac{1}{[f(t, x_0, x_0)]^2} \{ x_1 f(0, x_0, x_0) \\ &\quad - [f_1(0, x_0, x_0) + f_2(0, x_0, x_0) x_1 + f_3(0, x_0, x_0) x_1 \vartheta'(0)] \} \end{aligned} \tag{3.3}$$

for $f_1 = \frac{\partial f}{dt}$, $f_2 = \frac{\partial f}{dx}$ *and* $f_3 = \frac{\partial f}{dy}$.

Now we are in a position to apply the main result of the paper to the initial value problem (P_2). In view of Lemma 1 and the fact that

$$\sup_{t \in J} \int_0^t (t-s) \gamma(s) ds = \int_0^T (T-s) \gamma(s) ds,$$

from Theorem 1 for $Q(t) = c_0 + c_1 t$, $t \in [0, T]$ and $H(t, s) = t - s$, $(t, s) \in J^2$, we obtain the following proposition.

Proposition 4. *Assume that* (h_1) *and* (h_2) *hold. If there exists an $r > 0$ such that*

$$[\phi(r) + f_0] \sup_{t \in J} \left[|Q(t)| + \omega(r) \int_0^t (t-s) \gamma(s) ds \right] < r,$$

then the initial value problem (P_2) *has a solution on J.*

In case that f is Lipschitz, as the function $H(t, s) = \int_0^t (t-s) \gamma(s) ds$ is nondecreasing in t, from Corollary 4(ii) we obtain Proposition 5, below.

Proposition 5. *Assume that* (h_1^L) *and* (h_2) *hold. If there exists an $r > 0$ such that*

$$\frac{f_0 \left[\max_{t \in J} |c_0 + c_1 t| + \omega(r) \int_0^T (T-s) \gamma(s) ds \right]}{1 - \|k\| \left[\max_{t \in J} |c_0 + c_1 t| + \omega(r) \int_0^T (T-s) \gamma(s) ds \right]} \leq r$$

then the initial value problem (P_2) *has a solution on J, provided that*

$$\|k\| \left[\max_{t \in J} |c_0 + c_1 t| + \omega(r) \int_0^T (T-s) \gamma(s) ds \right] < 1.$$

3.3. **A higher-order differential i.v.p..** It is not difficult to see that results similar to the ones presented in the previous subsection may be obtained by applying Theorems 1-4 to an initial vlaue problem concerning higher order differential equations with delays. We state only the results obtained by applying Theorems 1 and 4.

Let us consider the initial value problem

$$(\mathrm{P}_n) \qquad \begin{cases} \frac{d^n}{dt^n}\left[\frac{x(t)}{f(t,x(t),x(\vartheta(t)))}\right] = g(t,x(t),x(\eta(t))), \quad t\in J \\ x(0)=x_0, \quad x'(0)=x_1, \, ..., \, x^{(n-1)}(0)=x_{n-1} \end{cases}$$

where f *and* g *are real-valued functions defined on* $J\times\mathbb{R}^2$, *with* $f(t,x,y)\neq 0$ *on* $J\times\mathbb{R}^2$, ϑ, $\eta\in C(J,J)$ *with* $\theta(t)\leq t$, $\eta(t)\leq t$ *for all* $t\in J$, *and* x_0, x_1, ..., x_{n-1} *are real numbers.*

Integrating $n-$times the differential equation in (P_n) on $[0,t]$ for $t\in J$ and taking into consideration the initial values in (P_n), we can see that the initial value problem (P_n) is equivalent to an integral equation of the form

$$\frac{x(t)}{f\left(t,x(t),x(\vartheta(t))\right)} = c_0+c_1t+...+c_{n-1}t^{n-1}+\int_0^t (t-s)^{n-1}g(s,x(s),x(\eta(s)))ds$$

i.e., an integral equation of the form

$$(\mathrm{I}_n) \quad x(t) = f\left(t,x(t),x(\vartheta(t))\right)\times$$

$$\times\left[c_0+c_1t+...+c_{n-1}t^{n-1}+\int_0^t (t-s)^{n-1}g(s,x(s),x(\eta(s)))ds\right], \quad t\in J$$

for suitable real constants $c_0,...,c_{n-1}$ depending on the initial conditions $x_0,...,x_{n-1}$ and the function f.

Clearly (I_n) is an equation of the form (1.3) with $Q(t)=c_0+c_1t+...+c_{n-1}t^{n-1}$, $t\in J$ and $H(t,s)=\frac{1}{n!}(t-s)^{n-1}$, $(t,s)\in J^2$. Note that the consistency condition (1.4) is fulfilled with $\frac{x_0}{f(0,x_0,x_0)}=c_0$. Observing that the function H is nondecreasing in its first argument, from Theorems 1 and 4 we have the following propositions.

Proposition 6. *Assume that* (h_1) *and* (h_2) *hold. If there exists an* $r>0$ *such that*

$$[\phi(r)+f_0]\sup_{t\in J}\left[|Q(t)|+\frac{\omega(r)}{n!}\int_0^t (t-s)^{n-1}\gamma(s)ds\right]<r,$$

then the initial value problem (P_n) *has a solution on* J.

Proposition 7. *Assume that* (h_1^L) *and* (h_2) *hold. If there exists an* $r>0$ *such that*

$$\frac{f_0\left[\sup\limits_{t\in J}\left|\sum\limits_{i=0}^n c_it^i\right|+\frac{\omega(r)}{n!}\int_0^T(T-s)^{n-1}\gamma(s)ds\right]}{1-\|k\|\left[\sup\limits_{t\in J}\left|\sum\limits_{i=0}^n c_it^i\right|+\frac{\omega(r)}{n!}\int_0^T(T-s)^{n-1}\gamma(s)ds\right]}<r$$

then the initial value problem (P_n) *has a solution on* J *provided that*

$$\|k\|\left[\sup_{t\in J}\left|\sum_{i=0}^{n}c_it^i\right|+\frac{\omega(r)}{n!}\int_0^T(T-s)^{n-1}\gamma(s)ds\right]<1.$$

3.4. **Integrodifferential equations.** As already mentioned, the integral equation (1.3) includes initial value problems concerning integrodifferential equations, such as the problems (1.1) and (1.2). For example, integrating the integrodifferential equation in (1.1) and taking into consideration the initial value at 0, we see that (1.1) is included to the integral equation

$$x(t)=f\left(t,x(t)\right)\left[\frac{x_0}{f\left(0,x_0\right)}+\int_0^t(t-s)g(t,x(s))ds\right],\quad t\in J$$

under the additional hypothesis that $f\left(t,y\right)\neq 0$ for $(t,y)\in J\times\mathbb{R}$ (see [5]). Clearly, the last equation can be regarded as a special case of (1.3) by taking $Q(t)=\frac{x_0}{f(0,x_0)}$, $t\in J$ and $H(t,s)=t-s$, $(t,s)\in J^2$. There is no difficulty to see that (1.3) includes initial value problems of the type

$$(\mathrm{ID}_n)\qquad\begin{cases}\frac{d^n}{dt^n}\left[\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right]=\int_0^t H(t,s)g(t,x(s),x(\eta(s)))ds, & t\in J\\ x(0)=x_0,\quad x'(0)=x_1,\ \dots\ ,\ x^{(n-1)}(0)=x_{n-1}\end{cases}$$

where the functions f, H, g, η and θ are subject to the same assumptions as in (1.3). Application of the main results of the paper to the case of such a problem is left to the reader.

4. DISCUSSION

The generality of the integral equation (1.3) allow us to obtain results concerning a great variety of initial value problems ivolving integral, differential as well as integrodifferential equations. For example, a number of existence results can be obtained by applying Theorems 2, 3 and 3* to the ivp (P_n) or Theorems 1-4 to the i.v.p. (ID_n), e.t.c.. It is also noted that our results extend and generalize several known results by considering delayed arguments, thus allowing the functions f and g depend on x not only at t but, also, at some previous time.

On comparison with other results already appeared in the literature, the technique developed in this paper enables us to relax several of the assumptions usually posed in existence results concerning initial value problems closely related to (1.3), such as the ones considered in [2-7]. For example, the requirement that the function f is Lipschitz may be relaxed to assuming that f is $D-$Lipschitzian and the assumptions on the function Ω (which dominates the function g) may also be diminished as several requirements (such as continuity or monotonicity) are taken away by simply considering the supremum of Ω on the triangle limited by the lines $y=x$, $x=r$ and the nonnegative half-axis.

As it concerns the role of the function g, it may be noticed that, in some cases, we can successfully deal with functions that are not necessarily Lipschitz on all of their domain, but may have a convenient Lipschitz-type behavior on neighborhoods of arbitrarily large reals. For the function f, it should be mentioned that even in the case that we assume that f is a $D-$function, we may still have the chance to obtain existence results without the $D-$function ϕ being necessarily sublinear or Lipschitz with constant less than unity. In Theoerm 3^* the functions f and g contribute to an easily verified condition that yields existence of solutions to (1.3). The common action of f and g is revealed through the behavior of the product $\phi\omega$. It is the behavior of $\phi\omega$ at infinity that may annihilate some bounded quantities: indeed, $|Q|$ and f_0 appearing in (C) in Theorem 1 are not present in condition (C_1) in Proposition 1. Note, also, that in Corollary 1 some appropriate behavior of the function $\phi\omega$ at infinity guarantees the existence of solutions to (1.3) on an interval $[0, T]$ for T arbitrarily large.

Finally, we may note that the technique developed in this paper may be applied to integrodifferential equations with a finite number of delays yielding existence results similar to the ones presented in this paper. A question yet to be answered is whether results of some interest may be derived by applying this technique to integrodifferential equations with more general type of delays.

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