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The persistence of elliptic lower dimensional tori with prescribed frequency for Hamiltonian systems

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Abstract. In this paper we consider the persistence of lower dimensional tori of a class of analytic perturbed Hamiltonian system,

$$H = \langle \omega(\xi), I \rangle + \frac{1}{2}\Omega_0 \cdot (u^2 + v^2) + P(\theta, I, z, \bar{z}; \xi)$$

and prove that if the frequencies (ω_0, Ω_0) satisfy some non-resonance condition and the Brouwer degree of the frequency mapping $\omega(\xi)$ at ω_0 is nonzero, then there exists an invariant lower dimensional invariant torus, whose frequencies are a small dilation of ω_0.

Keywords: Hamiltonian system, KAM iteration, invariant tori, non-degeneracy condition.

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1 Introduction

In this paper we consider small perturbations of an analytic Hamiltonian in a normal form

$$N = \langle \omega(\xi), I \rangle + \frac{1}{2}\Omega_0 \cdot (u^2 + v^2),$$

on a phase space

$$(\theta, I, z, \bar{z}) \in \mathcal{P} = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R},$$

where $\mathbb{T}^n$ is the usual n-dimensional torus and the tangential frequencies $\omega(\xi) = (\omega_1, \ldots, \omega_n)$ are parameters dependent on $\xi \in D \subset \mathbb{R}^n$ with D a bounded simply connected open domain. The associated symplectic form is

$$\sum_{j=1}^{n} d\theta_j \wedge dI_j + du \wedge dv.$$

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The Hamiltonian equations of motion of N are

$$\dot{\theta} = \omega(\xi), \qquad \dot{I} = 0, \qquad \dot{u} = \Omega_0 v, \qquad \dot{v} = -\Omega_0 u.$$

Thus for each $\xi \in D$, there exists an invariant n-dimensional torus $\mathbb{T}^n \times \{0\} \times \{0\} \subset \mathbb{R}^{2n} \times \mathbb{R}^2$ with *tangential frequencies* $\omega(\xi)$, which has an elliptic fixed point in the normal uv-space with *normal frequency* Ω_0. These tori are called *lower dimensional invariant tori*, split from resonant ones lying in the resonance zone constituted by both stochastic trajectories and regular orbits. The persistence of lower dimensional invariant tori has been widely studied. See many significant works [3, 4, 9, 11, 12, 14, 22].

The classical KAM theorem [1, 10, 13] asserts that, under Kolmogorov non-degeneracy condition, namely,

$$\det(\partial\omega/\partial p) \neq 0,$$

if the perturbation is sufficiently small, a Cantor family of n-dimensional Lagrangian invariant tori (so-called *maximal dimensional invariant tori*) persists with the frequencies ω satisfying Diophantine conditions:

$$|\langle k, \omega\rangle| \geq \frac{\alpha}{|k|^{\tau}}, \qquad 0 \neq k \in \mathbb{Z}^n.$$

When we consider the persistence of low dimensional invariant tori, the well known first and second Mel'nikov conditions [11, 12] are formulated to deal with the resonance between tangential and normal frequencies. The KAM theorem ensures that a large proportion of lower dimensional invariant tori (in the sense of Lebesgue measure) can survive during sufficiently small perturbations at the cost of removing a series of parameter sets with small measure, which gives rise to the inability of prescribing frequencies.

The classical KAM theorem is extended to the case of Rüssmann's non-degeneracy condition

$$a_1\omega_1(p) + a_2\omega_2(p) + \cdots + a_n\omega_n(p) \not\equiv 0 \quad \text{on } \bar{D}, \tag{1.1}$$

for all $(a_1, a_2, \cdots a_n) \in R^n \setminus \{0\}$. See [2, 6, 16, 17, 18, 21]. However, in the case of Rüssmann's non-degeneracy, generally speaking, we cannot expect any more information on the persistence of both maximal and lower dimensional invariant tori with a given Diophantine frequency vector without adding any other extra condition to the Hamiltonian, since the image of the frequency map may be on a sub-manifold.

Very recently, Sevryuk [20] obtained partial preservation of unperturbed frequencies of maximum invariant torus for perturbed Hamiltonian systems under Rüssmann's non-degeneracy condition, whose proof is based on external parameters and some Diophantine approximations properties.

Similarly, by introducing external parameters and applying the KAM method, Xu and You [23] showed the persistence of maximum invariant torus for a class of nearly integral Hamiltonian systems with a given Diophantine frequency vector $\omega(p_0)$ satisfying $\deg(\omega, D, \omega_0) \neq 0$ without assuming Kolmogorov non-degeneracy condition, just provided the perturbation is sufficiently small. Meanwhile, they also pointed out that, their results could not be generalized to the lower dimensional elliptic case.

In [4], Bourgain showed the persistence of lower dimensional invariant torus $\mathbb{T}^d \times \{0\} \times \{0\} \subseteq \mathbb{R}^{2d} \times \mathbb{R}^{2r}$ under Kolmogorov non-degeneracy condition by combining Nash–Moser type method, introduced and developed by Craig and Wayne [3, 7, 8] and KAM method. Furthermore, the author proved that for a fixed Diophantine frequency ω_0, the perturbed

Hamiltonian system admits a lower dimensional invariant torus, whose frequencies ω_* are a small dilation of ω_0 with the dilation factor λ, that is,

$$\omega_* = \lambda\omega_0, \qquad \lambda \in \mathbb{R},\ \lambda \approx 1,$$

which reveals some interesting dynamical behavior of object motions in the phase space that the frequencies of quasi-periodic motions winding around invariant tori always lie in a fixed direction, being a multiple of a given Diophantine vector.

Motivated by [4, 20, 23], in this paper, we aim at proving the persistence of elliptic lower dimensional invariant tori with prescribed frequencies for small perturbations $H = N + P$ of the Hamiltonian N. To be more precise, we will show that if frequencies (ω_0, Ω_0) satisfy some non-resonant conditions and the Brouwer degree of the frequency mapping $\omega(\xi)$ at ω_0 is nonzero, then there exists a lower dimensional invariant torus, whose frequencies are a small dilation of ω_0.

To present our main theorem quantitatively, we make some preliminaries and introduce some notations.

We first introduce complex conjugate variables

$$z = (u + \mathrm{i}v)/\sqrt{2}, \qquad \bar{z} = (u - \mathrm{i}v)/\sqrt{2}.$$

The corresponding symplectic form and Hamiltonian become $\sum d\theta_i \wedge dI_i + \mathrm{i}dz \wedge d\bar{z}$ and

$$H = \langle \omega(\xi), I \rangle + \Omega_0 \cdot z\bar{z} + P(\theta, I, z, \bar{z}; \xi), \tag{1.2}$$

respectively.

Denote a complex neighborhood of the torus $\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$ by

$$D(s,r) = \{(\theta, I) : |\mathrm{Im}\theta| < s, |I| < r^2, |z| + |\bar{z}| \le r\} \subset \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}.$$

Expand $P(\theta, I, z, \bar{z}; \xi)$ as Fourier series with respect to θ and we have

$$P(\xi; \theta, I) = \sum_{k \in \mathbb{Z}^n} P_k(I, z, \bar{z}; \xi)\, e^{\mathrm{i}\langle k, \theta \rangle}.$$

Define

$$\|P\|_{D(s,r)\times\Pi_\sigma} = \sum_{k,l} \|P_k\|_{r;\sigma}\, e^{s|k|},$$

where $\|P_k\|_{r;\sigma} = \sup_{|I|<r^2, |z|+|\bar{z}|\le r} \sup_{\xi \in \Pi_\sigma} |P_k(I, z, \bar{z}; \xi)|$.

Let

$$\Pi = \{\xi \in D : |\xi - \partial D| \ge \sigma\},$$

where $\sigma > r > 0$ is a small constant, and Π_σ a complex closed neighborhood of Π with the radius σ, that is,

$$\Pi_\sigma = \{\xi \in \mathbb{C}^n : |\xi - \Pi| \le \sigma\}.$$

For $\xi \in \Pi_\sigma$, denote by d the diameter of the image set of $\omega(\xi)$, and a cover of $\omega(\Pi)$ by

$$O = \left(\cup_{\xi\in\Pi} B(\omega(\xi), d)\right) \cap \mathbb{R}^n,$$

where $B(\omega, d) = \{\varpi \in \mathbb{C}^n : |\varpi - \omega| < d\}$. Define a positive constant L, such that $|\omega(\xi)| + 1 \le L$ for all $\xi \in D$.

For integer vectors $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}$ with $|l| \le 2$, we use the notation $|\cdot|$ to denote its $|\cdot|_1$ norm. Set $\mathcal{Z} = \{(k, l) \neq 0,\ |l| \le 2\} \subset \mathbb{Z}^n \times \mathbb{Z}$.

Theorem 1.1. *Suppose that the following Hamiltonian system*

$$H = \langle \omega(\xi), I \rangle + \Omega_0 \cdot z\bar{z} + P(\theta, I, z, \bar{z}; \xi) \tag{1.3}$$

is real analytic on $D(s,r) \times D$. Let $\omega_0 = \omega(\xi_0)$, $\xi_0 \in D$. Suppose that frequencies ω_0 and Ω_0 satisfy the following non-resonance condition:

$$|\langle k, \omega_0 \rangle + l \cdot \Omega_0| \geq \frac{\alpha}{A_k}, \qquad (k,l) \in \mathcal{Z},$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product and $A_k = 1 + |k|^\tau$ with $\tau \geq n+1$; and the Brouwer degree of the frequency mapping ω at ω_0 on D is not zero, i.e.

$$\deg(\omega, D, \omega_0) \neq 0.$$

Then there exists a sufficiently small constant $\epsilon > 0$, such that if $\|P\|_{D(s,r)\times\Pi_\sigma} \leq \epsilon$, (1.3) has an elliptic invariant torus with non-resonant frequencies $(\omega_*, \Omega_*) = (1+\mu_*)(\omega_0, \Omega_0)$, where μ_* is a small dilation depending on ϵ.

Remark 1.2. The above theorem can apply to the following example. Set $\omega(\xi) = \omega_0 + (\xi_1^{2d_1+1}, \ldots, \xi_n^{2d_n+1})$, where $d_1, \ldots, d_n$ are positive integers. Note that $\omega(\xi)$ does not satisfy the Kolmogorov non-degeneracy condition at $\xi = 0$ (only with Rüssmann's non-degeneracy condition satisfied). However, the previous KAM theorem cannot provide any information about the frequencies of invariant tori of perturbed systems. When applying Theorem 1.1, we know that the Hamiltonian system possesses an invariant torus along the prescribed direction ω_0.

Remark 1.3. In fact, the normal frequency Ω_0 of the system (1.3) can depend on the parameter ξ. But we should add certain restriction to the derivative of $\Omega_0(\xi)$ in order to make sure the shift of $\Omega_0(\xi)$ does not affect the Brouwer degree of $\omega(\xi)$ at ω_0. The extra restriction will be determined by the extent of degeneracy of $\omega(\xi)$. Here we do not explore this situation and assume Ω to be a constant.

Remark 1.4. In this paper we aim at the persistence of elliptic lower dimensional invariant tori with one normal frequency. In this case we come across essentially the first Mel'nikov condition, which can be solved by introducing external parameters and Brouwer degree assumption. Once two or more normal frequencies are involved, without any non-degeneracy condition we cannot manage the second Mel'nikov condition and preserve the frequencies at the same time.

2 Proof of the theorems

First we introduce an external parameter vector λ and consider the Hamiltonian

$$H(\theta, I, z, \bar{z}; \xi, \lambda) = \langle \omega(\xi) + \lambda, I \rangle + \Omega_0 \cdot z\bar{z} + P(\theta, I, z, \bar{z}; \xi). \tag{2.1}$$

In what follows we abbreviate $H(\theta, I, z, \bar{z}; \xi, \lambda)$ as $H(\cdot; \xi, \lambda)$. The method of introducing parameter was used in [19, 20] to deal with Rüssmann's non-degeneracy condition and remove degeneracy. The Hamiltonian system (2.1) then corresponds to (1.3) with $\lambda = 0$.

We subsequently give a KAM theorem for (2.1) with parameters (ξ, λ) and obtain an elliptic torus with prearranged frequencies direction. Topology degree theory ensures the existence of certain ξ such that $\lambda(\xi) = 0$, which implies the obtained invariant torus is actually one of the original perturbed Hamiltonian system.

For fixed Ω_0, define

$$\mathcal{O} = \left\{ \omega : |\langle k, \omega \rangle + l \cdot \Omega_0| \geq \frac{\alpha}{A_k}, \quad (k, l) \in \mathcal{Z} \right\}. \tag{2.2}$$

Let $M = \Pi_\sigma \times B(0, 2d+1)$. Then the Hamiltonian $H(\cdot; \xi, \lambda)$ is real analytic on $D(s, r) \times M$.

Theorem 2.1. *There exists a small $\epsilon > 0$ such that if*

$$\|P\|_{D(s,r)\times M} \leq \epsilon,$$

we have a Cantor-like family of analytic curves in M:

$$\Gamma_\varpi = \{(\xi, \lambda(\xi)) : \xi \in \Pi\},$$

which are implicitly determined by the following equation

$$\lambda + \omega(\xi) + g(\xi, \lambda) = (1 + \mu(\xi, \lambda))\varpi,$$

for $\varpi \in \mathcal{O}$, where $g(\xi, \lambda), \mu(\xi, \lambda)$ are C^∞ smooth on M with estimates

$$|g(\xi, \lambda)| \leq \frac{2\epsilon}{r}, \qquad |g_\lambda(\xi, \lambda)| + |g_\xi(\xi, \lambda)| \leq \frac{1}{2},$$

and

$$|\mu(\xi, \lambda) \cdot \Omega_0| \leq \frac{2\epsilon}{r^2}, \qquad |\mu_\lambda(\xi, \lambda)| + |\mu_\xi(\xi, \lambda)| \leq \frac{1}{4L},$$

and a parameterized family of symplectic mappings

$$\Phi(\cdot; \xi, \lambda) \colon D(s/2, r/2) \to D(s, r), \qquad (\xi, \lambda) \in \Gamma = \cup_{\varpi \in \mathcal{O}} \Gamma_\varpi,$$

where Φ is C^∞ smooth in (ξ, λ) on Γ in the sense of Whitney and analytic in $(\theta, I, z, \bar{z})$ on $D(s/2, r/2)$, such that for $(\xi, \lambda) \in \Gamma_\varpi$,

$$H(\cdot; \xi, \lambda) \circ \Phi(\cdot; \xi, \lambda) = N_*(\cdot; \xi, \lambda) + P_*(\cdot; \xi, \lambda),$$

where

$$N_*(\cdot; \xi, \lambda) = \langle \omega_*, I \rangle + \Omega_*(\xi, \lambda) z\bar{z},$$

with tangential frequencies

$$\omega_* = (1 + \mu(\xi, \lambda))\varpi, \qquad \Omega_* = (1 + \mu(\xi, \lambda))\Omega_0,$$

and

$$\partial_I^l \partial_u^p \partial_v^q P_* \big|_{I,u,v=0} = 0, \qquad 2|l| + |p + q| \leq 2.$$

Therefore, (2.1) *possesses an elliptic invariant torus $\Phi(\mathbb{T}^n \times \{0, 0, 0\}; \xi, \lambda)$ with tangential frequencies $\omega_* = (1 + \mu(\xi, \lambda))\varpi$ for each $(\xi, \lambda) \in \Gamma_\varpi$.*

Now we first use Theorem 2.1 to prove Theorem 1.1 and delay the proof of Theorem 2.2 later. In fact, let $\varpi = \omega_0$ and then we have an analytic curve $\Gamma_{\omega_0}\colon \xi \in \Pi \to \lambda(\xi)$, implicitly determined by the following equation

$$\lambda + \omega(\xi) + g(\xi,\lambda) = (1+\mu(\xi,\lambda))\omega_0.$$

The implicit function theorem shows

$$\lambda(\xi) = \omega_0 - \omega(\xi) + \hat{\lambda}(\xi), \qquad \forall \xi \in \Pi.$$

Moreover, if ϵ is sufficiently small, we have

$$|\hat{\lambda}(\xi)| \le \frac{2L}{|\Omega_0|}\cdot\frac{\epsilon}{r^2} \quad \text{and} \quad |\hat{\lambda}_\xi(\xi)| \le \frac{8L}{|\Omega_0|}\cdot\frac{\epsilon}{r^2}.$$

It follows from the assumption that

$$\deg(\omega_0 - \omega, \Pi, 0) \neq 0.$$

Therefore, if ϵ is sufficiently small, we have

$$\deg(\lambda, \Pi, 0) = \deg(\omega_0 - \omega, \Pi, 0) \neq 0.$$

Then there exists $\xi_* \in \Pi$ such that $\lambda(\xi_*) = 0$. The Hamiltonian system (2.1) with $H(\cdot;\xi_*) = H(\cdot;\xi_*,\lambda(\xi_*))$ has an elliptic invariant torus $\Phi(\mathbb{T}^n \times \{0,0,0\};\xi_*,\lambda(\xi_*))$ with tangential frequency $\big(1+\mu(\xi_*,\lambda(\xi_*))\big)\omega_0$.

Below we are to prove Theorem 2.1. In order to verify the Hamiltonian flow on the persisted tori winds along the prearranged direction ϖ, we tend to adjust tangential frequencies $w(\xi,\lambda)$ at each KAM step to guarantee the consistent direction; and for this goal the external parameter λ and internal parameter ξ are varying in decreasing domains.

The KAM iteration scheme mostly follows the classical papers [14, 15]. We also highlight a recent work by Berti and Biasco [5], which deals not only with various, weak small perturbations of elliptic tori to obtain the existence of KAM tori, but can apply to both our circumstance and PDEs with Hamiltonian structure. Consequently, we just provide admissible definition domain for (ξ,λ), and omit the other standard parts of KAM step, as readers can refer to [5, 14, 15] for concrete estimates.

KAM step. The following iteration lemma can be regarded as one KAM step. If the estimates (2.3)–(2.7) and (2.11) hold, then the assumptions **A1** and **A2** hold for H_+ and so the KAM step can be iterated infinitely.

Lemma 2.2 (Iteration lemma)**.** *Consider the following Hamiltonian*

$$H(\cdot;\xi,\lambda) = \langle w(\xi,\lambda), I\rangle + \Omega(\xi,\lambda)z\bar{z} + P(\cdot;\xi,\lambda),$$

where $w(\xi,\lambda) = \omega(\xi) + \lambda + g(\xi,\lambda)$ *and* $\Omega(\xi,\lambda) = \Omega_0 + \mu(\xi,\lambda)\Omega_0$. *Assume:*

(A1) *the Hamiltonian* H *is analytic on* $M \times D(s,r)$ *with* $\|P\|_{M\times D(s,r)} \le \epsilon$;

(A2) *the functions* g *and* μ *satisfy the following estimates:*

$$|g_\lambda(\xi,\lambda)| + |g_\xi(\xi,\lambda)| < \frac{1}{2}, \qquad \forall (\xi,\lambda) \in M, \tag{2.3}$$

$$|\mu(\xi,\lambda)| \le \frac{1}{4} \quad \text{and} \quad |\mu_\lambda(\xi,\lambda)| + |\mu_\xi(\xi,\lambda)| < \frac{1}{4L}, \qquad \forall (\xi,\lambda) \in M. \tag{2.4}$$

For each $\varpi \in \mathcal{O}$, the equation

$$w(\xi,\lambda) = \omega(\xi) + \lambda + g(\xi,\lambda) = (1+\mu(\xi,\lambda))\varpi$$

implicitly defines an analytic mapping

$$\lambda : \xi \in \Pi_\sigma \to \lambda(\xi) \in B(0, 2d+1)$$

such that $\Gamma_\varpi = \{(\xi,\lambda(\xi)) : \xi \in \Pi_\sigma\} \subset M$. Let $e^{-K\rho} = \frac{1}{6}\epsilon^{\frac{1}{2}}$, $h = \frac{\alpha}{4K^{\tau+1}}$ and $\delta = \frac{2}{3}h$. Thus,

$$U(\Gamma_\varpi, \delta) = \left\{(\xi,\lambda') \in \Pi_\sigma \times \mathbb{C}^n : |\lambda' - \lambda(\xi)| \le \delta\right\} \subset M.$$

Suppose

$$\epsilon < \min\left\{2^{-3}c\alpha r\rho^{\tau+n+1}, 2^{-16}c^2\right\}, \tag{2.5}$$

$$\epsilon < (32L)^{-1}|\Omega_0| r^2\delta, \tag{2.6}$$

$$\epsilon^{\frac{1}{2}} < (3c)^{-1}\alpha r\rho^{\tau+n+1}, \tag{2.7}$$

where the constant c is twice the largest constant appearing in the following iterative process and is independent of KAM steps. Set

$$s_+ = s - 5\rho, \qquad \eta^3 = (3c)^{-1}\epsilon^{\frac{1}{2}}, \qquad \rho_+ = \frac{1}{2}\rho, \qquad r_+ = \eta r, \qquad \epsilon_+ = \epsilon^{\frac{3}{2}},$$

where $0 < \rho < s/5$, and

$$M_+ = \left\{(\xi,\lambda') \in \mathbb{C}^n \times \mathbb{C}^n : \ \xi \in \Pi_{\sigma - \frac{1}{2}\delta}, (\xi,\lambda) \in \Gamma, |\lambda' - \lambda| \le \frac{1}{2}\delta\right\}, \tag{2.8}$$

where $\Gamma = \bigcup_{\varpi\in\mathcal{O}} \Gamma_\varpi$. Then for any $(\xi,\lambda) \in M_+$ there exists a symplectic mapping

$$\Phi(\cdot;\xi,\lambda) \colon D(s_+, r_+) \to D(s,r),$$

where Φ is real analytic on $D(s_+,r_+) \times M_+$, such that

$$H_+(\cdot;\xi,\lambda) = H(\cdot;\xi,\lambda)\circ\Phi(\cdot;\xi,\lambda) = \langle w_+(\xi,\lambda), I\rangle + \Omega_+(\xi,\lambda)\cdot z\bar{z} + P_+(\cdot;\xi,\lambda),$$

where

$$w_+(\xi,\lambda) = \omega(\xi) + \lambda + g(\xi,\lambda) + \hat{g}(\xi,\lambda),$$

and

$$\Omega_+(\xi,\lambda) = \Omega_0 + (\mu(\xi,\lambda) + \hat{\mu}(\xi,\lambda))\Omega_0$$

Furthermore, the following estimates hold.

(i) The new perturbation term P_+ satisfies $\|P_+\|_{D(s_+,r_+)\times M_+} \le \epsilon_+$. The mapping Φ has the following estimates:

$$\|W(\Phi - id)\|_{D(s_+,r_+)\times M_+} + \|W(\mathcal{D}\Phi - Id)W^{-1}\|_{D(s_+,r_+)\times M_+} \le \frac{c\epsilon}{\alpha r\rho^{\tau+n+1}},$$

where $\mathcal{D}$ is the differentiation operator with respect to $(\theta, I, z, \bar{z})$ and $W = \mathrm{diag}(\rho^{-1}I_n, r^{-2}I_n, r^{-1}, r^{-1})$ with I_n the n-th order unit matrix.

(ii) $\hat{g}$ *satisfies that*

$$|\hat{g}(\xi,\lambda)| \leq \frac{\epsilon}{r}, \qquad \forall(\xi,\lambda) \in M,$$

and

$$|\hat{g}_\lambda(\xi,\lambda)| + |\hat{g}_\xi(\xi,\lambda)| \leq \frac{2\epsilon}{r\delta}, \qquad \forall(\xi,\lambda) \in M_+;$$

$\hat{\mu}$ *satisfies that*

$$|\hat{\mu}(\xi,\lambda) \cdot \Omega_0| \leq \frac{\epsilon}{r^2}, \qquad \forall(\xi,\lambda) \in M,$$

and

$$|\hat{\mu}_\lambda(\xi,\lambda)| + |\hat{\mu}_\xi(\xi,\lambda)| \leq \frac{2\epsilon}{|\Omega_0| \cdot r^2\delta}, \qquad \forall(\xi,\lambda) \in M_+.$$

The equation

$$w_+(\xi,\lambda) = \omega(\xi) + \lambda + g_+(\xi,\lambda) = (1+\mu_+(\xi,\lambda))\varpi$$

implicitly determines an analytic mapping

$$\lambda_+ : \xi \in \Pi_{\sigma_+} \to \lambda_+(\xi) \in B(0, 2d+1) \quad \textit{with} \quad \sigma_+ = \sigma - \frac{1}{2}\delta,$$

satisfying

$$|\lambda_+(\xi) - \lambda(\xi)| \leq \frac{8L}{|\Omega_0|} \cdot \frac{\epsilon}{r^2} \leq \frac{1}{4}\delta \tag{2.9}$$

and

$$\Gamma^+_\varpi = \{(\xi, \lambda_+(\xi)) : \ \xi \in \Pi_{\sigma_+}\} \subset M_+. \tag{2.10}$$

Let $h_+ = \frac{\alpha}{4K_+^{\tau+1}}$ *and* $\delta_+ = \frac{2}{3}h_+$*, where* K_+ *satisfies* $e^{-K_+\rho_+} = \frac{1}{6}\epsilon^{\frac{1}{2}}$*. If*

$$\delta_+ < \frac{1}{4}\delta, \tag{2.11}$$

then for all $\varpi \in \mathcal{O}$ *we have* $U(\Gamma^+_\varpi, \delta_+) \subset M_+$*.*

Proof of the Iteration lemma. Assumption **(A2)** shows that $w(\xi,\lambda) = (1+\mu(\xi,\lambda))\varpi$ on Γ with $\varpi \in \mathcal{O}$. Noting (2.2) and $\Omega(\xi,\lambda) = (1+\mu(\xi,\lambda))\Omega_0$, then on Γ,

$$\begin{aligned} |\langle k, w(\xi,\lambda)\rangle + l \cdot \Omega(\xi,\lambda)| &= (1+\mu(\xi,\lambda)) \cdot |\langle k, \varpi\rangle + l \cdot \Omega_0| \\ &\geq (1 - |\mu(\xi,\lambda)|) \cdot \frac{\alpha}{A_k} \geq \frac{3}{4} \cdot \frac{\alpha}{A_k} \end{aligned} \tag{2.12}$$

for $(k,l) \in \mathcal{Z}$ and $|k| \leq K$.

Moreover, for $(\xi,\lambda) \in U(\Gamma,\delta)$, there exists $w_0 = (1+\mu(\xi,\lambda))\varpi_0$ with $\varpi_0 \in \mathcal{O}$ such that $|w - w_0| \leq h$. Thus, for $(\xi,\lambda) \in U(\Gamma,\delta)$, $(k,l) \in \mathcal{Z}$ and $|k| \leq K$,

$$\begin{aligned} |\langle k, w(\xi,\lambda)\rangle + l \cdot \Omega(\xi,\lambda)| &\geq |\langle k, w_0\rangle + l \cdot \Omega(\xi,\lambda)| - |k| \cdot |w - w_0| \\ &\geq \frac{3\alpha}{4A_k} - h \cdot |k| \geq \frac{\alpha}{2A_k}, \end{aligned} \tag{2.13}$$

where the last inequality follows from (2.12) and $h = \frac{\alpha}{4K^{\tau+1}}$.

Once the non-resonance condition (2.13) holds, we can simulate the proof of [5, Theorem 5.1] to conduct a detailed KAM step. The relevant estimates here are standard and analogous. The conclusion (i) holds subsequently.

Recall the small shift of frequencies $\hat{g}(\xi,\lambda) = P_{0100}(\xi,\lambda)$ and $\hat{\mu}(\xi,\lambda) = P_{0011}(\xi,\lambda)$; then the estimates of $\hat{g}$ and $\hat{\mu}$ hold obviously. Cauchy estimate also yields the estimates for $\hat{g}_\xi$ and $\hat{g}_\lambda$ in conclusion (ii). Set $g_+ = g + \hat{g}$ and $\mu_+ = \mu + \hat{\mu}$. Define M_+ as in (2.8). It follows from the closeness of the set $\mathcal{O}$ that M_+ is also closed. Note that $\mathrm{dist}(M_+, \partial M) \geq \delta/2$. Cauchy estimate again shows

$$|g_{+\lambda}(\xi,\lambda)| \leq \frac{1}{2}, \qquad |\mu_{+\lambda}(\xi,\lambda)| \leq \frac{1}{4L}, \qquad \forall (\xi,\lambda) \in M_+.$$

Implicit function theorem and (2.6) also imply that the equation

$$w_+(\xi,\lambda) = \omega(\xi) + \lambda + g_+(\xi,\lambda) = (1 + \mu_+(\xi,\lambda))\varpi$$

determines an analytic mapping

$$\lambda_+ : \xi \in \Pi_{\sigma_+} \to \lambda_+(\xi) \in B(0, 2d+1).$$

It is easy to see the estimates (2.9)–(2.10) hold. Inequality (2.11) yields $U(\Gamma^+_\varpi, \delta_+) \subset M_+$. Hence, the conclusion (ii) holds. □

Iteration and convergence. Now we choose some suitable parameters so that the above step can be iterated infinitely. At the initial step, let

$$s_0 = s, \qquad \rho_0 = \frac{1}{20}s, \qquad r_0 = r, \qquad \epsilon_0 = \epsilon, \qquad \sigma_0 = \sigma.$$

Let K_0 and η_0 satisfy $e^{-K_0\rho_0} = \frac{1}{6}\epsilon_0^{\frac{1}{2}}$ and $\eta_0^3 = \frac{1}{3c}\epsilon_0^{\frac{1}{2}}$, respectively. Furthermore, we choose ϵ_0 sufficiently small such that

$$\epsilon_0 \leq \left(2^{12\tau+12n+36}c^6\right)^{-1}, \qquad \epsilon_0 \cdot \left(\ln 6 - \ln \epsilon_0^{\frac{1}{2}}\right)^\tau < \left(2^{10}L\right)^{-1}|\Omega_0|\alpha r_0^2 \rho_0^\tau. \tag{2.14}$$

For $j \geq 0$, we define

$$h_j = \frac{\alpha}{4K_j^\tau}, \qquad \delta_j = \frac{2}{3}h_j, \qquad \sigma_{j+1} = \sigma_j - \frac{1}{2}\delta_j; \tag{2.15}$$

$$\rho_{j+1} = \frac{1}{2}\rho_j, \qquad r_{j+1} = \eta_j r_j, \qquad s_{j+1} = s_j - 5\rho_j; \tag{2.16}$$

$$\epsilon_{j+1} = \epsilon_j^{\frac{3}{2}}, \qquad e^{-K_{j+1}\rho_{j+1}} = \frac{1}{6}\epsilon_{j+1}^{\frac{1}{2}}, \qquad \eta_{j+1}^3 = \frac{1}{3c}\epsilon_{j+1}^{\frac{1}{2}}. \tag{2.17}$$

Then all the above parameters are well defined for j.

For conciseness, we merely provide the details concerning frequencies shift and admissible parameter domains, and recommend readers to refer to [5] for the other estimates.

Let $H_0 = H$ and $M_0 = \Pi_\sigma \times B(0, 2d+1)$. The iteration lemma introduces a monotonous decreasing sequence of closed sets $\{M_j\}$, and a sequence of symplectic mappings $\{\Phi_j(\cdot;\xi,\lambda)\}$ defined on $D(s_{j+1}, r_{j+1})$ for $(\xi,\lambda) \in M_{j+1}$.

Set $\Phi^j = \Phi_0 \circ \cdots \circ \Phi_{j-1}$ with $\Phi^0 = id$, and $H_j = H \circ \Phi^j = N_j + P_j$, where

$$N_j(\cdot;\xi,\lambda) = \langle w_j(\xi,\lambda), I\rangle + \Omega_j(\xi,\lambda) \cdot z\bar{z},$$

with $w_j(\xi,\lambda) = \omega(\xi) + \lambda + g_j(\xi,\lambda)$ and $\Omega_j(\xi,\lambda) = \Omega_0 + \mu_j(\xi,\lambda)\Omega_0$.

The iteration lemma shows, for $\varpi \in O_\alpha$, the equation

$$w_j(\xi,\lambda) = \omega(\xi) + \lambda + h_j(\xi,\lambda) = (1+\mu_j(\xi,\lambda))\varpi$$

on M_j implicitly defines an analytic mapping $\lambda = \lambda_j(\xi), \xi \in \Pi_{\sigma_j}$, whose graph in M_j forms an analytic curve Γ^j_ϖ. Denote by $\Gamma_j = \bigcup_{\varpi\in O_\alpha} \Gamma^j_\varpi$. Recall that

$$M_{j+1} = \left\{ (\xi,\lambda') \in \mathbb{C}^n \times \mathbb{C}^n : \xi \in \Pi_{\sigma_{j+1}}, |\lambda' - \lambda| \le \frac{1}{2}\delta_j, (\xi,\lambda) \in \Gamma_j \right\},$$

which yields $M_{j+1} \subset M_j$ and $\mathrm{dist}(M_{j+1}, \partial M_j) \ge \frac{1}{2}\delta_j$.

Note

$$\hat{g}_j(\xi,\lambda) = w_{j+1}(\xi,\lambda) - w_j(\xi,\lambda) \quad \text{and} \quad \hat{\mu}_j(\xi,\lambda) = \big(\Omega_{j+1}(\xi,\lambda) - \Omega_j(\xi,\lambda)\big)/\Omega_0.$$

Then for $(\xi,\lambda) \in M_j$, we arrive at

$$|\hat{g}_j(\xi,\lambda)| \le \frac{\epsilon_j}{r_j} \quad \text{and} \quad |\hat{\mu}_j(\xi,\lambda)\cdot\Omega_0| \le \frac{\epsilon_j}{r_j^2}.$$

Cauchy estimate shows, for $(\xi,\lambda) \in M_{j+1}$,

$$|\hat{g}_{j\xi}(\xi,\lambda)| + |\hat{g}_{j\lambda}(\xi,\lambda)| \le \frac{2\epsilon_j}{r_j\delta_j} \quad \text{and} \quad |\hat{\mu}_{j\xi}(\xi,\lambda)| + |\hat{\mu}_{j\lambda}(\xi,\lambda)| \le \frac{2}{|\Omega_0|}\cdot\frac{\epsilon_j}{r_j^2\delta_j}.$$

Furthermore, we have

$$|\lambda_{j+1}(\xi) - \lambda_j(\xi)| \le \frac{8L}{|\Omega_0|}\cdot\frac{\epsilon_j}{r_j^2\delta_j}, \qquad \forall (\xi,\lambda) \in M_{j+1}. \tag{2.18}$$

Based on the initial value and induction, it is easy to verify assumptions (2.5)–(2.7) in the iteration process. Noting (2.15)–(2.17) and the above estimates, we are able to verify (2.3), (2.4) and that all the sequences are Cauchy sequences. Hence, the defined variable sequences are ultimately convergent.

Let $D_* = D(0, \frac{1}{2}s)$, $M_* = \bigcap_{j\ge 0} M_j$ and $\sigma_* = \sigma - \frac{1}{2}\sum_{j=0}^{\infty}\delta_j$. Choose ϵ_0 sufficiently small such that $\delta_0 \le \sigma$, and then $\sigma_* \ge \sigma - \frac{2}{3}\delta_0 \ge \frac{1}{3}\sigma$. As a consequence, $\Pi_{\sigma_*} \subset \bigcap_{j\ge 0}\Pi_{\sigma_j}$.

Furthermore, let

$$\Phi = \lim_{j\to\infty}\Phi^j, \qquad \lambda(\xi) = \lim_{j\to\infty}\lambda_j(\xi); \qquad g(\xi,\lambda) = \lim_{j\to\infty} g_j(\xi,\lambda) \quad \text{and} \quad \mu(\xi,\lambda) = \lim_{j\to\infty}\mu_j(\xi,\lambda)$$

respectively, for $\xi \in \Pi_{\sigma_*}$ and $(\xi,\lambda) \in M_*$. Then we have the estimates for $g(\xi,\lambda)$ and $\mu(\xi,\lambda)$ for $(\xi,\lambda) \in M_*$ in Theorem 2.1.

Recall $\Gamma^j_\varpi = \{(\xi,\lambda_j(\xi)) : \xi \in \Pi_{\sigma_j}\} \subset M_j$ and λ_j is analytic on Π_{σ_*}. Then we obtain the analyticity of $\lambda(\xi)$ on Π_{σ_*} and

$$|\lambda(\xi) - \lambda_j(\xi)| \le \frac{\delta_j}{2},$$

by using (2.18). This indicates that

$$\Gamma^*_\varpi = \{(\xi,\lambda(\xi)) : \xi \in \Pi_{\sigma_*}\} \subset M_j \quad \text{and} \quad \Gamma^* = \bigcup_{\varpi\in\mathcal{O}} \Gamma^*_\varpi \subset M_j.$$

Consequently, $\Gamma^* \subset M_* = \bigcap_{j\geq 0} M_j$. For $(\xi,\lambda) \in \Gamma^*_\varpi$,

$$\lambda + \omega(\xi) + g(\xi,\lambda) = (1+\mu(\xi,\lambda))\varpi.$$

Note that M_* is not an open set. Hence, the smoothness of g, μ and P_* with respect to (ξ,λ) on M_* should be understood in the sense of Whitney. Applying Whitney extension theorem [24], we can extend g,μ and P_* to be C^∞ smooth on M, which only makes sense on M_*. Hence, we have completed the proof of Theorem 2.1.

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References

[1] V. I. Arnol'd, Proof of a theorem of A. N. Kolmogorov on the persistence of Quasi-perodic motions under small perturbations of the Hamiltonian (in Russian), *Uspehi Mat. Nauk* **18**(1963), No. 5 (113), 13–40. MR0163025

[2] A. D. Brjuno, Analytic form of differential equations. I, II (in Russian), *Trans. Moscow Math. Soc.* **25**(1971), 119–262. MR0377192

[3] J. Bourgain, Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE, *Internat. Math. Res. Notices* **11**(1994), 475–497. MR1316975; url

[4] J. Bourgain, On Melnikov's persistency problem, *Math. Res. Lett.* **4**(1997), 445–458. MR1470416; url

[5] M. Berti, L. Biasco, Branching of Cantor manifolds of elliptic tori and applications to PDEs, *Comm. Math. Phys.* **305**(2011), No. 3, 741–796. MR2819413; url

[6] C. Q. Cheng, Birkhoff–Kolmogorov–Arnold–Moser tori in convex Hamiltonian systems, *Comm. Math. Phys.* **177**(1996), No. 3, 529–559. MR1385075

[7] W. Craig, C. Wayne, Newton's method and periodic solutions of nonlinear wave equations, *Comm. Pure Appl. Math.* **46**(1993), No. 11, 1409–1501. MR1239318; url

[8] W. Craig, C. Wayne, Periodic solutions of nonlinear Schrödinger equations and the Nash-Moser method, in: *Hamiltonian mechanics (Toruń, 1993), NATO Adv. Sci. Inst. Ser. B Phys.* **331**, Plenum, New York, 1994, 103–122. MR1316671

[9] L. H. Eliasson, Perturbations of stable invariant tori for Hamiltonian systems, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **15**(1988), No. 1, 115–147. MR1001032

[10] A. N. Kolmogorov, Preservation of conditionally periodic motions for a small change in Hamilton's function (in Russian), *Dokl. Akad. Nauk* **98**(1954), 527–530. MR0550888

[11] V. K. MEL'NIKOV, On certain cases of conservation of almost periodic motions with a small change of the Hamiltonian function (in Russian), *Dokl. Akad. Nauk* **165**(1965), 1245–1248. MR0201753

[12] V. K. MEL'NIKOV, A certain family of conditionally periodic solutions of a Hamiltonian system (in Russian), *Dokl. Akad. Nauk* **181**(1968), 546–549. MR0233037

[13] J. MOSER, On invariant curves of area-preserving mappings of an annulus, *Nachr. Akad. Wiss. Gött. Math. Phys. Kl. II* **1962**, 1–20. MR0147741

[14] J. PÖCHEL, On elliptic lower dimensional tori in Hamiltonian systems, *Math. Z.* **202**(1989), 559–608. MR1022821; url

[15] J. PÖCHEL, A lecture on the classical KAM theorem, *Proc. Sympos. Pure Math.* **69**(2001), 707–732. MR1858551

[16] H. RÜSSMANN, Nondegeneracy in the perturbation theory of integrable dynamical systems, *Stochastics, algebra and analysis in classical and quantum dynamics*, Springer Netherlands, 1990, 211–223. MR1052709

[17] H. RÜSSMANN, Invariant tori in non-degenerate nearly integrable Hamiltonian systems, *Regul. Chaotic Dyn.* **6**(2001), 119–204. MR1843664; url

[18] H. RÜSSMANN, Addendum to: "Invariant tori in non-degenerate nearly integrable Hamiltonian systems" [Regul. Chaotic Dyn. **6** (2001), no. 2, 119–204; MR1843664], *Regul. Chaotic Dyn.* **10**(2005), 21–31. MR2136827; url

[19] M. B. SEVRYUK, KAM-stable Hamiltonians, *J. Dyn. Control Syst.* **1**(1995), No. 3, 351–366. MR1354540; url

[20] M. B. SEVRYUK, Partial preservation of frequencies in KAM theory, *Nonlinearity* **19**(2006), No. 5, 1099–1140. MR2221801; url

[21] J. XU, J. YOU, Q. QIU, Invariant tori of nearly integrable Hamiltonian systems with degeneracy, *Math. Z.* **226**(1997), No. 3, 375–386. MR1483538; url

[22] J. XU, J. YOU, Persistence of lower-dimensional tori under the first Melnikov's non-resonance condition, *J. Math. Pures Appl.* **80**(2001), No. 10, 1045–1067. MR1876763; url

[23] J. XU, J. YOU, Persistence of the non-twist torus in nearly integrable Hamiltonian systems, *Proc. Amer. Math. Soc.* **138**(2010), No. 7, 2385–2395. MR2607868; url

[24] H. WHITNEY, Analytical extensions of differentiable functions defined in closed sets, *Trans. Amer. Math. Soc.* **36**(1934), No. 1, 63–89. MR1501735; url