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Discontinuous almost periodic type functions, almost automorphy of solutions of differential equations with discontinuous delay and applications

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Abstract. In this work, using discontinuous almost periodic type functions, exponential dichotomy and the notion of Bi-almost automorphicity we give sufficient conditions to obtain a unique almost automorphic solution of a quasilinear system of differential equations with piecewise constant arguments. Finally, an application to the Lasota–Wazewska model with piecewise constant delayed argument is given.

Keywords: almost automorphic functions, difference equations, differential equation with piecewise constant argument, exponential dichotomy.

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1 Introduction

It is well known that delay differential equations have been successfully applied to diverse models in real life, especially in biology, physics, economics, etc. In 1977, A. D. Myshkis [33] proposed to study differential equations with discontinuous arguments as

$$x'(t) = g(t, x(t), x(h(t))),$$

where h is a piecewise constant deviating function of the form $h(t) = [t]$ or $h(t) = 2[\frac{t+1}{2}]$, with $[\cdot]$ the greatest integer function. Equations of this type are called frequently differential equations with a piecewise constant argument (DEPCA). The first consistent work on DEPCA was initiated in the year 1983 with the works of S. M. Shah and J. Wiener [44], one year later K. L. Cooke and J. Wiener in his work [16] studied DEPCA with delay. DEPCA have been shown to be important by their applications in medical, physical and other sciences (see for instance [4, 9, 15, 19, 31, 50] and some references therein), also in discretization problems [19, 26–29, 50], etc. These are strong reasons why DEPCA have had a huge development, see [13, 14, 20, 34, 35, 39, 40, 42, 48] (and references therein). The research in DEPCA has included qualitative properties of their solutions, like uniqueness, boundedness, periodicity, almost

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periodicity, pseudo almost periodicity, stability, etc. (see [1–4, 10, 34–36, 50, 52–56]). Recently, in 2006 the qualitative study of almost automorphic solutions for a DEPCA was considered [20, 45].

Our main goal in this article is to obtain sufficient conditions establishing the existence of a unique almost automorphic solution on $\mathbb{R}$ for the following DEPCA:

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t, y(t), y([t])), \tag{1.1}$$

where $A(t) \in M_{p\times p}(\mathbb{C})$, $B(t) \in M_{p\times p}(\mathbb{C})$ are almost automorphic matrices and $f \in BC(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p; \mathbb{C}^p)$ is an almost automorphic function which satisfies a condition of Lipschitz type. The study is developed using the discontinuous almost automorphic functions [1, 12], theory of exponential dichotomy [17, 28] and the Banach fixed point theorem.

In the following definition we express what is understood by a solution of a DEPCA.

Definition 1.1. A function $y(t)$ is a solution of the DEPCA (1.1) in the interval I, if this satisfies the following conditions:

i) $y(t)$ is continuous in all I.

ii) $y(t)$ is differentiable in all I, except possibly in the integer numbers $n \in I \cap \mathbb{Z}$ where there should be a lateral derivative.

iii) $y(t)$ satisfies the equation in all the interval $]n, n+1[$, $n \in \mathbb{Z}$ as well as is satisfied by the right hand side derivative in each $n \in \mathbb{Z}$.

We will show the existence of an almost automorphic solution defined on the whole axis $I = \mathbb{R}$.

Almost periodic solutions for the equation (1.1) have been studied in [54], while in [1, 52] pseudo almost periodic solutions for equations with delay which are slightly more general than (1.1). Using spectral theory of functions, T. Dat and N. Van Minh [45] studied, the classical Massera problem: the almost automorphicity of bounded solutions of the following abstract DEPCA:

$$y'(t) = B(t)y([t]) + f(t),$$

where $B(t) = B$ is a constant bounded operator on a general Banach space and f an almost automorphic function, while W. Dimbour in [20] studied the non-autonomous equation, for which $B(t)$ is an almost automorphic operator on a finite dimensional Banach space. Consequently, the study of equation (1.1) in the almost automorphic framework particularly include the equations treated in [20] and [45] in the case of a finite dimensional Banach space and naturally generalizes the work of [54].

While $y(\cdot)$ is an almost automorphic function, $y([\cdot])$ is not, however, its translations over $\mathbb{Z}$: $y([t] + n)$, $n \in \mathbb{Z}$, still have clear almost automorphic properties. Concretely, the function $y([\cdot])$ is a $\mathbb{Z}$-almost automorphic function. $\mathbb{Z}$-almost automorphic functions are discontinuous functions introduced in [12], which generalize the classical continuous almost automorphic ones (see Definition 2.3).

To study equation (1.1), we first pay attention to the linear nonhomogeneous DEPCA

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t) \tag{1.2}$$

on $\mathbb{C}^p$, with $A(\cdot), B(\cdot)$ almost automorphic matrix valued functions and f a $\mathbb{Z}$-almost automorphic function. The need of considering $\mathbb{Z}$-almost automorphic functions appears explicitly, even if f is almost automorphic, in the solution of system (1.2) with A and B triangular, see [1,12].

From the variation of constants formula, that a solution y of (1.2) on $\mathbb{R}$ satisfies, in the interval $[n, n+1[, n \in \mathbb{Z}$, the equation

$$y(t) = \Big(\Phi(t,n) + \int_n^t \Phi(t,u)B(u)\,du\Big)y(n) + \int_n^t \Phi(t,u)f(u)\,du \tag{1.3}$$

holds with $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ and $\Phi(t)$ a fundamental matrix solution of the system

$$x'(t) = A(t)x(t). \tag{1.4}$$

Since the solution y is continuous in $\mathbb{R}$, taking $t \to (n+1)^-$ in equation (1.3), we obtain the difference system

$$y(n+1) = C(n)y(n) + h(n), \quad n \in \mathbb{Z}, \tag{1.5}$$

where

$$C(n) = \Phi(n+1,n) + \int_n^{n+1} \Phi(n+1,u)B(u)\,du, \qquad h(n) = \int_n^{n+1} \Phi(n+1,u)f(u)\,du.$$

Already from (1.3) it is clear that a solution $y = y(t)$ of the DEPCA (1.2) is defined on $\mathbb{R}$ if and only if the matrix

$$I + \int_\tau^t \Phi(\tau,u)B(u)\,du, \tag{1.6}$$

is invertible for $t, \tau \in [n, n+1[$, $n \in \mathbb{Z}$, where I is the identity matrix, see [3,4,37,39]. This implies that the fundamental matrix

$$Z(t,n) = \Phi(t,n) + \int_n^t \Phi(t,u)B(u)\,du,$$

where $t \in [n, n+1[$ and $n \in \mathbb{Z}$, is also invertible and hence

$$C(n) = Z(n+1,n)$$

is invertible too.

Note that the discrete system

$$x(n+1) = C(n)x(n), \quad n \in \mathbb{Z}, \tag{1.7}$$

is obtained from the DEPCA linear system

$$x'(t) = A(t)x(t) + B(t)x([t]). \tag{1.8}$$

Since the discrete solution of (1.5) is the restriction on $\mathbb{Z}$ of the continuous solution for the DEPCA (1.2), both equations are strongly linked, showing the hybrid character of DEPCA. G. Papaschinopoulos has made important contributions to DEPCA [34–36], defining exponential dichotomy for the linear DEPCA system (1.8) when the discrete system (1.7) has it. We will prove that for y bounded, the discrete solution $y(n)$ of equation (1.5) is almost automorphic if and only if the continuous solution $y(t)$ is almost automorphic (Theorems 3.4, 3.6).

For that, we must establish sufficient conditions to obtain an almost automorphic solution to the non-autonomous difference equation (1.5), for which we will prove that the $p \times p$ matrix $C(n)$ and the sequence $h(n)$ are almost automorphics (Lemma 3.3). Then, we find an almost automorphic solution for the nonlinear DEPCA (1.1). Using $\mathbb{Z}$-almost automorphic functions, the problem of solving (1.1) becomes well posed (see [1]) and is more simple and clear (see Theorem 3.8 following and [20, Lemma 3], [45, Lemma 3.3]). As in [28,38,51], kernel functions with a Bi-property are very useful. Here, we will show the local Bi-almost automorphicity in the variables (t,u), with $(t,u) \in I \times I$, $(t,u) \in \mathbb{Z} \times \mathbb{Z}$ and $(t,u) \in \mathbb{Z} \times I$, for $I = [n, n+1[$.

The rest of the paper is organized as follows. In Section 2, we summarize some basic results on $\mathbb{Z}$-almost automorphic functions, discrete almost automorphic equations and some basic definitions which will be useful in the other sections. In Section 3, we study the existence of the almost automorphic solution of the linear non-autonomous DEPCA (1.2) and its extension to (1.1). Finally, in Section 4, we apply our theory to obtain a unique almost automorphic solution to the classical model of Lasota–Wazewska [22,49] with piecewise constant delay.

2 $\mathbb{Z}$-almost automorphic functions and difference equations.

The space of $\mathbb{Z}$-almost automorphic functions was introduced in the paper [12]. Here we recall the definition and some of its fundamental properties. Also we summarize a result on almost automorphic sequence solution of non-autonomous difference equations which is important in the study of DEPCA.

In this paper $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integer and real numbers, respectively, $|\cdot|$ represents any norm on $\mathbb{C}^p$, $\mathbb{X}$ and $\mathbb{Y}$ will be Banach spaces and $BC(\mathbb{Y};\mathbb{X})$ will denote the Banach space of bounded and continuous functions from $\mathbb{Y}$ to $\mathbb{X}$ with the uniform convergence norm.

Definition 2.1. A function $f \in BC(\mathbb{R};\mathbb{X})$ is said to be almost automorphic if given any sequence $\{s'_n\}$ of real numbers, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a function $\tilde{f}$, such that the following pointwise limits holds:

$$\lim_{n\to\infty} f(t+s_n) = \tilde{f}(t), \qquad \lim_{n\to\infty} \tilde{f}(t-s_n) = f(t), \quad t \in \mathbb{R}. \tag{2.1}$$

If in Definition 2.1, the limits are uniform on $\mathbb{R}$ (in which case, (2.1) is reduced to the first limit), f is called almost periodic (in the sense of Bochner). The space of almost automorphic functions is denoted by $AA(\mathbb{R};\mathbb{X})$. Similarly, $AP(\mathbb{R};\mathbb{X})$ denotes the space of the almost periodic functions.

Definition 2.2. A function $f \in BC(\mathbb{R} \times \mathbb{Y};\mathbb{X})$ is said to be almost automorphic in compact subsets of $\mathbb{Y}$, if given any compact set $K \subset \mathbb{Y}$ and a sequence $\{s'_n\}$ of real numbers, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a function $\tilde{f}$, such that the following pointwise limits hold:

$$\lim_{n\to\infty} f(t+s_n, x) = \tilde{f}(t,x), \qquad \lim_{n\to\infty} \tilde{f}(t-s_n, x) = f(t,x), \quad t \in \mathbb{R},\ x \in K.$$

The space of these functions is denoted by $AA(\mathbb{R} \times \mathbb{Y};\mathbb{X})$. The limits in Definition 2.2 are understood as pointwise in $t \in \mathbb{R}$ and uniform on $x \in K$. The spaces $AP(\mathbb{R};\mathbb{X})$, $AA(\mathbb{R};\mathbb{X})$ and $AA(\mathbb{R} \times \mathbb{Y};\mathbb{X})$ become Banach spaces under the uniform convergence norm. Important properties of these functional spaces are exposed in the references [6–8,18,21,23–25,41,46,55].

Let us denote by $B(\mathbb{R};\mathbb{C}^p)$ the Banach space of bounded functions under the uniform convergence norm and consider $BPC(\mathbb{R};\mathbb{C}^p)$ the space of functions in $B(\mathbb{R};\mathbb{C}^p)$, continuous in $\mathbb{R}\backslash\mathbb{Z}$ with finite lateral limits in $\mathbb{Z}$. Note that $BC(\mathbb{R};\mathbb{C}^p) \subset BPC(\mathbb{R};\mathbb{C}^p)$.

Definition 2.3 ([12])**.** A function $f \in BPC(\mathbb{R};\mathbb{C}^p)$ is said to be $\mathbb{Z}$-almost automorphic if for any sequence of integer numbers $\{s'_n\} \subseteq \mathbb{Z}$ there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$ such that the pointwise limits in (2.1) hold.

When the convergence in Definition 2.3 is uniform, f is called $\mathbb{Z}$-almost periodic. We denote the sets of almost automorphic (resp. periodic) functions by $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ (resp. $\mathbb{Z}AP(\mathbb{R};\mathbb{C}^p)$, see [1]). $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ becomes Banach space with the uniform convergence norm, see [12].

Lemma 2.4 ([12])**.** *Let $f \in AA(\mathbb{R}\times\mathbb{C}^p\times\mathbb{C}^p;\mathbb{C}^p)$ and uniformly continuous on compact subsets of $\mathbb{C}^p\times\mathbb{C}^p$, $\psi \in AA(\mathbb{R};\mathbb{C}^p)$, then $f(t,\psi(t),\psi([t])) \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$.*

Lemma 2.5 ([12])**.** *Let f be a continuous $\mathbb{Z}$-almost automorphic function. If f is uniformly continuous on $\mathbb{R}$, then f is almost automorphic.*

Remark 2.6. If we denote the space of continuous periodic functions from $\mathbb{R}$ to $\mathbb{C}^p$ by $P(\mathbb{R};\mathbb{C}^p)$ and the discontinuous ones in $\mathbb{Z}$ by $\mathbb{Z}P(\mathbb{R};\mathbb{C}^p)$, the following diagram of inclusions holds

$$\begin{array}{ccccc} P(\mathbb{R};\mathbb{C}^p) & \longrightarrow & AP(\mathbb{R};\mathbb{C}^p) & \longrightarrow & AA(\mathbb{R};\mathbb{C}^p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}P(\mathbb{R};\mathbb{C}^p) & \longrightarrow & \mathbb{Z}AP(\mathbb{R};\mathbb{C}^p) & \longrightarrow & \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p). \end{array}$$

Note the special meaning of $f([\cdot]) \in \mathbb{Z}P(\mathbb{R};\mathbb{C}^p)$ for $f(\cdot) \in P(\mathbb{R};\mathbb{C}^p)$.

Since DEPCA naturally considers the study of difference equations, we summarize a result for the non-autonomous difference equation (1.5), assuming that $C(n)$ and $h(n)$, $n \in \mathbb{Z}$, are discrete almost automorphic. Previously, we need the following definitions.

Definition 2.7. A function $f\colon \mathbb{Z} \to \mathbb{X}$ is said to be discrete almost automorphic, if for any sequence $\{s'_n\} \subseteq \mathbb{Z}$, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$, such that the following pointwise limits

$$\lim_{n\to+\infty} f(k+s_n) =: \tilde{f}(k), \qquad \lim_{n\to+\infty} \tilde{f}(k-s_n) = f(k), \quad k \in \mathbb{Z}$$

holds.

We denote the vector space of discrete almost automorphic functions by $AA(\mathbb{Z},\mathbb{X})$ which becomes a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ with the sup-norm [5,11,47].

Definition 2.8 ([12])**.** A function $H\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{X}$ is said to be discrete Bi-almost automorphic, if for any sequence $\{s'_n\} \subseteq \mathbb{Z}$, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$, such that we have the following pointwise limits

$$\begin{aligned} &\lim_{n\to+\infty} H(k+s_n, m+s_n) =: \tilde{H}(k,m), \qquad k,m \in \mathbb{Z}, \\ &\lim_{n\to+\infty} \tilde{H}(k-s_n, m-s_n) = H(k,m), \qquad k,m \in \mathbb{Z}. \end{aligned}$$

Since the function matrix $C(n)$, $n \in \mathbb{Z}$, of the equation (1.5) is invertible, we can take $Y(n)$, $n \in \mathbb{Z}$, as an invertible fundamental matrix solution of the discrete system (1.7) and define the following [28,55].

Definition 2.9. The equation (1.7) has an exponential dichotomy with parameters (α, K, P), if there are positive constants α, K and a projection P such that

$$|G(m,n)| \leq Ke^{-\alpha|m-n|}, \quad m,n \in \mathbb{Z},$$

where

$$G(m,n) := \begin{cases} Y(m)PY^{-1}(n), & \text{if } m \geq n, \\ -Y(m)(I-P)Y^{-1}(n), & \text{if } m < n \end{cases}$$

is the discrete Green function. If in addition, G is discrete Bi-almost automorphic, we will say that (1.7) has an (α, K, P)-exponential dichotomy with discrete Bi-almost automorphic Green function.

We obtain the following result.

Theorem 2.10. *Let $h \in AA(\mathbb{Z}, \mathbb{C}^p)$ and suppose that the difference equation (1.7) has an (α,K,P)-exponential dichotomy with discrete Bi-almost automorphic Green function $G(\cdot,\cdot)$. Then the unique almost automorphic solution of (1.5) takes the form:*

$$x(n) = \sum_{k \in \mathbb{Z}} G(n, k+1)h(k). \tag{2.2}$$

Moreover,

$$|x(n)| \leq K(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\|h\|_\infty, \quad n \in \mathbb{Z}.$$

An explicit example of a nonautonomous difference equation with Bi-almost automorphic exponential dichotomy is given and used in Section 4.

3 Almost automorphic solutions for non-autonomous DEPCA.

In this section, we study the almost automorphic solution of the equation (1.1). Firstly, we study the non-homogeneous DEPCA (1.2).

Lemma 3.1. *Let $A(\cdot)$, $B(\cdot)$, $f(\cdot)$ be locally integrable and bounded functions. Then, every bounded solution of (1.2) is uniformly continuous.*

Proof. Let $y(\cdot)$ be a bounded solution of (1.2), since $A(\cdot), B(\cdot)$ and $f(\cdot)$ are also bounded, there is a constant $M_0 > 0$, such that $\sup_{u \in \mathbb{R}} |A(u)y(u) + B(u)y([u]) + f(u)| \leq M_0$. A combination between the continuity of y and the fundamental theorem of calculus gives us

$$|y(t) - y(s)| \leq \left| \int_s^t (A(u)y(u) + B(u)y([u]) + f(u))\, du \right| \leq M_0|t-s|.$$

□

In the rest of the paper, the matrices $A(\cdot)$ and $B(\cdot)$ will be almost automorphics. Then, for $A(\cdot)$, given any sequence $\{s'_n\} \subset \mathbb{R}$ there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a matrix $\tilde{A}(\cdot)$ such that

$$\lim_{n\to\infty} A(t+s_n) =: \tilde{A}(t), \qquad \lim_{n\to\infty} \tilde{A}(t-s_n) = A(t). \tag{3.1}$$

Let Φ be a fundamental matrix solution of the system (1.4) and let Ψ be a fundamental matrix solution of the system

$$\zeta'(t) = \tilde{A}(t)\zeta(t). \tag{3.2}$$

Let us define the functions

$$\Phi(t,s) := \mathit{\Phi}(t)\mathit{\Phi}^{-1}(s), \qquad \Phi_n(t,s) := \Phi(t+s_n, s+s_n).$$

Then, the equation

$$x'(t) = A(t+s_n)x(t) \tag{3.3}$$

has the fundamental matrix solution $\Phi_n(t,0)$ and the equation

$$\zeta'(t) = \tilde{A}(t-s_n)\zeta(t)$$

has the fundamental matrix solution $\Psi_n(t,0)$.

With this notation, we obtain the local Bi-almost automorphicity of $\Phi(t,s)$.

Lemma 3.2. *Let us take a sequence $\{s'_n\} \subset \mathbb{R}$, a positive real number ℓ, and $t,s \in \mathbb{R}$ with $0 < t-s \leq \ell$. Then, there exists a subsequence $\{s_n\} \subseteq \{s'_n\}$, such that (3.1) holds, and:*

a) *There exists a constant $k_0 > 0$, such that for all $n \in \mathbb{N}$* $|\Phi(t,s)| \leq k_0$, $|\Psi(t,s)| \leq k_0$ *and* $|\Phi_n(t,s)| \leq k_0$, $|\Psi_n(t,s)| \leq k_0$.

b) (Bi-almost automorphicity of $\Phi(t,s)$) *For any $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for $n \geq N$, $|\Phi_n(t,s) - \Psi(t,s)| \leq \epsilon k'_0$ and $|\Psi_n(t,s) - \Phi(t,s)| \leq \epsilon k'_0$, for $k'_0 > 0$ a constant.*

Proof. Since $A(\cdot)$ is an almost automorphic matrix, for the sequence $\{s'_n\} \subset \mathbb{R}$, there exist a subsequence $\{s_n\} \subseteq \{s'_n\}$ and a matrix function $\tilde{A}(\cdot)$, such that (3.1) holds. Let $\mathit{\Phi}(\cdot)$ be a fundamental matrix solution of (1.4), then the matrix $\mathit{\Phi}^{-1}(t)$ satisfies $x'(t) = -x(t)A(t)$. Therefore

$$\mathit{\Phi}^{-1}(s) - \mathit{\Phi}^{-1}(t) = \int_s^t \mathit{\Phi}^{-1}(u)A(u)\,du.$$

a) From the last equality, we have

$$\Phi(t,s) = I + \int_s^t \Phi(t,u)A(u)\,du. \tag{3.4}$$

Therefore $|\Phi(t,s)| \leq |I| + \int_s^t |\Phi(t,u)|\,du\|A\|_\infty$, and the Gronwall–Bellman lemma gives us

$$|\Phi(t,s)| \leq |I|e^{(t-s)\|A\|_\infty} \leq |I|e^{\ell\|A\|_\infty} = k_0.$$

The same argument is used with $\Psi(t,s)$, $\Phi_n(t,s)$, and $\Psi_n(t,s)$, for all $n \in \mathbb{Z}$.
b) Similar to (3.4), we get

$$\Psi(t,s) = I + \int_s^t \Psi(t,u)\tilde{A}(u)\,du \qquad \text{and} \qquad \Phi_n(t,s) = I + \int_s^t \Phi_n(t,u)A(u+\xi_n)\,du.$$

Then

$$\begin{aligned}|\Phi_n(t,s) - \Psi(t,s)| &\leq \int_s^t |\Phi_n(t,u)A(u+s_n) - \Psi(t,u)\tilde{A}(u)|\,du\\ &\leq \int_s^t |\Phi_n(t,u) - \Psi(t,u)|\|A\|_\infty\,du\\ &\quad+ \int_s^t |\Psi(t,u)||A(u+s_n) - \tilde{A}(u))|\,du.\end{aligned}$$

Now, due to (3.1) and a), given $\epsilon > 0$, we can take n large enough such that

$$|\Phi_n(t,s) - \Psi(t,s)| \le \ell k_0 \epsilon + \int_s^t |\Phi_n(t,u) - \Psi(t,u)| \|A\|_\infty \, du,$$

from which, the Gronwall–Bellman inequality gives us

$$|\Phi_n(t,s) - \Psi(t,s)| \le k_0' \epsilon, \quad k_0' = \ell k_0 e^{\|A\|_\infty \ell}.$$

With the same argument we can prove that $|\Psi_n(t,s) - \Phi(t,s)| \le k_0' \epsilon$. □

Now, we use the $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times I$ and $I \times \mathbb{Z}$ Bi-almost automorphicity of $\Phi(t,s)$.

Lemma 3.3. *We have:*

a) *The matrix $D(n) = \Phi(n+1, n)$ is discrete almost automorphic.*

b) *For $B \in AA(\mathbb{R}; M_{p\times p}(\mathbb{R}))$ and $f \in \mathbb{Z}AA(\mathbb{R}; \mathbb{C}^p)$; $H(n) = \int_n^{n+1} \Phi(n+1,u)B(u)du$ and $h(n) = \int_n^{n+1} \Phi(n+1,u) f(u)\, du$ are discrete almost automorphics.*

c) *Moreover, the functions*

$$\Phi(t, [t]), \qquad \int_{[t]}^t \Phi(t,u)B(u)\, du, \qquad \int_{[t]}^t \Phi(t,u) f(u)\, du,$$

are $\mathbb{Z}$-almost automorphics.

Proof. Let $\{s_m'\} \subseteq \mathbb{Z}$ be an arbitrary sequence, then there exists a subsequence $\{s_m\} \subseteq \{s_m'\}$ satisfying b) of Lemma 3.2 and

$$\lim_{m\to\infty} f(u + s_m) =: \tilde{f}(u), \qquad \lim_{m\to\infty} \tilde{f}(u - s_m) = f(u), \quad u \in \mathbb{R}. \tag{3.5}$$

a) Note that $D(n + s_m) = \Phi(n + 1 + s_m, n + s_m)$. Consider the sequence $\tilde{D}(n) = \Psi(n+1, n)$, then part b) of Lemma 3.2 implies

$$\lim_{m\to+\infty} D(n + s_m) = \tilde{D}(n).$$

In the same manner, Lemma 3.2 implies that $\lim_{m\to+\infty} \tilde{D}(n - s_m) = D(n)$.
b) We only prove the assertion $h \in AA(\mathbb{Z}; \mathbb{C}^p)$. Note that

$$\begin{aligned} h(n + s_m) &= \int_{n+s_m}^{n+1+s_m} \Phi(n+1+s_m, u) f(u)\, du \\ &= \int_n^{n+1} \Phi(n+1+s_m, u+s_m) f(u+s_m)\, du. \end{aligned}$$

Defining the limit sequence $\tilde{h}(n) = \int_n^{n+1} \Psi(n+1,u)\tilde{f}(u)\, du$, due to part b) of Lemma 3.2 and (3.5) we have

$$\lim_{m\to+\infty} h(n + s_m) = \tilde{h}(n).$$

Analogously, $\lim_{m\to+\infty} \tilde{h}(n - s_m) = h(n)$.
c) This statement follows in a similar way. □

From condition (1.6), the solutions of the DEPCA (1.2) are defined on $\mathbb{R}$.

Theorem 3.4. *Let $A(t), B(t)$ be almost automorphic matrices. Let $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and $y(t)$ be a bounded solution of (1.2), then $y(t)$ is almost automorphic if and only if $y(n)$ is discrete almost automorphic.*

Proof. We note that, if the bounded solution $y(t)$ is almost automorphic, its restriction to $\mathbb{Z}$ is discrete almost automorphic. Now we prove the other implication. Since $y(t)$ is a bounded solution of (1.2), it is uniformly continuous (see Lemma 3.1) and hence due to Lemma 2.5, it will be almost automorphic if it is $\mathbb{Z}$-almost automorphic. Then we must prove that the almost automorphicity of $\{y(n)\}_{n\in\mathbb{N}}$ implies that y is in $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$. First we have that $y([\cdot])$ is $\mathbb{Z}$-almost automorphic. On the other hand, due to (1.3), the solution $y(t)$ of (1.2) satisfies

$$y(t) = \left(\Phi(t,[t]) + \int_{[t]}^{t} \Phi(t,u)B(u)\,du\right) y([t]) + \int_{[t]}^{t} \Phi(t,u)f(u)\,du. \tag{3.6}$$

Moreover, from Lemma 3.3, every term on the right-hand side of (3.6) is $\mathbb{Z}$-almost automorphic. Then y is $\mathbb{Z}$-almost automorphic. □

Remark 3.5. The notion of $\mathbb{Z}$-almost automorphic function has simplified very much the proof of this theorem as can be seen in [20, Lemma 3] and [45, Lemma 3.3]. Obviously, Theorem 3.4 can be extended to $\mathbb{Z}$-almost automorphic matrices $A(\cdot), B(\cdot)$.

Theorem 3.6. *Let $A(t), B(t)$ be almost automorphic matrices, $f \in \mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$ and suppose that (1.7) has a Bi-almost automorphic exponential dichotomy. Then (1.2) has a unique almost automorphic solution.*

Proof. By using the variation of constants formula, we know that a solution $y(t)$ of (1.2) satisfies the expression (3.6) on $[n, n+1[$, and also, for $t = n$, the difference equation (1.5). Since the discrete equation (1.7) has an exponential dichotomy with discrete Bi-almost automorphic Green function, Theorem 2.10 guarantees that (1.5) has a unique bounded solution $y(n)$, $n \in \mathbb{Z}$, which is discrete almost automorphic. Therefore, from Theorem 3.4, $y(t)$ is almost automorphic. Suppose that there exists another solution, say $y_1(t)$, of (1.2) then y_1 satisfies (1.5); therefore for all $n \in \mathbb{Z}$ we have $y_1(n) = y(n)$, from that and the integral representation (3.6) we conclude that the solutions y and y_1 coincide in the real line. □

For the final statements of this section, we will say that $f \in BC(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p;\mathbb{C}^p)$ is M-Lipschitz, if there exists a positive constant M such that

$$|f(t,x,y) - f(t,z,w)| \le M(|x-y| + |z-w|), \quad \forall t \in \mathbb{R}, \ \ \forall (x,y),(z,w) \in \mathbb{C}^p \times \mathbb{C}^p.$$

Lemma 3.7. *Let $A(t), B(t)$ be almost automorphic matrix functions, $f \in AA(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p;\mathbb{C}^p)$ be M-Lipschitz and ψ a $\mathbb{Z}$-almost automorphic function. Then the sequence*

$$\int_{n}^{n+1} \Phi(n+1,u) f(u,\psi(u),\psi(n))\,du$$

is discrete almost automorphic.

Proof. From Lemma 3.3, it is sufficient to prove that the function $g_\psi \colon t \to f(t,\psi(t),\psi([t]))$ belongs to $\mathbb{Z}AA(\mathbb{R};\mathbb{C}^p)$; which is a consequence of Lemma 2.4. □

Theorem 3.8. *Let $A(t), B(t)$ be almost automorphic matrices, $f \in AA(\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p;\mathbb{C}^p)$ be M-Lipschitz. Suppose, in addition, that (1.7) has a Bi-almost automorphic exponential dichotomy with parameters (α,K,P). Then there exists $M^* > 0$ such that if $0 < M < M^*$ the equation (1.1) has a unique almost automorphic solution.*

Proof. Let $\psi \in AA(\mathbb{R};\mathbb{C}^p)$ and consider the differential equation

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t,\psi(t),\psi([t])). \tag{3.7}$$

Note that the function $f(t,\psi(t),\psi([t]))$ is not necessarily almost automorphic, but $\mathbb{Z}$-almost automorphic, due to Lemma 2.4. Theorem 3.6 implies that (3.7) has a unique almost automorphic solution y_ψ. Moreover, we know that for $t \in [n,n+1[,\ n \in \mathbb{Z}$, y_ψ satisfies

$$y_\psi(t) = \Big(\Phi(t,n) + \int_n^t \Phi(t,u)B(u)\,du\Big)y_\psi(n) + \int_n^t \Phi(t,u)f(u,\psi(u),\psi(n))\,du,$$

where $y_\psi(n)$ is the unique discrete almost automorphic solution of the difference equation

$$y_\psi(n+1) = C(n)y_\psi(n) + h_\psi(n), \quad n \in \mathbb{Z}, \tag{3.8}$$

with

$$\begin{aligned} C(n) &= \Phi(n+1,n) + \int_n^{n+1} \Phi(n+1,u)B(u)\,du, \\ h_\psi(n) &= \int_n^{n+1} \Phi(n+1,u)f(u,\psi(u),\psi(n))\,du. \end{aligned}$$

From Theorem 2.10, the unique discrete almost automorphic solution $y_\psi(n)$ verifies the estimate

$$|y_\psi(n)| \le K(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\|h_\psi\|_\infty, \quad \forall n \in \mathbb{Z}. \tag{3.9}$$

Consider $S\colon AA(\mathbb{R};\mathbb{C}^p) \to AA(\mathbb{R};\mathbb{C}^p)$ the operator defined by

$$(S\psi)(t) = y_\psi(t).$$

From Theorem 3.6, this is a well defined operator, since for each $\psi \in AA(\mathbb{R};\mathbb{C}^p)$, $S\psi$ is the unique almost automorphic solution of (3.7). Condition (1.6) allows the existence of $S\psi$ on $\mathbb{R}$. Since f is M-Lipschitz, given $\psi_1,\psi_2 \in AA(\mathbb{R};\mathbb{C}^p)$, from (3.9) we obtain

$$\begin{aligned} \|y_{\psi_1} - y_{\psi_2}\|_\infty &\le K(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\|h_{\psi_1} - h_{\psi_2}\|_\infty \\ &\le 2k_0KM(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\|\psi_1 - \psi_2\|_\infty. \end{aligned}$$

This permits us to have, for $t \in [n,n+1[,\ n \in \mathbb{Z}$, the estimate

$$\begin{aligned} |S\psi_1(t) - S\psi_2(t)| \le{}& \left|\left(\Phi(t,n) + \int_n^t \Phi(t,u)B(u)\,du\right)\left(y_{\psi_1}(n) - y_{\psi_2}(n)\right)\right| + \\ &+ \left|\int_n^t \Phi(t,u)(f(u,\psi_1(u),\psi_1(n)) - f(u,\psi_2(u),\psi_2(n)))\,du\right| \\ \le{}& (k_0 + \|B\|_\infty k_0)|y_{\psi_1}(n) - y_{\psi_2}(n)| \\ &+ k_0M\int_n^t (|\psi_1(u) - \psi_2(u)| + |\psi_1(n) - \psi_2(n)|)\,du \\ \le{}& (1+\|B\|_\infty)k_0\|y_{\psi_1} - y_{\psi_2}\|_\infty + 2k_0M\|\psi_1 - \psi_2\|_\infty \\ \le{}& 2k_0^2KM(1+\|B\|_\infty)(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\|\psi_1 - \psi_2\|_\infty + 2k_0M\|\psi_1 - \psi_2\|_\infty \\ \le{}& \left(2k_0^2K(1+\|B\|_\infty)(1+e^{-\alpha})(1-e^{-\alpha})^{-1} + 2k_0\right)M\|\psi_1 - \psi_2\|_\infty. \end{aligned}$$

Therefore, taking

$$M^* = \frac{1}{2k_0^2 K(1 + ||B||_\infty)(1 + e^{-\alpha})(1 - e^{-\alpha})^{-1} + 2k_0},$$

for every $M \in]0, M^*[$, the operator S is contractive and the conclusion follows from the Banach fixed point theorem. □

The use of $\mathbb{Z}$-almost automorphicity has allowed the well-posedness of DEPCA (1.1) an also has simplified the treatment and the proof of almost automorphicity of the solution y (see [20, Lemma 3] and [45, Lemma 3.3]).

4 The Lasota–Wazewska model with piecewise constant delay

The Lasota–Wazewska model is an autonomous differential equation of the form

$$y'(t) = -\delta y(t) + pe^{-\gamma y(t-\tau)}, \quad t \geq 0. \tag{4.1}$$

It was discovered by Wazewska-Czyzewska and Lasota [49] and is used to describe the survival of red blood cells in the blood of an animal. In this equation, $y(t)$ describes the number of red blood cells in the time t, $\delta > 0$ is the probability of death of a red blood cell; p, γ are positive constants related with the production of red blood cells per unity of time and τ is the time required to produce a red blood cell.

In this section, we study the following model with piecewise constant argument:

$$y'(t) = -\delta(t)y(t) + p(t)f(y([t])), \tag{4.2}$$

where $\delta(\cdot)$, $p(\cdot)$ are positive almost automorphic functions, $0 < \delta_- = \inf_{s\in\mathbb{R}} \delta(s)$ and $f(\cdot)$ is a positive γ-Lipschitz function. Equation (4.2) is used to model several situations in real life [26,27,32] and for $f(y) = e^{-\gamma y}$, (4.2) represents a piecewise constant argument version of Lasota–Wazewska model [22], see [30].

The principal goal is the following theorem.

Theorem 4.1. *In the above conditions, for γ sufficiently small, equation (4.2) has a unique almost automorphic solution.*

Let $\psi(t)$ be a real almost automorphic function and consider the equation

$$y'(t) = -\delta(t)y(t) + p(t)f(\psi([t])). \tag{4.3}$$

Then, in the interval $[n, n+1[, n \in \mathbb{N}$, the solution for the equation (4.3) satisfies

$$y(t) = \exp\left(-\int_n^t \delta(s)\, ds\right) y(n) + f(\psi(n)) \int_n^t \exp\left(-\int_u^t \delta(s)\, ds\right) p(u)du.$$

Due to continuity of the solution, if $t \to (n+1)^-$ we obtain the difference equation

$$y(n+1) = C(n)y(n) + g(n, \psi(n)), \tag{4.4}$$

where

$$C(n) := \exp\left(-\int_n^{n+1} \delta(s)\, ds\right),$$
$$g(n, \psi(n)) := f(\psi(n)) \int_n^{n+1} \exp\left(-\int_u^{n+1} \delta(s)\, ds\right) p(u)\, du.$$

Lemma 4.2. *The equation* (4.4) *is discrete almost automorphic.*

Proof. Since the function f is continuous, the composite $f(\psi(n))$ is discrete almost automorphic. From Lemma 3.3, it follows that $C(n)$ and

$$\int_n^{n+1} \exp\left(-\int_u^{n+1} \delta(s)\, ds\right) p(u)\, du$$

are discrete almost automorphics and $g(n, \psi(n))$ too. The lemma holds. □

Lemma 4.3. *The equation* (4.4) *has a unique discrete almost automorphic solution.*

Proof. Since $\delta_- > 0$, the homogeneous equation associated to (4.4) has an exponential dichotomy, hence its bounded solution is

$$y_\psi(n) = \sum_{k=-\infty}^{n} G(n, k+1) g(k, \psi(k)),$$

where G is the associated discrete Green function:

$$G(n, k+1) := \prod_{j=k+1}^{n} C(j) = \prod_{j=k+1}^{n} \exp\left(-\int_j^{j+1} \delta(s) ds\right) = \exp\left(-\int_{k+1}^{n+1} \delta(s) ds\right).$$

According to Theorem 2.10, to prove that y_ψ is almost automorphic, we only need to verify that the Green function is discrete Bi-almost automorphic. In fact, let $\{\xi_i'\}$ be an arbitrary sequence of integer numbers, since $\delta(\cdot)$ is almost automorphic, there exist a subsequence $\{\xi_i\} \subseteq \{\xi_i'\}$ and a function $\tilde{\delta}$ such that the following pointwise limits hold:

$$\lim_{i \to +\infty} \delta(s + \xi_i) = \tilde{\delta}(s), \qquad \lim_{i \to +\infty} \tilde{\delta}(s - \xi_i) = \delta(s), \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} G(n + \xi_i, k + 1 + \xi_i) &= \exp\left(-\int_{k+1+\xi_i}^{n+1+\xi_i} \delta(s)\, ds\right) \\ &= \exp\left(-\int_{k+1}^{n+1} \delta(s + \xi_i)\, ds\right). \end{aligned}$$

From the Lebesgue dominated convergence theorem, we obtain

$$\lim_{i \to +\infty} G(n + \xi_i, k + 1 + \xi_i) = \exp\left(-\int_{k+1}^{n+1} \tilde{\delta}(s)\, ds\right) =: \tilde{G}(n, k+1).$$

The proof of the limit $\lim_{i \to +\infty} \tilde{G}(n - \xi_i, k + 1 - \xi_i) = G(n, k+1)$ follows in a similar way. □

Following Theorem 3.4, we obtain

Lemma 4.4. *Let $y(\cdot)$ be a bounded solution of equation* (4.3). *Then $y(\cdot)$ is almost automorphic if and only if the sequence $y(n)$ is discrete almost automorphic.*

Now we can conclude Theorem 4.1 with the same arguments used in Theorem 3.8.

The final statement of this section involves in equation (4.2) the explicit function $f(y) = e^{-\gamma y}$, $\gamma > 0$.

Corollary 4.5. *Let γ be small enough. Then, the piecewise constant delayed Lasota–Wazewska model:*

$$y'(t) = -\delta(t) y(t) + p(t) e^{-\gamma y([t])},$$

has a unique almost automorphic solution.

The above results can be extended for δ and p $\mathbb{Z}$-almost automorphic functions.

5 Final observation

It is not obvious to extend the exponential dichotomy for the difference equation (1.7) for the DEPCAG (1.8). We could consider an intuitively direct definition given by the existence of a projection Π_* and positive constants M and α such that

$$\begin{aligned} |Z(t,t_0)\Pi_* Z(s,t_0)^{-1}| &\le M e^{-\alpha(t-s)}, && \text{if } t \ge s \\ |Z(t,t_0)(I-\Pi_*)Z(s,t_0)^{-1}| &\le M e^{\alpha(t-s)}, && \text{if } t \le s. \end{aligned} \tag{5.1}$$

However, if we take $A(t) = 0$, and $B(t) = \operatorname{diag}(\lambda_1(t), \lambda_2(t))$, where

$$\lambda_1(t) = -\frac{2}{\pi} + \sin(2\pi t),$$

and $\lambda_2(t) = -\lambda_1(t)$, then

$$\int_{[t]}^{t} \lambda_1(\xi)\, d\xi = -\frac{1}{2\pi}\left(4(t-[t]) - 1 + \cos(2\pi(t-[t]))\right)$$

and

$$\int_{[t]}^{t} \lambda_2(\xi)\, d\xi = \frac{1}{2\pi}\left(4(t-[t]) - 1 + \cos(2\pi(t-[t]))\right).$$

So, the exponential dichotomy on the difference equation (1.7) which can be written as given in Definition 2.9 is satisfied for $\Pi = \operatorname{diag}(1,0)$ but there is no Π_* such that condition (5.1) is satisfied. Indeed, when we take $t - [t] < \frac{1}{2}$ then

$$\int_{[t]}^{t} \lambda_1(\xi)\, d\xi > 0$$

and when we take $\frac{1}{2} < t - [t] < 1$ the sign of $\int_{[t]}^{t} \lambda_1(\xi)\, d\xi$ changes. The same thing but with contrary sign happens to $\int_{[t]}^{t} \lambda_2(\xi)\, d\xi$. Moreover,

$$\int_{[t]}^{t} \lambda_1(\xi)\, d\xi = \int_{[t]}^{t} \lambda_2(\xi)\, d\xi = 0,$$

if $t - [t] = \frac{1}{2}$.

Notice that a dichotomy condition on the ordinary differential equation (1.4) implies an exponential dichotomy on the difference equation (1.7) [34, Proposition 2] when $|B(t)|$ is small enough. However, an exponential dichotomy for the difference equation on (1.7) is not a necessary condition for an exponential dichotomy for the ordinary differential system (1.4). In fact, let us consider, $A(t) = 0$ and $B(t) = \operatorname{diag}\left(-\frac{3}{2}, \frac{1}{2}\right)$. Then the exponential dichotomy for difference system (1.7) is satisfied, with no exponential dichotomy for the ordinary differential system (1.4).

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