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Infinitely many weak solutions for a mixed boundary value system with $(p_1, \ldots, p_m)$-Laplacian

Diego Averna[✉] and **Elisabetta Tornatore**

Dipartimento di Matematica e Informatica, Università degli Studi di Palermo,
Via Archirafi 34, Palermo, 90123, Italy

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Abstract. The aim of this paper is to prove the existence of infinitely many weak solutions for a mixed boundary value system with $(p_1, \ldots, p_m)$-Laplacian. The approach is based on variational methods.

Keywords: critical points, variational methods, infinitely many solutions, p-Laplacian.

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1 Introduction

The aim of this paper is to establish the existence of infinitely many weak solutions for the following mixed boundary value system with $(p_1, \ldots, p_m)$-Laplacian.

$$\begin{cases} -(|u_1'|^{p_1-2}u_1')' = \lambda F_{u_1}(t, u_1, \ldots, u_m) & \text{in }]0,1[\\ \quad \vdots \\ -(|u_m'|^{p_m-2}u_m')' = \lambda F_{u_m}(t, u_1, \ldots, u_m) & \text{in }]0,1[\\ u_i(0) = u_i'(1) = 0 & i = 1, \ldots, m \end{cases} \tag{1.1}$$

where $m \geq 2$, $p_i > 1$ $(1 \leq i \leq m)$, λ is a positive real parameter, $F\colon [0,1] \times \mathbb{R}^m \to \mathbb{R}$ is a C^1-Carathéodory function such that $F(t, 0, \ldots, 0) = 0$ for every $t \in [0,1]$ and moreover we suppose that for every $\rho > 0$

$$\sup_{|(x_1,\ldots,x_m)| \leq \rho} |F_{u_i}(t, x_1, \ldots, x_m)| \in L^1([0,1]), \qquad i = 1, \ldots, m.$$

Here F_{u_i} denotes the partial derivatives of F respect on u_i $(i = 1, \ldots, m)$.

Among the papers which have dealt with the nonlinear mixed boundary value problems we cite [1,3,10,13].

We investigate the existence of infinitely many weak solutions for system (1.1) by using Theorem 1.1. This theorem is a refinement, due to Bonanno and Molica Bisci, of the variational principle of Ricceri [12, Theorem 2.5] and represents a smooth version of an infinitely many critical point theorem obtained in [5, Theorem 2.1].

✉ Corresponding author. Email: diego.averna@unipa.it

Theorem 1.1. *Let X be a reflexive Banach space, $\Phi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional, $\Psi\colon X \to \mathbb{R}$ is sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional, λ is a positive real parameter.*

Put, for each $r > \inf_X \Phi$

$$\begin{gathered}\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)},\\ \gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X\Phi)^+} \varphi(r).\end{gathered} \tag{1.2}$$

One has

- *(a) For every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X.*
- *(b) If $\gamma < \infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternatives hold: either*
  - *(b_1) $\Phi - \lambda\Psi$ possesses a global minimum, or*
  - *(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*
- *(c) If $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternatives hold: either*
  - *(c_1) there is a global minimum of Ψ which is a local minimum of $\Phi - \lambda\Psi$, or*
  - *(c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$ which weakly converges to a global minimum of Φ.*

Many authors proved the existence of infinitely many solutions by using the theorem above for different problems see for example [2,4–9,11].

The paper is arranged as follows. At first we prove the existence of an unbounded sequence of weak solutions of system (1.1) under some hypotheses on the behaviour of potential F at infinity (see Theorem 3.1). And as a consequence, we obtain the existence of infinitely many weak solutions for autonomous case (see Corollary 3.4).

2 Preliminaries

Let us introduce notation that will be used in the paper. Let

$$X_p = \{u \in W^{1,p}([0,1]), \quad u(0) = 0\}, \quad p \geq 1$$

be the Sobolev space with the norm defined by

$$\|u\|_p = \left(\int_0^1 |u'(t)|^p\, dt\right)^{\frac{1}{p}}$$

for every $u \in X_p$, that is equivalent to the usual one.

It is well known that $(X_p, \|\cdot\|_p)$ is compactly embedded in $(C^0([0,1]), \|\cdot\|_\infty)$ and one has

$$\|u\|_\infty \leq \|u\|_p \qquad \forall u \in X_p. \tag{2.1}$$

Now, let X be the Cartesian product of m Sobolev spaces X_{p_i}, i.e. $X = \prod_{i=1}^m X_{p_i}$ endowed with the norm

$$\|u\| := \sum_{i=1}^m \|u_i\|_{p_i}$$

for all $u = (u_1, \dots, u_m) \in X$.

A function $u = (u_1, \dots, u_m) \in X$ is said a weak solution to system (1.1) if

$$\int_0^1 \sum_{i=1}^m |u_i'(t)|^{p_i-2} u_i'(t) v_i'(t)\, dt = \lambda \int_0^1 \sum_{i=1}^m F_{u_i}(t, u_1(t), \dots, u_m(t)) v_i(t)\, dt$$

for every $v = (v_1, \dots, v_m) \in X$.

In order to study system (1.1), we will use the functionals $\Phi, \Psi \colon X \to \mathbb{R}$ defined by putting

$$\Phi(u) := \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \qquad \Psi(u) := \int_0^1 F(t, u_1(t), \dots, u_m(t))\, dt \tag{2.2}$$

for every $u = (u_1, \dots, u_m) \in X$.

Clearly, Φ is coercive, weakly sequentially lower semicontinuous and continuously Gâteaux differentiable and the Gâteaux derivative at point $u = (u_1, \dots, u_m) \in X$ is defined by

$$\Phi'(u)(v) = \int_0^1 \sum_{i=1}^m |u_i'(t)|^{p_i-2} u_i'(t) v_i'(t)\, dt$$

for every $v = (v_1, \dots, v_m) \in X$. On the other hand Ψ is well defined, weakly upper sequentially semicontinuous, continuously Gâteaux differentiable and the Gâteaux derivative at point $u = (u_1, \dots, u_m) \in X$ is defined by

$$\Psi'(u)(v) = \int_0^1 \sum_{i=1}^m F_{u_i}(x, u_1(t), \dots, u_m(t)) v_i(t)\, dt$$

for every $v = (v_1, \dots, v_m) \in X$.

A critical point for the functional $I_\lambda := \Phi - \lambda\Psi$ is any $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0 \qquad \forall v \in X.$$

Hence, the critical points for functional $I_\lambda := \Phi - \lambda\Psi$ are exactly the weak solutions to system (1.1).

A function $u \colon [0,1] \to \mathbb{R}^m$ is said a solution to system (1.1) if $u \in C^1([0,1], \mathbb{R}^m)$, $|u_i'|^{p_i-2}u_i'$ is $AC([0,1])$ $(i = 1, \dots, m)$ and the system (1.1) is satisfied a.e.

Standard methods show that solutions to system (1.1) coincide with weak ones when F is a C^1 function.

Now, put

$$A = \liminf_{r \to +\infty} \frac{\int_0^1 \max_{\xi \in Q(r)} F(t, \xi_1, \dots, \xi_m)\, dt}{r^s}, \tag{2.3}$$

where $s = \min_{1 \le i \le m}\{p_i\}$, $Q(r) = \{\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m : \sum_{i=1}^m |\xi_i| \le r\}$.

$$B = \limsup_{|\xi| \to +\infty,\ \xi \in \mathbb{R}_+^m} \frac{\int_{\frac{1}{2}}^1 F(t, \xi_1, \dots, \xi_m)\, dt}{\sum_{i=1}^m |\xi_i|^{p_i}}, \tag{2.4}$$

$$\lambda_1 = \frac{1}{B}, \qquad \lambda_2 = \frac{1}{\left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s A}, \tag{2.5}$$

we suppose $\lambda_1 = 0$ if $B = +\infty$, and $\lambda_2 = +\infty$ if $A = 0$,

$$\bar{k} = \max_{1\le i\le m}\left\{\frac{2^{p_i-1}}{p_i}\right\}. \tag{2.6}$$

3 Main results

Our main result is the following theorem.

Theorem 3.1. *Assume that*

(i_1) $F(t,x) \ge 0$ *for every* $(t,x) \in [0,1]\times\mathbb{R}^m_+$, *where* $\mathbb{R}^m_+ = \{x = (x_1,\dots,x_m) \in \mathbb{R}^m : x_i \ge 0,\ i = 1,\dots,m\}$;

(i_2)

$$\liminf_{r\to+\infty} \frac{\int_0^1 \max_{\xi\in Q(r)} F(t,\xi_1,\dots,\xi_m)\,dt}{r^s} < \frac{1}{\left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s} \limsup_{|\xi|\to+\infty,\xi\in\mathbb{R}^m_+} \frac{\int_{\frac{1}{2}}^1 F(t,\xi_1,\dots,\xi_m)\,dt}{\sum_{i=1}^m |\xi_i|^{p_i}},$$

where $Q(r) = \{\xi = (\xi_1,\dots,\xi_m) \in \mathbb{R}^m : \sum_{i=1}^m |\xi_i| \le r\}$ *and* $s = \min_{1\le i\le m}\{p_i\}$.

Then, for each $\lambda \in]\lambda_1,\lambda_2[$, *where* λ_1, λ_2 *are given by* (2.5), *the system* (1.1) *has a sequence of weak solutions which is unbounded in* X.

Proof. Our goal is to apply Theorem 1.1 (b). Consider the Sobolev space X and the operators defined in (2.2). Pick $\lambda \in]\lambda_1,\lambda_2[$.

Let $\{c_n\}$ be a real sequence such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \max_{\xi\in Q(c_n)} F(t,\xi_1,\dots,\xi_m)}{c_n^s} = A.$$

Put

$$r_n = \frac{c_n^s}{\left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s}$$

for all $n \in \mathbb{N}$.

Taking into account (2.1), one has $\sum_{i=1}^m |v_i(t)| < c_n$ where $v = (v_1,\dots,v_m) \in X$ such that $\sum_{i=1}^m \frac{\|v_i\|_{p_i}^{p_i}}{p_i} < r_n$.

Hence, for all $n \in \mathbb{N}$, one has

$$\begin{aligned}\varphi(r_n) &= \inf_{(u_1,\dots,u_m)\in\Phi^{-1}(]-\infty,r_n[)} \frac{\sup_{(v_1,\dots,v_m)\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v_1,\dots,v_m) - \Psi(u_1,\dots,u_m)}{r_n - \Phi(u_1,\dots,u_m)} \\ &\le \frac{\sup_{(v_1,\dots,v_m)\in\Phi^{-1}(]-\infty,r_n[)} \int_0^1 F(t,v_1(t),\dots,v_m(t))\,dt}{r_n} \\ &\le \left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s \frac{\int_0^1 \max_{\xi\in Q(c_n)} F(t,\xi_1,\dots,\xi_m)\,dt}{c_n^s},\end{aligned}$$

therefore, since from (i_2) one has $A < \infty$, we obtain

$$\gamma := \liminf_{n\to\infty} \varphi(r_n) \le \left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s A < \infty.$$

Now, fix $\lambda \in]\lambda_1, \lambda_2[$, we claim that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Let $\{\xi_n = (\xi_{in})_{i=1,\dots,m}\}$ be a real sequence such that $\lim_{n\to\infty} |\xi_n| = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_{\frac{1}{2}}^1 F(t,\xi_{1n},\dots,\xi_{mn})dt}{\sum_{i=1}^m |\xi_{in}|^{p_i}} = B. \tag{3.1}$$

For all $n \in \mathbb{N}$ define

$$\omega_{in}(t) = \begin{cases} 2\xi_{in} t & \text{if t} \in [0, \frac{1}{2}[\\ \xi_{in} & \text{if t} \in [\frac{1}{2}, 1] \end{cases} \qquad i = 1,\dots,m$$

clearly, $\omega_n = (\omega_{1n},\dots,\omega_{mn}) \in X$ and

$$\Phi(\omega_n) = \sum_{i=1}^m \frac{1}{p_i}\|\omega_{in}\|_{p_i}^{p_i} \le \overline{k}\sum_{i=1}^m |\xi_{in}|^{p_i} \tag{3.2}$$

where $\overline{k}$ is given by (2.6).

Taking into account (i_1), we have

$$\int_0^1 F(t,\omega_n(t))\,dt \ge \int_{\frac{1}{2}}^1 F(t,\xi_{1n},\dots,\xi_{mn})\,dt. \tag{3.3}$$

Then, by using (3.2) and (3.3) for all $n \in \mathbb{N}$ we have

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) \le \overline{k}\sum_{i=1}^m |\xi_{in}|^{p_i} - \lambda\int_{\frac{1}{2}}^1 F(t,\xi_{1n},\dots,\xi_{mn})dt. \tag{3.4}$$

Now, if $B < \infty$, we fix $\epsilon \in \left]\frac{\overline{k}}{\lambda B}, 1\right[$, from (3.1) there exists $\nu_\epsilon \in \mathbb{N}$ such that

$$\int_{\frac{1}{2}}^1 F(t,\xi_{1n},\dots,\xi_{mn})\,dt > \epsilon B\sum_{i=1}^m |\xi_{in}|^{p_i} \qquad \forall n > \nu_\epsilon$$

therefore

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) \le \left(\overline{k} - \lambda\epsilon B\right)\sum_{i=1}^m |\xi_{in}|^{p_i} \qquad \forall n > \nu_\epsilon$$

by the choice of ϵ, one has

$$\lim_{n\to\infty}(\Phi(\omega_n)-\lambda\Psi(\omega_n))=-\infty.$$

On the other hand, if $B=+\infty$, we fix

$$M>\frac{\bar{k}}{\lambda}$$

from (3.1) there exists $\nu_M\in\mathbb{N}$ such that

$$\int_{\frac{1}{2}}^{1}F(t,\xi_{1n},\ldots,\xi_{mn})dt>M\sum_{i=1}^{m}|\xi_{in}|^{p_i}\qquad\forall n>\nu_M$$

therefore

$$\Phi(\omega_n)-\lambda\Psi(\omega_n)\leq\left(\bar{k}-\lambda M\right)\sum_{i=1}^{m}|\xi_{in}|^{p_i}\qquad\forall n>\nu_M$$

by the choice of M, one has

$$\lim_{n\to\infty}(\Phi(\omega_n)-\lambda\Psi(\omega_n))=-\infty.$$

Hence, our claim is proved.

Since all assumptions of Theorem 1.1 (b) are verified, the functional $I_\lambda=\Phi-\lambda\Psi$ admits a sequence $\{u_n\}$ of critical points such that $\lim_{n\to\infty}\|u_n\|=+\infty$ and the conclusion is achieved. □

Remark 3.2. In Theorem 3.1 we can replace $r\to+\infty$ by $r\to 0^+$, applying in the proof part (c) of Theorem 1.1 instead of (b). In this case a sequence of pairwise distinct weak solutions to the system (1.1) which converges uniformly to zero is obtained.

Remark 3.3. We consider the system

$$\begin{cases}-(|u_1'|^{p_1-2}u_1')'+|u_1|^{p_1-2}u_1=\lambda F_{u_1}(t,u_1,\ldots,u_m) & \text{in }]0,1[\\ \quad\vdots & \\ -(|u_m'|^{p_m-2}u_m')'+|u_m|^{p_m-2}u_m=\lambda F_{u_m}(t,u_1,\ldots,u_m) & \text{in }]0,1[\\ u_i(0)=u_i'(1)=0 & i=1,\ldots,m\end{cases}\tag{3.5}$$

by using the usual norm

$$\|u\|_{p_i}=\left(\int_0^1|u(t)|^{p_i}\,dt+\int_0^1|u'(t)|^{p_i}\,dt\right)^{\frac{1}{p_i}}$$

in X_{p_i}, and the constant $\bar{k}=\max_{1\leq i\leq m}\left\{\frac{2+p_i+2^{p_i}(p_i+1)}{2p_i(p_i+1)}\right\}$ we can prove in a very similar way to that used to prove Theorem 3.1, that for each $\lambda\in]\lambda_1,\lambda_2[$, with λ_1 and λ_2 given by (2.5), the system (3.5) has a sequence of weak solutions which is unbounded in X .

Now, we point out a special case of Theorem 3.1.

Corollary 3.4. *Let $f,g\colon\mathbb{R}^2\to\mathbb{R}$ be two positive continuous functions such that the differential 1-form $\omega=f(x,y)\,dx+g(x,y)\,dy$ is integrable and let F be a primitive of ω with $F(0,0)=0$. Fix $p,q>1$ with $p\leq q$ assume that*

$$\liminf_{r\to+\infty}\frac{F(r,r)}{r^p}=0,\qquad \limsup_{r\to+\infty}\frac{F(r,r)}{r^q}=+\infty.$$

Then, the system

$$\begin{cases}-(|u'|^{p-2}u')'=f(u,v) & \text{in I }=]0,1[\\ -(|v'|^{q-2}v')'=g(u,v) & \text{in I }=]0,1[\\ u(0)=u'(1)=0 \\ v(0)=v'(1)=0\end{cases}$$

possesses a sequence of pairwise distinct solutions which is unbounded in X.

Proof. Since f and g are positive one has that $\max_{(\xi,\eta)\in Q(r)}F(\xi,\eta)\le F(r,r)$ for every $r\in\mathbb{R}_+$.
Therefore

$$\liminf_{r\to+\infty}\frac{\int_0^1\max_{(\xi,\eta)\in Q(r)}F(\xi,\eta)\,dt}{r^p}\le\liminf_{r\to+\infty}\frac{F(r,r)}{r^p}=0$$

on the other hand, we have

$$+\infty=\frac{1}{2}\limsup_{r\to+\infty}\frac{F(r,r)}{r^q}\le\limsup_{r\to+\infty}\frac{F(r,r)}{r^p+r^q}\le\limsup_{\sqrt{\xi^2+\eta^2}\to+\infty,\,(\xi,\eta)\in\mathbb{R}^2_+}\frac{F(\xi,\eta)}{\xi^p+\eta^q},$$

then we have $\lambda_1=0$ and $\lambda_2=+\infty$ and all assumptions of Theorem 3.1 are satisfied and the proof is complete. □

Now, we present one example that illustrates our result.

Example 3.5. Consider $p=q=4$ and the function $F\colon\mathbb{R}^2\to\mathbb{R}$ defined by

$$F(x,y)=\begin{cases}x^2y^2e^{2(\sin\log x+1)}e^{2(\sin\log y+1)} & \text{if } x>0,\ y>0,\\ 0 & \text{otherwise.}\end{cases}$$

We denote by $f(x,y)$ and $g(x,y)$ the partial derivatives of F respect on x and y respectively

$$f(x,y)=\begin{cases}2xy^2e^{2(\sin\log x+1)}e^{2(\sin\log y+1)}[1+\cos\log x] & \text{if } x>0,\ y>0,\\ 0 & \text{otherwise;}\end{cases}$$

$$g(x,y)=\begin{cases}2x^2ye^{2(\sin\log x+1)}e^{2(\sin\log y+1)}[1+\cos\log y] & \text{if } x>0,\ y>0,\\ 0 & \text{otherwise.}\end{cases}$$

Since f and g are non negative one has that $\max_{(x,y)\in Q(r)}F(x,y)\le F(r,r)$ for every $r\in\mathbb{R}_+$. By a simple computation, we obtain

$$\liminf_{r\to+\infty}\frac{\max_{(x,y)\in Q(r)}F(x,y)}{r^4}\le\liminf_{r\to+\infty}\frac{F(r,r)}{r^4}=1$$

$$\limsup_{\sqrt{x^2+y^2}\to+\infty,(x,y)\in\mathbb{R}^2_+}\frac{F(x,y)}{2(x^4+y^4)}=\frac{e^8}{4}.$$

Hence, from Theorem 3.1, for each $\lambda \in]\frac{4}{e^8}, \frac{1}{2^6}[$ the system

$$\begin{cases} -(|u'|^2u')' = \lambda f(u,v) & \text{in } I =]0,1[\\ -(|v'|^2v')' = \lambda g(u,v) & \text{in } I =]0,1[\\ u(0) = u'(1) = 0 \\ v(0) = v'(1) = 0 \end{cases}$$

has a sequence of solutions which is unbounded in $X = X_4 \times X_4$.

References

[1] R. I. Avery, C. J. Chyan, J. Henderson, Twin solutions of boundary value problems for ordinary differential equations and finite difference equations, *Comput. Math. Appl.* **42**(2001), 695–704. MR1838025; url

[2] D. Averna, S. M. Buccellato, E. Tornatore, On a mixed boundary value problem involving the p-Laplacian, *Matematiche (Catania)* **66**(2011), 93–104. MR2827188; url

[3] D. Averna, R. Salvati, Three solutions for a mixed boundary value problem involving the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **298**(2004), 245–260. MR2086544; url

[4] G. Bonanno, G. D'Aguì, On the Neumann problem for elliptic equations involving the p-Laplacian, *J. Math. Anal. Appl.* **358**(2009), 223–228. MR2532500; url

[5] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009**, Art. ID 670675, 20 pp. MR2487254; url

[6] G. Bonanno, G. Molica Bisci, D. O'Regan, Infinitely many weak solutions for a class of quasilinear elliptic systems, *Math. Comput. Modelling* **52**(2010), 152–160. MR2645927; url

[7] G. Bonanno, E. Tornatore, Infinitely many solutions for a mixed boundary value problem, *Ann. Polon. Math* **99**(2010), 285–293. MR2720222; url

[8] P. Candito, Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities, *Proc. Edinb. Math. Soc. (2)* **45**(2002), 397–409. MR1912648; url

[9] G. D'Aguì, Infinitely many solutions for a double Sturm–Liouville problem, *J. Glob. Optim.* **54**(2012), 619–625. MR2988202; url

[10] H. Lü, D. O'Regan, C. Zhong, Multiple positive solutions for the one-dimensional singular p-Laplacian, *Appl. Math. Comput.* **133**(2002), 407–422. MR1924626; url

[11] S. Marano, D. Motreanu, Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving p-Laplacian, *J. Differential Equations* **182**(2002), 108–120. MR1912071; url

[12] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113**(2000), 401–410. MR1735837; url

[13] R. Salvati, Multiple solutions for a mixed boundary value problem, *Math. Sci. Res. J.* **7**(2003), 275–283. MR1996739