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Existence results for a two point boundary value problem involving a fourth-order equation

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Abstract. We study the existence of non-zero solutions for a fourth-order differential equation with nonlinear boundary conditions which models beams on elastic foundations. The approach is based on variational methods. Some applications are illustrated.

Keywords: fourth-order equations, critical points, variational methods

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1 Introduction

In this paper, we consider the following fourth-order problem

$$\begin{cases} u^{(iv)}(x) = \lambda f(x, u(x)) \quad \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = \mu g(u(1)), \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and λ, μ are positive parameters. The problem ($P_{\lambda,\mu}$) describes the static equilibrium of a flexible elastic beam of length 1 when, along its length, a load f is added to cause deformation. Precisely, conditions $u(0) = u'(0) = 0$ mean that the left end of the beam is fixed and conditions $u''(1) = 0$, $u'''(1) = \mu g(u(1))$ mean that the right end of the beam is attached to a bearing device, given by the function g.

Existence and multiplicity results for this kinds of problems has been extensively studied. In particular, by using a variational approach, the existence of three solutions for the problems ($P_{\lambda,1}$) and ($P_{\lambda,\lambda}$) has been established respectively in [6] and in [4]. Moreover, in [8] the author obtained the existence of at least two positive solutions for the problem ($P_{1,1}$). Finally, we point out that the problem ($P_{\lambda,\mu}$) can be also studied by iterative methods (see for instance [7])

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and, for fourth order equations subject to conditions of different type, we refer, for instance, to [3, 5] and references therein.

In this paper we will deal with the existence of one non-zero solution for the problem ($P_{\lambda,\mu}$). Precisely, using a variational approach, under conditions involving the antiderivatives of f and g, we will obtain two precise intervals of the parameters λ and μ for which the problem ($P_{\lambda,\mu}$) admits at least one non-zero classical solution (see Theorem 3.1). As a way of example, we present here a special case of our results.

Theorem 1.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Then, for each $\lambda \in \left]0, \frac{1}{10\int_0^2 f(t)\,dt}\right[$ the problem*

$$\begin{cases} u^{(iv)}(x) = \lambda f(u(x)) & \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = \sqrt{|u(1)|} \end{cases}$$

admits at least one non-zero classical solution.

We explicitly observe that in Theorem 1.1, assumptions on the behavior of f, as for instance asymptotic conditions at zero or at infinity, are not requested, whereby f is a totally arbitrary function.

The paper is arranged as follows. In Section 2, we recall some basic definitions and our main tool (Theorem 2.2), which is a local minimum theorem established in [1]. Finally, Section 3 is devoted to our main results. Precisely, under a suitable behaviour of f and for parameters μ small enough, the existence of a non-zero solution for ($P_{\lambda,\mu}$) is obtained (Theorem 3.1) and a variant is highlighted (Theorem 3.3). Moreover, some consequences are pointed out (Corollaries 3.4 and 3.5) and a concrete example of application is given (Example 3.7).

2 Basic definitions and preliminary results

We consider the space

$$X := \{u \in H^2([0,1]) : u(0) = u'(0) = 0\}$$

where $H^2([0,1])$ is the Sobolev space of all functions $u\colon [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0,1])$. X is a Hilbert space with inner product

$$\langle u, v \rangle := \int_0^1 u''(t)v''(t)\,dt$$

and norm

$$\|u\| := \left(\int_0^1 (u''(t))^2\,dt\right)^{\frac{1}{2}},$$

which is equivalent to the usual norm $\int_0^1 (|u(t)|^2 + |u'(t)|^2 + |u''(t)|^2)\,dt$. Moreover, the inclusion $X \hookrightarrow C^1([0,1])$ is compact (see [6]) and it results

$$\|u\|_{C^1([0,1])} := \max\left\{\|u\|_\infty, \|u'\|_\infty\right\} \le \|u\| \tag{2.1}$$

for each $u \in X$. We consider the functionals $\Phi, \Psi_{\lambda,\mu}\colon X \to \mathbb{R}$ defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2$$

and

$$\Psi_{\lambda,\mu}(u) := \int_0^1 F(x, u(x))\, dx + \frac{\mu}{\lambda} G(u(1))$$

for each $u \in X$ and for each $\lambda, \mu > 0$ where $F(x, \xi) := \int_0^\xi f(x,t)\, dt$ and $G(\xi) := \int_0^\xi g(t)\, dt$ for each $x \in [0,1]$, $\xi \in \mathbb{R}$. By standard arguments, Φ is sequentially weakly lower semicontinuous and coercive. Moreover, Φ and $\Psi_{\lambda,\mu}$ are in $C^1(X)$ and their Fréchet derivatives are respectively

$$\langle \Phi'(u), v \rangle = \int_0^1 u''(x) v''(x)\, dx$$

and

$$\left\langle \Psi'_{\lambda,\mu}(u), v \right\rangle = \int_0^1 f(x, u(x)) v(x)\, dx + \frac{\mu}{\lambda} g(u(1)) v(1)$$

for each $u, v \in X$. In [6] the authors proved that Φ' admits a continuous inverse on X^* and Ψ' is compact. In particular, in Lemma 2.1 of [6] it has been shown that, for each $\lambda, \mu > 0$, the critical points of the functional

$$I_{\lambda,\mu} := \Phi - \lambda \Psi_{\lambda,\mu}$$

are solutions for problem ($P_{\lambda,\mu}$).

In order to obtain solutions for the problem ($P_{\lambda,\mu}$), we make use of a recent critical point result, where a novel type of Palais–Smale condition is applied (see Theorem 3.1 of [1]). We recall it.

Definition 2.1. Let Φ and Ψ two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r (in short $(P.S.)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(α) $\{I(u_n)\}$ is bounded;

(β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;

(γ) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;

has a convergent subsequence.

The following theorem is a particular case of Theorem 5.1 of [1] and it is the main tool of the next section.

Theorem 2.2 (Theorem 2.3 of [2]). *Let X be a real Banach space, $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that:*

(a_1) $\dfrac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \dfrac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a_2) *for each*

$$\lambda \in \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)^{[r]}$ condition.

Then, for each

$$\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[,$$

there is $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ such that $I'_\lambda(x_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, r[)$.

3 Existence of one solution

Before introducing the main result, we define some notation. With $\alpha \geq 0$, we put

$$F^{\alpha} := \int_0^1 \max_{|\xi| \leq \alpha} F(x, \xi)\, dx$$

and

$$G^{\alpha} := \max_{|\xi| \leq \alpha} G(\xi)\cdot$$

Theorem 3.1. *Assume that*

(f_1) *there exist $\delta, \gamma \in \mathbb{R}$, with $0 < \delta < \gamma$, such that*

$$\frac{F^{\gamma}}{\gamma^2} < \frac{1}{8\pi^4} \left(\frac{3}{2}\right)^3 \frac{\int_{\frac{3}{4}}^1 F(x, \delta)\, dx}{\delta^2}$$

(f_2) $F(x,t) \geq 0$ *for almost every $x \in [0,1]$ and for all $t \in [0, \delta]$.*

Then, for each

$$\lambda \in \Lambda_{\delta,\gamma} := \left] 4\pi^4 \left(\frac{2}{3}\right)^3 \frac{\delta^2}{\int_{\frac{3}{4}}^1 F(x, \delta)\, dx}, \frac{\gamma^2}{2F^{\gamma}} \right[,$$

and for each $g \colon \mathbb{R} \to \mathbb{R}$ continuous, there exists $\eta_{\lambda,g} > 0$, where

$$\eta_{\lambda,g} = \begin{cases} \dfrac{\gamma^2 - 2\lambda F^{\gamma}}{2G^{\gamma}} & \text{if } G(\delta) \geq 0 \\ \min\left\{ \dfrac{\gamma^2 - 2\lambda F^{\gamma}}{2G^{\gamma}}, \dfrac{4\pi^4\delta^2 - \lambda\left(\frac{3}{2}\right)^3 \int_{\frac{3}{4}}^1 F(x,\delta)\, dx}{\left(\frac{3}{2}\right)^3 G(\delta)} \right\} & \text{if } G(\delta) < 0, \end{cases} \tag{3.1}$$

such that for each $\mu \in]0, \eta_{\lambda,g}[$ the problem ($P_{\lambda,\mu}$) admits at least one non-zero solution u_λ such that $\|u_\lambda\|_\infty, \|u'_\lambda\|_\infty < \gamma$.

Proof. Fix $\lambda \in \Lambda_{\delta,\gamma}$. We observe that $\eta_{\lambda,g} > 0$. Indeed, if $G(\delta) \geq 0$, then $G^{\gamma} \geq 0$ and by $\lambda \in \Lambda_{\delta,\gamma}$ it follows that $\gamma^2 - 2\lambda F^{\gamma} > 0$. Hence $\eta_{\lambda,g} > 0$. Let $G(\delta) < 0$. We have by $\lambda \in \Lambda_{\delta,\gamma}$ that $4\pi^4\left(\frac{2}{3}\right)^3 \frac{\delta^2}{\int_{3/4}^1 F(x,\delta) dx} < \lambda$, which implies $4\pi^4\delta^2 - \lambda\left(\frac{3}{2}\right)^3 \int_{3/4}^1 F(x,\delta)\, dx < 0$. Hence $\eta_{\lambda,g} > 0$, in this case as well.

Now, fix $g \colon \mathbb{R} \to \mathbb{R}$ continuous, $\mu \in]0, \eta_{\lambda,g}[$ and consider the space X. Our aim is to apply Theorem 2.2 to the functionals $\Phi, \Psi_{\lambda,\mu}$ defined above. To this end, we fix $r = \frac{\gamma^2}{2}$.

The properties of the functionals Φ and $\Psi_{\lambda,\mu}$ ensure that the functional $I_{\lambda,\mu} = \Phi - \lambda\Psi_{\lambda,\mu}$ verifies $(P.S.)^{[r]}$ condition for each $r, \lambda, \mu > 0$ (see Proposition 2.1 of [1]) and so condition (a_2) of Theorem 2.2 is verified.

Denote by $\bar{v}$ the function of X defined by

$$\bar{v}(x) = \begin{cases} 0 & x \in \left[0, \frac{3}{8}\right], \\ \delta \cos^2\left(\frac{4\pi x}{3}\right) & x \in \left]\frac{3}{8}, \frac{3}{4}\right[, \\ \delta & x \in \left[\frac{3}{4}, 1\right], \end{cases} \tag{3.2}$$

for which it results

$$\Phi(\bar{v}) = 4\pi^4\delta^2\left(\frac{2}{3}\right)^3. \tag{3.3}$$

Taking into account that $\bar{v}(x) \in [0,\delta]$ for each $x \in \left[\frac{3}{8}, \frac{3}{4}\right]$, condition (f_2) ensures that

$$\int_0^{\frac{3}{4}} F(x,\bar{v}(x))\,dx \geq 0$$

and

$$\int_{\frac{3}{4}}^1 F(x,\delta)\,dx \geq 0,$$

which implies

$$\Psi_{\lambda,\mu}(\bar{v}) = \int_0^1 F(x,\bar{v}(x))\,dx + \frac{\mu}{\lambda}G(\delta) \geq \int_{\frac{3}{4}}^1 F(x,\delta)\,dx + \frac{\mu}{\lambda}G(\delta).$$

This ensures that

$$\frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})} \geq \frac{\int_{\frac{3}{4}}^1 F(x,\delta)\,dx + \frac{\mu}{\lambda}G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}. \tag{3.4}$$

For each $u\colon \Phi(u) = \frac{||u||^2}{2} \leq r$, by (2.1) one has

$$\|u\| \leq \gamma = \sqrt{2r}$$

and

$$\|u\|_\infty \leq \gamma.$$

It results

$$\Psi_{\lambda,\mu}(u) = \int_0^1 F(x,u(x))\,dx + \frac{\mu}{\lambda}G(u(1)) \leq F^\gamma + \frac{\mu}{\lambda}G^\gamma$$

for each $u \in \Phi^{-1}(]-\infty,r])$. This leads to

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi_{\lambda,\mu}(u) \leq \frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma. \tag{3.5}$$

Now, taking into account (f_1), if $G(\delta) \geq 0$, then, it results

$$\frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma < \frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\eta_{\lambda,g}}{\lambda}G^\gamma = \frac{1}{\lambda}$$

and

$$\frac{1}{\lambda} < \frac{1}{4\pi^4\delta^2}\left(\frac{3}{2}\right)^3\int_{\frac{3}{4}}^1 F(x,\delta)\,dx \leq \frac{1}{4\pi^4\delta^2}\left(\frac{3}{2}\right)^3\left(\int_{\frac{3}{4}}^1 F(x,\delta)\,dx + \frac{\mu}{\lambda}G(\delta)\right).$$

If $G(\delta) < 0$, taking into account that

$$\mu < \eta_{\lambda,g} = \min\left\{\frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma}, \frac{4\pi^4\delta^2 - \lambda\left(\frac{3}{2}\right)^3\int_{\frac{3}{4}}^1 F(x,\delta)\,dx}{\left(\frac{3}{2}\right)^3 G(\delta)}\right\}, \tag{3.6}$$

it results

$$\frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma < \frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\eta_{\lambda,g}}{\lambda}G^\gamma \leq \frac{1}{\lambda}$$

if $G^\gamma > 0$, and $\frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma < \frac{1}{\lambda}$ if $G^\gamma = 0$.

Moreover, again from (3.6),

$$\frac{1}{\lambda} < \frac{1}{4\pi^4\delta^2}\left(\frac{3}{2}\right)^3 \int_{\frac{3}{4}}^{1} F(x,\delta)\,dx + \frac{\mu}{\lambda}\frac{1}{4\pi^4\delta^2}\left(\frac{3}{2}\right)^3 G(\delta).$$

In all cases, taking into account (3.4) and (3.5), we have

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi_{\lambda,\mu}(u) < \frac{1}{\lambda} < \frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})}.$$

Moreover, we observe that from $\delta < \gamma$, taking (f_1) into account, we obtain $\sqrt{8\pi^4\left(\frac{2}{3}\right)^3}\delta < \gamma$. In fact, arguing by a contradiction, if we assume $\delta < \gamma \leq \sqrt{8\pi^4\left(\frac{2}{3}\right)^3}\delta$, we obtain

$$\frac{F^\gamma}{\gamma^2} \geq \frac{1}{\pi^4}\left(\frac{3}{4}\right)^3 \frac{\int_{\frac{3}{4}}^{1} F(x,\delta)\,dx}{\delta^2}$$

and this is an absurd by (f_1). Therefore, we have $\Phi(\bar{v}) = 4\pi^4\delta^2\left(\frac{2}{3}\right)^3 < \frac{\gamma^2}{2} = r$ and the condition (a_1) of Theorem 2.2 is verified.

Moreover, since

$$\lambda \in \Lambda_{\delta,\gamma} \subseteq \left]\frac{\Phi(\bar{v})}{\Psi_{\lambda,\mu}(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi_{\lambda,\mu}(u)}\right[,$$

Theorem 2.2 guarantees the existence of a local minimum point u_λ for the functional I_λ such that

$$0 < \Phi(u_\lambda) < r$$

and so u_λ is a nontrivial classical solution of problem $(P_{\lambda,\mu})$ such that $\|u_\lambda\|_\infty, \|u'_\lambda\|_\infty < \gamma$. □

Remark 3.2. We observe that in Theorem 3.1 we read $\frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma} = +\infty$ when $G^\gamma = 0$.

By reversing the roles of λ and μ, we obtain the following result.

Theorem 3.3. *Assume that*

(g_1) *there exist* $\delta, \gamma \in \mathbb{R}$ *with* $0 < \delta < \gamma$*:*

$$\frac{G^\gamma}{\gamma^2} < \frac{1}{8\pi^4}\left(\frac{3}{2}\right)^3 \frac{G(\delta)}{\delta^2}.$$

Then for each $\mu \in \Gamma_{\delta,\gamma} := \left]4\pi^4\left(\frac{2}{3}\right)^3 \frac{\delta^2}{G(\delta)}, \frac{\gamma^2}{2G^\gamma}\right[$*, and for each* $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ L^1*-Carathéodory function verifying condition* (f_2) *of Theorem 3.1, there exists* $\theta_{\mu,f} > 0$*, where*

$$\theta_{\mu,f} := \frac{\gamma^2 - 2\mu G^\gamma}{2F^\gamma},$$

such that for each $\lambda \in]0, \theta_{\mu,f}[$ *the problem* $(P_{\lambda,\mu})$ *admits at least one non-zero solution* u *such that* $\|u\|_\infty, \|u'\|_\infty < \gamma$*.*

Proof. Fix $\mu \in \Gamma_{\delta,\gamma}$ and $\lambda \in]0, \theta_{\mu,f}[$. Put

$$\tilde{\Psi}_{\lambda,\mu}(u) := \frac{\lambda}{\mu}\int_0^1 F(x,u(x))\,dx + G(u(1)), \qquad \tilde{I}_{\lambda,\mu}(u) := \Phi(u) - \mu\tilde{\Psi}_{\lambda,\mu}(u),$$

for all $u \in X$. Clearly, one has $\tilde{I}_{\lambda,\mu} = I_{\lambda,\mu}$.

Now, let $\bar{v}$ the function as given in (3.2) and $r = \frac{\gamma^2}{2}$. Arguing as in the proof of Theorem 3.1 (see (3.4) and (3.5)) we obtain

$$\frac{\tilde{\Psi}_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})} \geq \frac{\frac{\lambda}{\mu}\int_{\frac{3}{4}}^1 F(x,\delta)\,dx + G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} \tag{3.7}$$

and

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\tilde{\Psi}_{\lambda,\mu}(u) \leq \frac{2}{\gamma^2}\frac{\lambda}{\mu}F^\gamma + \frac{2}{\gamma^2}G^\gamma. \tag{3.8}$$

Therefore, from (3.7) we obtain

$$\frac{\tilde{\Psi}_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})} \geq \frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} > \frac{1}{\mu}$$

and from (3.8) it follows that

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\tilde{\Psi}_{\lambda,\mu}(u) < \frac{2}{\gamma^2}\frac{\theta_{\mu,f}}{\mu}F^\gamma + \frac{2}{\gamma^2}G^\gamma = \frac{1}{\mu}.$$

Moreover, from (g_1), arguing as in the proof of Theorem 3.1, one has $\Phi(\bar{v}) < r$. So, assumption (a_1) of Theorem 2.2 is verified and

$$\mu \in \left]\frac{\Phi(\bar{v})}{\tilde{\Psi}_{\lambda,\mu}(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r}\tilde{\Psi}_{\lambda,\mu}(u)}\right[,$$

for which $\Phi - \mu\tilde{\Psi}_{\lambda,\mu}$ admits a non-zero critical point and the conclusion is obtained. □

Now, we present some consequences of previous results.

Corollary 3.4. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous and non negative function such that*

(f_1'') $\limsup_{t\to 0^+} \frac{F(t)}{t^2} = +\infty$.

Then, for each $\gamma > 0$, $\lambda \in \left]0, \frac{\gamma^2}{2F(\gamma)}\right[$, for each $g\colon \mathbb{R} \to \mathbb{R}$ continuous and nonnegative and for each $\mu \in \left]0, \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)}\right[$, the problem

$$\begin{cases} u^{(iv)}(x) = \lambda f(u(x)) & \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = \mu g(u(1)) \end{cases} \tag{$\tilde{P}_{\lambda,\mu}$}$$

admits at least one non-zero classical solution u such that $\|u\|_\infty, \|u'\|_\infty < \gamma$.

Proof. Fix $\gamma > 0$, $\lambda \in \left]0, \frac{\gamma^2}{2F(\gamma)}\right[$, $g\colon \mathbb{R} \to \mathbb{R}$ continuous and nonnegative and $\mu \in \left]0, \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)}\right[$.

Condition (f_2) of Theorem 3.1 is verified. Moreover, by (f_1''), there exists $0 < \bar{\delta} < \gamma$ such that

$$\frac{F(\bar{\delta})}{\bar{\delta}^2} > \frac{16\pi^4(\frac{2}{3})^3}{\lambda}.$$

Taking into account that $\lambda \in]0, \frac{\gamma^2}{2F(\gamma)}[$, it results

$$\frac{F(\gamma)}{\gamma^2} < \frac{1}{2\lambda} < \frac{F(\bar{\delta})}{\bar{\delta}^2}\left(\frac{3}{2}\right)^3 \frac{1}{16\pi^4}$$

and so condition (f_1) of Theorem 3.1 is verified. Since g is nonnegative, $\eta_{\lambda,g} = \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)}$ and the conclusion follows easily. □

Clearly, arguing as in the proof of Corollary 3.4, from Theorem 3.3 we obtain the following result.

Corollary 3.5. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$. Then, for each $\gamma > 0$, for each $\mu \in \left]0, \frac{\gamma^2}{2G(\gamma)}\right[$, for each nonnegative continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and for each $\lambda \in \left]0, \frac{\gamma^2 - 2\mu G(\gamma)}{2F(\gamma)}\right[$, the problem $(\tilde{P}_{\lambda,\mu})$ admits at least one non-zero classical solution u such that $\|u\|_\infty, \|u'\|_\infty < \gamma$.*

Remark 3.6. Theorem 1.1 in the Introduction is an immediate consequence of Corollary 3.5. Indeed, it is enough to pick $g(t) = \sqrt{|t|}$ for all $t \in \mathbb{R}$ and $\gamma = 2$, so that one has $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$, $\mu = 1 < \frac{2^2}{G(2)}$ and $\lambda < \frac{1}{10F(2)} < \frac{12-8\sqrt{2}}{6F(2)} = \frac{\gamma^2 - 2\mu G(\gamma)}{2F(\gamma)}$.

Example 3.7. Let us take $\delta = 1/2$, $\gamma = 22$ and $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(u) := \begin{cases} 0, & u < 0, \\ u - u^2, & 0 \le u \le 1, \\ 0, & u > 1. \end{cases}$$

Then, by Theorem 3.1, for each $\lambda \in]1385.4, 1452[$ and each $g\colon \mathbb{R} \to \mathbb{R}$ continuous there exists $\eta_{\lambda,g} > 0$ such that for each $\mu \in]0, \eta_{\lambda,g}[$, the problem $(P_{\lambda,\mu})$ admits at least one non-zero solution u_λ with $\|u\|_\infty, \|u'\|_\infty < 22$.

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