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CONVEX SOLUTIONS OF SYSTEMS ARISING FROM MONGE-AMPÈRE EQUATIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We establish two criteria for the existence of convex solutions to a boundary value problem for weakly coupled systems arising from the Monge-Ampère equations. We shall use fixed point theorems in a cone.

Key words and phrases: Convex solutions, Monge-Ampère equations, cone.
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1 Introduction

In this paper we consider the existence of convex solutions to the Dirichlet problem for the weakly coupled system

$$\begin{aligned} &\Big(\big(u_1'(t)\big)^N\Big)' = Nt^{N-1}f(-u_2(t)) \quad \text{in } \ 0<t<1,\\ &\Big(\big(u_2'(t)\big)^N\Big)' = Nt^{N-1}g(-u_1(t)) \quad \text{in } \ 0<t<1,\\ &u_1'(0)=u_2'(0)=0, \ \ u_1(1)=u_2(1)=0, \end{aligned} \tag{1.1}$$

where $N \geq 1$. A nontrivial convex solution of (1.1) is negative on [0,1). Such a problem arises in the study of the existence of convex radial solutions to the Dirichlet problem for the system of the Monge-Ampère equations

$$\begin{cases} \det D^2 u_1 = f(-u_2) \ \text{ in } \ B,\\ \det D^2 u_2 = g(-u_1) \ \text{ in } \ B,\\ u_1 = u_2 = 0 \ \text{ on } \ \partial B, \end{cases} \tag{1.2}$$

where $B = \{x \in \mathbb{R}^N : |x| < 1\}$ and $\det D^2 u_i$ is the determinant of the Hessian matrix $(\frac{\partial^2 u_i}{\partial x_m \partial x_n})$ of u_i. For how to reduce (1.2) to (1.1), one may see Hu and the author [5].

The Dirichlet problem for a single unknown variable Monge-Ampère equations

$$\begin{cases} \det D^2 u = f(-u) \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases} \tag{1.3}$$

in general domains in $\mathbb{R}^n$ may be found in Caffarelli, Nirenberg and Spruck [1]. Kutev [7] investigated the existence of strictly convex radial solutions of (1.3) when $f(-u) = (-u)^p$. Delano [3] treated the existence of convex radial solutions of (1.3) for a class of more general functions, namely $\lambda \exp f(|x|, u, |\nabla u|)$.

The author [10] and Hu and the author [5] showed that the existence, multiplicity and nonexistence of convex radial solutions of (1.3) can be determined by the asymptotic behaviors of the quotient $\frac{f(u)}{u^N}$ at zero and infinity.

In this paper we shall establish the existence of convex radial solutions of the weakly coupled system (1.1) in superlinear and sublinear cases. First, introduce the notation

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x^N}, \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x^N},$$

and

$$g_0 = \lim_{x \to 0^+} \frac{g(x)}{x^N}, \quad g_\infty = \lim_{x \to \infty} \frac{g(x)}{x^N}.$$

We shall show that if (1.1) is superlinear, or $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$, (1.1) is sublinear, or $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then (1.1) has a convex solution.

Our main results are:

Theorem 1.1 *Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous.*
(a). If $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$, then (1.1) has a convex solution.
(b). If $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then (1.1) has a convex solution.

2 Preliminaries

With a simple transformation $v_i = -u_i, i = 1, 2$ (1.1) can be brought to the following equation

$$\begin{cases} \Big(\big(-v_1'(r)\big)^N\Big)' = Nr^{N-1} f(v_2), \ \ 0 < r < 1, \\ \Big(\big(-v_2'(r)\big)^N\Big)' = Nr^{N-1} g(v_1), \ \ 0 < r < 1, \\ v_i'(0) = v_i(1) = 0, \ \ i = 1, 2. \end{cases} \tag{2.4}$$

Now we treat positive concave classical solutions of (2.4).

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.1 ([2, 4, 6]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) If $\|Tx\| \geq \|x\|$ *for* $x \in \partial K_r$*, then*

$$i(T, K_r, K) = 0.$$

(ii) If $\|Tx\| \leq \|x\|$ *for* $x \in \partial K_r$*, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.4), let X be the Banach space $C[0,1] \times C[0,1]$ and, for $(v_1, v_2) \in X$,

$$\|(v_1, v_2)\| = \|v_1\| + \|v_2\|$$

where $\|v_i\| = \sup_{t\in[0,1]}|v_i(t)|$. Define K to be a cone in X by

$$K = \{(v_1, v_2) \in X : v_i(t) \geq 0,\ t \in [0,1], \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} v_i(t) \geq \frac{1}{4}\|v_i\|, i = 1, 2\}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{(v_1, v_2) \in K : \|(v_1, v_2)\| < r\}.$$

Note that $\partial\Omega_r = \{(v_1, v_2) \in K : \|(v_1, v_2)\| = r\}$.

Let $\mathbf{T} : K \to X$ be a map with components (T^1, T^2), which are defined by

$$\begin{aligned} T^1(v_1, v_2)(r) &= \int_r^1 \Big(\int_0^s N\tau^{N-1} f(v_2(\tau))d\tau\Big)^{\frac{1}{n}} ds,\ r \in [0,1], \\ T^2(v_1, v_2)(r) &= \int_r^1 \Big(\int_0^s N\tau^{N-1} g(v_1(\tau))d\tau\Big)^{\frac{1}{n}} ds,\ r \in [0,1]. \end{aligned} \tag{2.5}$$

It is straightforward to verify that (2.4) is equivalent to the fixed point equation

$$\mathbf{T}(v_1, v_2) = (v_1, v_2) \quad \text{in} \quad K.$$

Thus, if $(v_1, v_2) \in K$ is a positive fixed point of $\mathbf{T}$, then $(-v_1, -v_2)$ is a convex solution of (1.1). Conversely, if (u_1, u_2) is a convex solution of (1.1), then $(-u_1, -u_2)$ is a fixed point of $\mathbf{T}$ in K.

The following lemma is a standard result due to the concavity of u. We prove it here only for completeness.

Lemma 2.2 *Let* $u \in C^1[0,1]$ *with* $u(t) \geq 0$ *for* $t \in [0,1]$*. Assume that* $u'(t)$ *is nonincreasing on* $[0,1]$*. Then*

$$u(t) \geq \min\{t, 1-t\}||u||, \quad t \in [0,1],$$

where $||u|| = \sup_{t\in[0,1]} u(t)$*. In particular,*

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \frac{1}{4}||u||.$$

PROOF Since $u'(t)$ is nonincreasing, we have for $0 \leq t_0 < t < t_1 \leq 1$,

$$u(t) - u(t_0) = \int_{t_0}^{t} u'(s)ds \geq (t - t_0)u'(t)$$

and

$$u(t_1) - u(t) = \int_{t}^{t_1} u'(s)ds \leq (t_1 - t)u'(t),$$

from which, we have

$$u(t) \geq \frac{(t_1 - t)u(t_0) + (t - t_0)u(t_1)}{t_1 - t_0}.$$

Considering the above inequality on $[0, \sigma]$ and $[\sigma, 1]$, we have

$$u(t) \geq t||u|| \quad \text{for} \quad t \in [0, \sigma],$$

and

$$u(t) \geq (1 - t)||u|| \quad \text{for} \quad t \in [\sigma, 1],$$

where $\sigma \in [0, 1]$ such that $u(\sigma) = ||u||$. Hence,

$$u(t) \geq \min\{t, 1 - t\}||u||, \quad t \in [0, 1].$$

□

Lemma 2.3 can be verified by the standard procedures.

Lemma 2.3 *Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous. Then $\mathbf{T}(K) \subset K$ and $\mathbf{T} : K \to K$ is a compact operator and continuous.*

Let

$$\Gamma = \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \Big(\int_{\frac{1}{4}}^{s} N\tau^{N-1} d\tau \Big)^{\frac{1}{N}} ds > 0.$$

Lemma 2.4 *Let $(v_1, v_2) \in K$ and $\eta > 0$. If*

$$f(v_2(t)) \geq (\eta v_2(t))^N \quad \textit{for} \quad t \in [\frac{1}{4}, \frac{3}{4}],$$

or

$$g(v_1(t)) \geq (\eta v_1(t))^N \quad \textit{for} \quad t \in [\frac{1}{4}, \frac{3}{4}],$$

then

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|v_2\|,$$

or

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|v_1\|,$$

respectively.

PROOF Note, from the definition of $\mathbf{T}(v_1, v_2)$, that $T^i(v_1, v_2)(0)$ is the maximum value of $T^i(v_1, v_2)$ on [0,1]. It follows that

$$
\begin{aligned}
\|\mathbf{T}(v_1, v_2)\| &\geq \sup_{t\in[0,1]} |T^1(v_1, v_2)(t)| \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \Big(\int_{\frac{1}{4}}^{s} N\tau^{N-1} f(v_2(\tau))d\tau \Big)^{\frac{1}{N}} ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \Big(\int_{\frac{1}{4}}^{s} N\tau^{N-1} (\eta v_2(\tau))^N d\tau \Big)^{\frac{1}{N}} ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \Big(\int_{\frac{1}{4}}^{s} N\tau^{N-1} (\frac{\eta}{4}\|v_2\|)^N d\tau \Big)^{\frac{1}{N}} ds \\
&= \Gamma\eta\|v_2\|.
\end{aligned}
$$

Similarly,

$$\|\mathbf{T}(v_1, v_2)\| \geq \sup_{t\in[0,1]} |T^2(v_1, v_2)(t)| \geq \Gamma\eta\|v_1\|.$$

□

We define new functions $\hat{f}(t), \hat{g}(t) : [0, \infty) \to [0, \infty)$ by

$$\hat{f}(t) = \max\{f(v) : 0 \leq v \leq t\}, \;\; \hat{g}(t) = \max\{g(v) : 0 \leq v \leq t\}.$$

Note that $\hat{f}_0 = \lim_{t\to 0} \frac{\hat{f}(t)}{t^N}$, $\hat{f}_\infty = \lim_{t\to\infty} \frac{\hat{f}(t)}{t^N}$ and $\hat{g}_0, \hat{g}_\infty$ can be defined similarly.

Lemma 2.5 *[9] Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous. Then*

$$\hat{f}_0 = f_0, \;\; \hat{f}_\infty = f_\infty,$$

and

$$\hat{g}_0 = g_0, \;\; \hat{g}_\infty = g_\infty.$$

Lemma 2.6 *Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous. Let $r > 0$. If there exists an $\varepsilon > 0$ such that*

$$\hat{f}(r) \leq (\varepsilon r)^N, \hat{g}(r) \leq (\varepsilon r)^N,$$

then

$$\|\mathbf{T}(v_1, v_2)\| \leq 2\varepsilon\|(v_1, v_2)\| \;\text{ for }\; (v_1, v_2) \in \partial\Omega_r.$$

PROOF From the definition of T, for $(v_1, v_2) \in \partial\Omega_r$, we have

$$\begin{aligned} \|\mathbf{T}(v_1, v_2)\| &= \sum_{i=1}^{2} \sup_{t\in[0,1]} |T^i(v_1, v_2)(t)| \\ &\leq (\int_0^1 N\tau^{N-1} f(v_2(\tau))d\tau)^{\frac{1}{N}} + (\int_0^1 N\tau^{N-1} g(v_1(\tau))d\tau)^{\frac{1}{N}} \\ &\leq (\int_0^1 N\tau^{N-1} \hat{f}(r)d\tau)^{\frac{1}{N}} + (\int_0^1 N\tau^{N-1} \hat{g}(r)d\tau)^{\frac{1}{N}} \\ &\leq (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} \varepsilon r + (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} \varepsilon r \\ &\leq 2\varepsilon \|(u_1, u_2)\|. \end{aligned}$$

□

3 Proof of Theorem 1.1

PROOF Part (a). It follows from Lemma 2.5 that $\hat{f}_0 = 0, \hat{g}_0 = 0$. Therefore, we can choose $r_1 > 0$ so that $\hat{f}^i(r_1) \leq (\varepsilon r_1)^N, \hat{g}^i(r_1) \leq (\varepsilon r_1)^N$ where the constant $\varepsilon > 0$ satisfies

$$\varepsilon < \frac{1}{2}.$$

We have by Lemma 2.6 that

$$\|\mathbf{T}(v_1, v_2)\| \leq 2\varepsilon\|(v_1, v_2)\| < \|(v_1, v_2)\| \quad \text{for} \quad (v_1, v_2) \in \partial\Omega_{r_1}.$$

Now, since $f_\infty = \infty, g_\infty = \infty$, there is an $\hat{H} > 0$ such that

$$f(v) \geq (\eta v)^N, g(v) \geq (\eta v)^N$$

for $v \geq \hat{H}$, where $\eta > 0$ is chosen so that

$$\frac{1}{2}\Gamma\eta > 1.$$

Let $r_2 = \max\{2r_1, 8\hat{H}\}$. If $(v_1, v_2) \in \partial\Omega_{r_2}$, there exists one of $i = 1$ or $i = 2$ such that $\sup_{t\in[0,1]} v_i \geq \frac{1}{2}r_2$. Without loss of generality, assume that $\sup_{t\in[0,1]} v_1 \geq \frac{1}{2}r_2$. Then

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} v_1(t) \geq \frac{1}{4} \sup_{t\in[0,1]} v_1 \geq \frac{1}{8}r_2 \geq \hat{H},$$

which implies that

$$g(v_1(t)) \geq (\eta v_1(t))^N \text{ for t} \in [\frac{1}{4}, \frac{3}{4}].$$

It follows from Lemma 2.4 that

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|v_1\| > \frac{1}{2}\Gamma\eta r_2 \geq r_2 = \|(v_1, v_2)\|.$$

By Lemma 2.1,

$$i(\mathbf{T}, \Omega_{r_1}, K) = 1 \text{ and } i(\mathbf{T}, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that

$$i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1.$$

Thus, $i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \neq 0$, which implies $\mathbf{T}$ has a fixed point $(v_1, v_2) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ by the existence property of the fixed point index. The fixed point $(-v_1, -v_2) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (1.1).

Part (b). since $f_0 = \infty, g_0 = \infty$, there is an $H > 0$ such that

$$f(v) \geq (\eta v)^N, g(v) \geq (\eta v)^N$$

for $0 < v \leq H$, where $\eta > 0$ is chosen so that

$$\Gamma\eta > 1.$$

If $(v_1, v_2) \in \partial\Omega_{r_1}$, then

$$f(v_2(t)) \geq (\eta v_2)^N, \quad g(v_1(t)) \geq (\eta v_1)^N \text{for } t \in [0, 1].$$

Lemma 2.4 implies that

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|(v_1, v_2)\| > \|(v_1, v_2)\| \quad \text{for} \quad (v_1, v_2) \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. It follows from Lemma 2.5 that $\hat{f}_\infty = 0$ and $\hat{g}_\infty = 0$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \leq (\varepsilon r_2)^N, \quad \hat{g}^i(r_2) \leq (\varepsilon r_2)^N,$$

where the constant $\frac{1}{2} > \varepsilon > 0$. Thus, we have by Lemma 2.6 that

$$\|\mathbf{T}(v_1, v_2)\| \leq 2\varepsilon\|(v_1, v_2)\| < \|(v_1, v_2)\| \quad \text{for} \quad (v_1, v_2) \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}, \Omega_{r_1}, K) = 0 \quad \textit{and} \quad i(\mathbf{T}, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}$ has a fixed point (v_1, v_2) in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$. And $(-v_1, -v_2)$ is the desired convex solution of (1.1). □

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References

[1] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation, *Comm. Pure Appl. Math.* **37** (1984), 369-402.

[2] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[3] Ph. Delano, Radially symmetric boundary value problems for real and complex elliptic Monge-Ampère equations, *J. Differential Equations* **58** (1985), 318-344.

[4] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, 1988.

[5] S. Hu and H. Wang, Convex solutions of boundary value problems arising from Monge-Ampère equations, *Discrete and Continuous Dynamical Systems* **16** (2006), 705-720.

[6] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[7] N. D. Kutev, Nontrivial solutions for the equations of Monge-Ampère type, *J. Math. Anal. Appl.* **132** (1988), 424-433.

[8] P. L. Lions, Two remarks on Monge-Ampère equations, *Ann. Mat. Pura Appl.* **142** (4) (1985), 263-275 (1986).

[9] H. Wang, On the number of positive solutions of nonlinear systems, *J. Math. Anal. Appl.* **281** (2003) 287-306.

[10] H. Wang, Convex Solutions of boundary value problems, *J. Math. Anal. Appl.* **318** (2006) 246-252.