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POSITIVE SOLUTIONS OF SOME HIGHER ORDER NONLOCAL BOUNDARY VALUE PROBLEMS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We show how a unified method, due to Webb and Infante, of tackling many nonlocal boundary value problems, can be applied to nonlocal versions of some recently studied higher order boundary value problems. In particular, we give some explicit examples and calculate the constants that are required by the theory.

Key words and phrases: Positive solution, boundary value problem, nonlocal boundary conditions, fixed point index.
AMS (MOS) Subject Classifications: 34B18, 34B15

1 Introduction

We will discuss some generalizations of a higher order boundary value problem (BVP)

$$u^{(n)}(t) + \lambda g(t) f(t, u(t)) = 0,\ t \in (0, 1), \tag{1.1}$$

with some boundary conditions (BCs) that are nonlocal versions of some BCs studied by M. El-Shahed [2]. In [2] the following three sets of BCs were studied.

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 2 \le k \le n-1,\ \ u'(1) = 0; \tag{1.2}$$

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 1 \le k \le n-2,\ \ u'(1) = 0; \tag{1.3}$$

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 1 \le k \le n-2,\ \ u''(1) = 0. \tag{1.4}$$

The existence of at least one positive solution was shown for λ in certain intervals, defined in terms of the behaviour of $f(u)/u$ as $u \to 0^+$ and as $u \to \infty$, by using the well-known Krasnosel'skiĭ fixed point theorems of cone compression and cone expansion. This methodology was previously used, for example, in [15, 23] on some fourth order and third order problems respectively with some other BCs. For some other work on similar BVPs see, for example, [1, 3, 4, 5], and for some higher order systems, see [7].

In the present paper, we discuss the higher order equation (1.1), under weaker assumptions on f and g, with nonlocal versions of the BCs written above. We obtain results on the existence of one positive solution by utilizing results of Webb and Lan [21] involving comparison with the principal eigenvalue of a related linear problem and show that these results can be *sharp*. We also show that, in some cases, a previous theory using other constants does not apply.

In particular, we give some explicit examples and calculate some constants named $m, M(a,b)$ that are commonly used on such problems. It is interesting to note that, in the case $g(t) \equiv 1$, for an arbitrary n we can compute explicitly the constant m and the *optimal* value of $M(a,b)$; that is, we determine a, b so that $M(a,b)$ is minimal for each of the three local problems.

We then use the theory worked out by Webb and Infante in [18, 19] to study the same equation with some quite general nonlocal BCs. For example, corresponding to (1.2) we can treat the BCs

$$u(0) = 0, u^{(k)}(0) = 0,\ 2 \leq k \leq n-1, \ \ u'(1) = \alpha[u], \tag{1.5}$$

where α is a linear functional on $C[0,1]$; that is, $\alpha[u] = \int_0^1 u(t)dA(t)$, a Stieltjes integral. We similarly obtain sharp existence results for this case. This case covers multi-point BCs, where $\alpha[u] = \sum_{i=1}^m a_i\, u(\eta_i)$, and also integral BCs, where $\alpha[u] = \int_0^1 a(s)u(s)\, ds$, in a single framework.

Nonlocal BCs have been extensively studied in recent years, see the survey article [14] for many references. Nonlocal BCs defined by Stieltjes integrals were studied in [8, 9, 10, 13] where it was assumed that the measure dA is positive. In contrast, the method of [18, 19, 20] does not require that $\alpha[u] \geq 0$ for all $u \geq 0$; that is, we can allow dA to be a signed measure. In particular, in the multi-point case we can have coefficients a_i of both signs. We give examples where we explicitly calculate the constants required by the theory.

2 Preliminaries

We will study the problem

$$u^{(n)}(t) + \lambda g(t) f(t, u(t)) = 0,\ t \in (0,1), \tag{2.1}$$

with one of the BCs

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 2 \leq k \leq n-1, \ \ u'(1) = \alpha[u], \tag{2.2}$$

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 1 \leq k \leq n-2, \ \ u'(1) = \alpha[u], \tag{2.3}$$

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 1 \leq k \leq n-2, \ \ u''(1) = \alpha[u], \tag{2.4}$$

where $\alpha[u]$ is given by a Riemann-Stieltjes integral

$$\alpha[u] = \int_0^1 u(s)dA(s), \tag{2.5}$$

with A a function of bounded variation. This is quite natural because such a functional $\alpha[u]$ is a linear functional on $C[0,1]$, and it includes sums and integrals as special cases. We do not suppose that $\alpha[u] \geq 0$ for *all* $u \geq 0$ but we allow dA to be a *signed* measure. The condition $\alpha[u] \geq 0$ is only imposed on a positive *solution* u. The set-up of [19] allows us to study also more general BCs with two nonlocal terms, for example we could readily treat the BCs

$$u(0) = \alpha_1[u], \;\; u^{(k)}(0) = 0, \; 2 \leq k \leq n-1, \;\; u'(1) = \alpha_2[u].$$

Using [20] we could similarly easily handle the case when *any number* of the BCs has a nonlocal part. We only treat the case of one nonlocal term here, firstly for simplicity, and secondly because, in our approach, the main effort has to be directed at the local problems (1.1)-(1.2), (1.1)-(1.3), (1.1)-(1.4); then the theory of [19, 20] can be applied. The problem is then reduced to calculating the constants that occur in the theory. This has been done in a number of papers. For second order equations with one nonlocal term with a variety of BCs see [18, 19], for some some typical fourth order problems see [20, 22], and for fourth order conjugate type BCs see [17].

We will apply the standard methodology of seeking solutions as fixed points of the integral operator

$$Su(t) := \int_0^1 G(t,s)g(s)f(s,u(s))\,ds, \tag{2.6}$$

where G is the Green's function corresponding to each BVP consisting of the equation (2.1) with the respective BCs (2.2), (2.3), (2.4).

We use the well known cone

$$K_0 = \{u \in P : \min_{t\in[a,b]} u(t) \geq c\|u\|\},$$

where $[a,b]$ is some subset of $[0,1]$ and $c > 0$. K_0 is a well-known type of cone, first used by Krasnosel'skiĭ, see e.g. [12] and D. Guo, see e.g. [6].

The following condition is a key one that allows use of the cone K_0. We use the same label as [19] for convenience.

(C_2) There exist a subinterval $[a,b] \subseteq [0,1]$, a measurable function Φ, and a constant $c = c(a,b) \in (0,1]$ such that

$$\begin{aligned} &G(t,s) \leq \Phi(s) \;\text{ for }\; t \in [0,1] \text{ and almost every (a.e.) } s \in [0,1], \\ &G(t,s) \geq c\Phi(s) \text{ for }\; t \in [a,b] \text{ and a.e. }\; s \in [0,1]. \end{aligned}$$

In fact, we will determine functions Φ and c so that $c(t)\Phi(s) \le G(t,s) \le \Phi(s)$ for $s,t \in [0,1]$. Since we find c satisfying $c(t) > 0$ for $t \in (0,1)$, the subinterval $[a,b]$ can be chosen arbitrarily in $(0,1)$.

We shall assume throughout, and without further mention, that $g\Phi \in L^1$ (so g can have pointwise singularities at arbitrary points of $[0,1]$) and satisfies the non-degeneracy condition $\int_a^b g(s)\Phi(s)\,ds > 0$. We also assume that f satisfies Carathéodory conditions.

When $\alpha[u] \equiv 0$ the Green's function is denoted $G_0(t,s)$. For each of the BCs above G_0 is easily found and is as follows, see for example [2].

$$G_{0,1}(t,s) := \frac{t(1-s)^{n-2}}{(n-2)!} - \frac{(t-s)^{n-1}}{(n-1)!}H(t-s), \tag{2.7}$$

$$G_{0,2}(t,s) := \frac{t^{n-1}(1-s)^{n-2}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!}H(t-s), \tag{2.8}$$

$$G_{0,3}(t,s) := \frac{t^{n-1}(1-s)^{n-3}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!}H(t-s), \tag{2.9}$$

where $H(t) := \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0, \end{cases}$ is the Heaviside function.

To treat a nonlocal BC such as

$$u(0) = 0,\ u^{(k)}(0) = 0,\ 2 \le k \le n-1,\ \ u'(1) = \alpha[u],$$

we make use of the function γ defined to be the solution of the equation $\gamma^{(n)}(t) = 0$ with the corresponding BCs

$$\gamma(0) = 0,\ \gamma^{(k)}(0) = 0,\ 2 \le k \le n-1,\ \ \gamma'(1) = 1.$$

We need the following 'positivity' assumptions on the 'boundary term', again using the same labels as in [19].

(C_5) A is a function of bounded variation, and $\mathcal{G}_A(s) := \int_0^1 G_0(t,s)\,dA(t)$ satisfies $\mathcal{G}_A(s) \ge 0$ for a. e. $s \in [0,1]$.

(C_7) The functional α satisfies $0 \le \alpha[\gamma] = \int_0^1 \gamma(t)dA(t) < 1$.

Remark 2.1 *Because $(C_5), (C_7)$ are integral (or sum) conditions, not pointwise ones, we can allow some sign changing measures dA.*

Under these conditions, it is shown in [18] that the Green's function G for each nonlocal problem is given by

$$G(t,s) = \frac{\gamma(t)}{1-\alpha[\gamma]}\,\mathcal{G}_A(s) + G_0(t,s). \tag{2.10}$$

For the BCs given above the functions γ_i satisfy the respective BCs

$$\gamma_1(0) = 0,\ \gamma_1^{(k)}(0) = 0,\ 2 \le k \le n-1,\ \ \gamma_1'(1) = 1; \tag{2.11}$$

$$\gamma_2(0) = 0,\ \gamma_2^{(k)}(0) = 0,\ 1 \le k \le n-2,\ \ \gamma_2'(1) = 1; \tag{2.12}$$

$$\gamma_3(0) = 0,\ \gamma_3^{(k)}(0) = 0,\ 1 \le k \le n-2,\ \ \gamma_3''(1) = 1. \tag{2.13}$$

Hence $\gamma_1(t) = t$, $\gamma_2(t) = \dfrac{t^{n-1}}{n-1}$ and $\gamma_3(t) = \dfrac{t^{n-1}}{(n-1)(n-2)}$.

A major advantage of the technique developed in [18, 19, 20] is that it is only necessary to verify the key positivity assumption (C_2) for the simpler Green's function G_0, corresponding to the problem with no nonlocal terms, obtaining a constant c_0. It is then shown that (C_2) holds for the full Green's function G. Moreover, in each of the cases studied here we have $c = c_0$.

We are able to allow sign changing measures by working in the following cone:

$$K := \{u \in P : \min_{t\in[a,b]} u(t) \ge c\|u\|,\ \alpha[u] \ge 0\}. \tag{2.14}$$

Note that $\gamma \in K$ so $K \neq \{0\}$, and $K = K_0 \cap \{u \in P : \alpha[u] \ge 0\}$. It is shown in [18] that $S : P \to K$ and known fixed point index results can then be applied to S.

We use connections with the related linear operator

$$Lu(t) := \int_0^1 G(t,s)g(s)u(s)\,ds. \tag{2.15}$$

Then L is a compact linear operator in $C[0,1]$ and, by (C_2), the spectral radius $r(L)$ of L satisfies $r(L) > 0$. By the Krein-Rutman theorem, L has an eigenfunction $\varphi \in P\backslash\{0\}$ corresponding to the principal eigenvalue $r(L)$; we suppose that $\|\varphi\| = 1$. Since L maps P into K, we have $\varphi \in K$. We set $\mu_1 := 1/r(L)$, and call it the *principal characteristic value* of L; it is often called the principal eigenvalue of the corresponding BVP.

For $r > 0$ we define the following extended real numbers:

$$\begin{aligned}
\overline{f}(u) &:= \sup_{t\in[0,1]} f(t,u),\ \ \underline{f}(u) := \inf_{t\in[0,1]} f(t,u);\\
f^0 &:= \limsup_{u\to 0+} \overline{f}(u)/u,\ \ f_0 := \liminf_{u\to 0+} \underline{f}(u)/u;\\
f^\infty &:= \limsup_{u\to\infty} \overline{f}(u)/u,\ \ f_\infty := \liminf_{u\to\infty} \underline{f}(u)/u;\\
f^{0,r} &:= \sup\{f(t,u)/r : 0 \le t \le 1,\ 0 \le u \le r\},\\
f_{r,r/c} &:= \inf\{f(t,u)/r : a \le t \le b,\ r \le u \le r/c\}.
\end{aligned}$$

We use the following constants:

$$m := \Big(\sup_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds\Big)^{-1},\ M(a,b) := \Big(\inf_{t\in[a,b]} \int_a^b G(t,s)g(s)\,ds\Big)^{-1}. \tag{2.16}$$

Fixed point index results given in [21] can be used in a standard way to get multiplicity results. The important point of [21] was to prove these results in sufficient generality and for *non-symmetric* kernels.

Theorem 2.1 *Assume that, whenever we have the condition $\mu_1 < f_\infty$, the condition (C_2) holds for an arbitrary $[a,b] \subset (0,1)$. Then S has at least one positive fixed point in K if one of the following conditions holds.*
(S_1) $0 \le f^0 < \mu_1$ and $\mu_1 < f_\infty \le \infty$.
(S_2) $\mu_1 < f_0 \le \infty$ and $0 \le f^\infty < \mu_1$.
S has at least two positive fixed points in K if one of the following conditions holds.
(D_1) $0 \le f^0 < \mu_1$, $f_{r,r/c} > M$ for some $r > 0$, and $0 \le f^\infty < \mu_1$.
(D_2) $\mu_1 < f_0 \le \infty$, $f^{0,r} < m$ for some $r > 0$, and $\mu_1 < f_\infty \le \infty$.
S has at least three positive fixed points in K if either (T_1) or (T_2) below holds.
(T_1) There exist $0 < r_1 < cr_2 < \infty$, such that

$$0 \le f^0 < \mu_1, \quad f_{\rho_1,\rho_1/c} > M, \quad f^{0,\rho_2} < m, \quad \mu_1 < f_\infty \le \infty.$$

(T_2) There exist $0 < r_1 < r_2 < \infty$, such that

$$\mu_1 < f_0 \le \infty, \quad f^{0,r_1} < m, \quad f_{r_2,r_2/c} > M, \quad 0 \le f^\infty < \mu_1.$$

For the proofs see [21] and Theorems 4.1 and 4.2 of [18], or Theorem 4.1 of [19].

The results using (S_1), (S_2) are sharp. Instead of using the sharp conditions such as $f_0 > \mu_1$, $f^\infty < \mu_1$ in (S_2), the stronger conditions $f_0 > M$, $f^\infty < m$ can be used; similarly for the conditions in (S_1). It was shown in [21] that one always has $m \le \mu_1 \le M$ and the inequalities are strict if the corresponding eigenfunction is not constant.

It is routine to extend the list of conditions in order to show the existence of four, five, or an arbitrary finite number of fixed points, under increasingly restrictive conditions on f. We do not write the obvious statements.

The restrictions on f are weaker if $[a,b]$ is chosen so that $M(a,b)$ is as small as possible: the height to be exceeded by the graph of f is less. Also, for a given $[a,b]$ the restrictions on f are weaker when c is chosen as large as possible, since the length of interval on which f has to be large is reduced.

Remark 2.2 *Using conditions (S_1) and (S_2) with μ_1 can give existence results when using the stronger conditions with m and M (even the optimal M) might not apply. Example 3.1 below illustrates this fact.*

In the case $g(t) \equiv 1$, for an arbitrary n and for each of the three local problems we can compute explicitly the constant m and the optimal value of $M(a,b)$; that is, we determine a, b so that $M(a,b)$ is minimal.

Positive solutions do not exist if the nonlinearity does not cross the ‘principal eigenvalue’ of the differential equation. This is a sharper nonexistence result than has been used in [2, 15]. To state the result we need a concept due to Krasnosel’skiĭ, [11, 12].

Definition 2.1 *We say that L is u_0-positive on a cone K, if there exists $u_0 \in K \setminus \{0\}$, such that for every $u \in K \setminus \{0\}$ there are positive constants $k_1(u), k_2(u)$ such that*

$$k_1(u)u_0(t) \le Lu(t) \le k_2(u)u_0(t), \ \text{for every} \ t \in [0,1].$$

It is shown in [16] that many nonlocal BVPs have corresponding linear operators that are u_0-positive. The following nonexistence result is shown in [20] for quite general nonlocal problems, with a short proof.

Theorem 2.2 *Suppose there is $\varepsilon > 0$ such that one of the following conditions hold.*

(i) $f(t,u) \le (\mu_1 - \varepsilon)u$, *for all* $u > 0$ *and almost all* $t \in [0,1]$.

(ii) $f(t,u) \ge (\mu_1 + \varepsilon)u$, *for all* $u > 0$ *and almost all* $t \in [0,1]$.

If (i) holds then 0 is the unique fixed point of S in K. If (ii) holds and L_S is u_0-positive for some $u_0 \in K \setminus \{0\}$ then 0 is the only possible fixed point of S in K.

3 The First BVP

We will study the BVP

$$u^{(n)}(t) + \lambda g(t) f(t,u(t)) = 0, \ t \in (0,1), \tag{3.1}$$

with the BCs

$$u(0) = 0, u^{(k)}(0) = 0, \ 2 \le k \le n-1, \ \ u'(1) = \alpha[u]. \tag{3.2}$$

3.1 The First Local BVP

We need to establish certain properties of the local problem, when $\alpha[u] \equiv 0$. The Green's function for this problem is

$$G(t,s) = \frac{t\,(1-s)^{n-2}}{(n-2)!} - \frac{(t-s)^{n-1}}{(n-1)!}\,H(t-s). \tag{3.3}$$

We will use the following properties.

Lemma 3.1 *For $n \ge 3$, The Green's function $G(t,s)$ satisfies the inequalities*

$$c_0(t)\Phi(s) \le G(t,s) \le \Phi(s), \ 0 \le s,t \le 1, \tag{3.4}$$

for $\Phi(s) = G(1,s) = \dfrac{(1-s)^{n-2}(s+n-2)}{(n-1)!}$, $\ c_0(t) = t$.

Remark 3.1 *Lemma 3 of [2] has the same upper bound but has the lower bound with* $c_0(t) = \dfrac{n-2}{n-1}t$. *His proof uses uses some inequalities without trying for optimal conditions (and has some misprints). We seek an optimal bound so we give our method for completeness.*

Proof. We write

$$G_1(t,s) = \frac{t(1-s)^{n-2}}{(n-2)!} - \frac{(t-s)^{n-1}}{(n-1)!},\ 0 \le s \le t; \quad G_2(t,s) = \frac{t(1-s)^{n-2}}{(n-2)!},\ s \ge t. \tag{3.5}$$

Since

$$\frac{\partial G_2}{\partial t} > 0 \ \text{ and } \ \frac{\partial G_1}{\partial t} = \frac{(1-s)^{n-2}}{(n-2)!} - \frac{(t-s)^{n-2}}{(n-2)!} > 0$$

we see that $G(t,s) \le G(1,s) =: \Phi(s)$.

We now want to show that $\dfrac{G(t,s)}{\Phi(s)} \ge c_0(t)$. We first show that $\dfrac{G_2(t,s)}{\Phi(s)} \ge c_0(t)$ for $t \le s \le 1$. This is satisfied if

$$\frac{c_0(t)}{t} \le \frac{n-1}{s+n-2} \ \text{ for } t \le s \le 1.$$

The minimum on the right occurs when $s = 1$, so we need $c_0(t)) \le \dfrac{t(n-1)}{1+n-2} = t$.

A similar, but more complicated, argument shows that $G_1(t,s)/\Phi(s) \ge c_0(t)$ for $0 \le s \le t$ when $c_0(t) \le \dfrac{t(n-1)}{t+n-2}$. Thus $c_0(t) = \min\Big\{t, \dfrac{t(n-1)}{t+n-2}\Big\} = t$.

For the case $g(t) \equiv 1$, we now compute the constant m and the optimal value of $M(a,b)$, that is, we determine a, b so that $M(a,b)$ is minimal.

We have $\displaystyle\int_0^1 G(t,s)\,ds = \frac{t}{(n-1)!} - \frac{t^n}{n!}$, and the maximum of this expression occurs when $t = 1$, hence $m = \dfrac{n!}{n-1}$.

For $a < b$ we have by direct integration

$$\int_a^b G(t,s)\,ds = \frac{1}{n!}\big[t\,n\big((1-a)^{n-1} - (1-b)^{n-1}\big) - (t-a)^n\big].$$

The sign of the derivative shows that this is an increasing function of t so the minimum occurs at $t = a$. Let

$$R(a,b) := \frac{a}{(n-1)!}\big[(1-a)^{n-1} - (1-b)^{n-1}\big].$$

The quantity $R(a,b)$ is an increasing function of b so its maximum is when $b = 1$. The function $R(a,1) = \dfrac{a}{(n-1)!}(1-a)^{n-1}$ has a maximum when $a = 1/n$. Hence the optimal (minimal) value of $M(a,b)$ is $M(1/n,1) = n^n(n-1)!/(n-1)^{n-1}$.

Some numerical examples illustrate the size of these constants.
$n = 3,\;\; m = 3,\;\; M = 27/2 = 13.5.$
$n = 4,\;\; m = 8,\;\; M = 512/9 \approx 56.8889.$
$n = 5,\;\; m = 30,\; M = 9375/32 = 292.96875.$

We can also compute these constants when $g \not\equiv 1$ but we cannot give explicit formulae. We do this is our first example below.

We reconsider Example 1 of [2] to show how our results sharpen the ones there. Similar types of nonlinear terms have been used before in examples; see for example, Example 5.1 of [23], Example 3.5 of [15] and Example 1 of [5].

Example 3.1 *We consider the fifth order problem*

$$u^{(5)}(t) + \lambda(5t+2)\big(8+\sin(u(t))\big)\frac{7u^2(t)+u(t)}{u(t)+1} = 0,\; t \in (0,1), \tag{3.6}$$

$$u(0) = 0, u''(0) = 0, u'''(0) = 0, u^{(4)}(0) = 0, u'(1) = 0. \tag{3.7}$$

Here we have $g(t) = 5t+2$ and $f(u) = (8+\sin(u))\dfrac{7u^2+u}{u+1}$. It is readily shown that

$$f^0 = f_0 = 8,\;\; f^\infty = 63,\;\; f_\infty = 49.$$

Also, $8u \le f(u) \le 63u$ for all $u \ge 0$. By calculation, with the aid of Maple, we find $m = 720/73 \approx 9.863$, the smallest M calculated is $M((\sqrt{337}-7)/48, 1) \approx M(0.2366, 1) \approx 75.681$, and with a numerical program we find $\mu_1 \approx 32.8755$. Hence, by Theorem 2.1, we have the following conclusions.

There is at least one positive solution if $8\lambda < \mu_1$ and $49\lambda > \mu_1$; that is, there is a positive solution if $\lambda \in (0.6709, 4.1094)$.

There does not exist a positive solution if either $8\lambda > \mu_1$ or $63\lambda < \mu_1$; that is, if $\lambda < 0.5218$ or $\lambda > 4.1094$ no positive solution exists.

The results in [2] are: for $\lambda \in (1.0595, 1.23288)$ there is a positive solution, for $\lambda < 0.156556$ and for $\lambda > 6.46154$ no positive solution exists.

This shows that our results improve those of [2] and can give sharp estimates. The result of [2] and one of [15] use constants called A, B which in our notation are defined to be

$$A := \int_0^1 c(s)\Phi(s)g(s)\,ds,\;\; B := \int_0^1 \Phi(s)g(s)\,ds.$$

It is clear that $1/m \le B$, so $1/B \le m \le \mu_1$. It is shown in Theorem 4.2 of [16] that $1/A \ge \mu_1$, so using these constants rather than μ_1 will always give a poorer estimate on λ.

If we had tried to use the more stringent conditions $\lambda f^0 < m$ and $\lambda f_\infty > M$ we would need $8\lambda < 9.863$ and $49\lambda > 75.681$ and there are no λ satisfying both inequalities, so the theory that uses the constant M is ineffective on this example. However, the point of using m, M is that they can be used together with the behaviour of f on *bounded* intervals, not solely on the behaviour of $f(u)/u$ near 0 and ∞ alone.

We could give similar examples for other values of n, for example, for $n = 4$ and $g(t) \equiv 1$ the corresponding constants are $m = 8,\ \ M = 512/9 \approx 56.8889, \mu_1 \approx 24.352$.

3.2 The First Nonlocal BC

We now consider the nonlocal problem

$$u^{(n)}(t) + \lambda g(t) f(t, u(t)) = 0,\ t \in (0, 1), \tag{3.8}$$

with the BCs

$$u(0) = 0, u^{(k)}(0) = 0,\ 2 \le k \le n - 1,\ \ u'(1) = \alpha[u], \tag{3.9}$$

where $\alpha[u]$ is a linear functional on $C[0, 1]$ give by a Stieltjes integral $\alpha[u] = \int_0^1 u(s)\,dA(s)$ with A a function of bounded variation.

For this problem the function γ is, as seen above, $\gamma(t) = t$. The Green's function is

$$G(t, s) = \frac{\gamma(t)}{1 - \alpha[\gamma]}\,\mathcal{G}_A(s) + G_0(t, s), \tag{3.10}$$

where $\mathcal{G}_A(s) = \int_0^1 G_0(t, s)dA(t)$ for

$$G_0(t, s) = \frac{t(1 - s)^{n-2}}{(n - 2)!} - \frac{(t - s)^{n-1}}{(n - 1)!} H(t - s).$$

Since we have verified that G_0 satisfies the key condition (C_2), it follows from the form of γ and from [18, 19] that G also satisfies these conditions with the same function $c(t)$. The theory therefore is directly applicable.

We give an example for the 4th-order equation with a 4-point BVP with coefficients of both signs.

Example 3.2 *Consider the BVP*

$$u^{(4)}(t) = \lambda\, u(t)\,\frac{1 + 3u(t)}{1 + u(t)},\ t \in (0, 1),$$
$$u(0) = 0, u''(0) = 0,\ u'''(0) = 0,\ u'(1) = 2u(1/2) - u(3/4).$$

We check that $\mathcal{G}_A(s) = 2G_0(1/2, s) - G_0(3/4, s) \ge 0$ and that $\alpha[\gamma] = 2\gamma(1/2) - \gamma(3/4) = 1/4 \in [0, 1)$. By calculation, with the aid of Maple, we find $m = 4608/881 \approx 5.23$, the smallest M calculated is $M(0.246, 1) \approx 41.3197$, and with a numerical program we find $\mu_1 \approx 17.5707$.

We have $f(u)/u = (1 + 3u)/(1 + u)$ and we see that $f_0 = f^0 = 1, f_\infty = f^\infty = 3$, and $u \le f(u) \le 3f(u)$ for all $u \ge 0$. This gives the following conclusions:

If $\mu_1/3 < \lambda < \mu_1$, that is, $5.857 < \lambda < 17.5707$, then the problem has at least one positive solution.

If either $\lambda < \mu_1/3 \approx 5.857$, or $\lambda > \mu_1 \approx 17.5707$, then the problem has no positive solution.

This shows that the estimates are sharp.

We could easily give other examples where two or more positive solutions exist using Theorem 2.1.

4 The Second BC

The second BVP we study is

$$u^{(n)}(t) + \lambda g(t) f(t, u(t)) = 0, \ t \in (0, 1), \tag{4.1}$$

with the BCs

$$u(0) = 0, u^{(k)}(0) = 0, \ 1 \leq k \leq n-2, \ \ u'(1) = \alpha[u]. \tag{4.2}$$

The Green's function for the local problem, when $\alpha[u] \equiv 0$ is

$$G(t, s) := \frac{t^{n-1}(1-s)^{n-2}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!} H(t-s). \tag{4.3}$$

The following properties hold.

Lemma 4.1 *For $n \geq 3$, The Green's function $G(t,s)$ satisfies the inequalities*

$$c_0(t)\Phi(s) \leq G(t, s) \leq \Phi(s), \ 0 \leq s, t \leq 1, \tag{4.4}$$

for $\Phi(s) = G(1, s) = \dfrac{s(1-s)^{n-2}}{(n-1)!}, \ \ c_0(t) = t^{n-1}.$

We omit this proof, it is readily shown using the same method as in Lemma 3.1; it is already shown in [2] (with a misprinted = sign).

We now compute the constants m and the smallest $M(a, b)$ when $g(t) \equiv 1$.

We have $\displaystyle\int_0^1 G(t, s)ds = \frac{t^{n-1}}{(n-1)(n-1)!} - \frac{t^n}{n!}$, the maximum occurs when $t = 1$ and hence $m = n!(n-1)$.

For $a < b$ we compute

$$R(a, b, t) := \int_a^b G(t, s)ds = \frac{t^{n-1}}{(n-1)(n-1)!}\big[(1-a)^{n-1} - (1-b)^{n-1}\big] - \frac{(t-a)^n}{n!}.$$

The sign of the derivative $\partial R/\partial t$ shows that this is an increasing function of t so the minimum occurs at $t = a$. Let

$$R(a, b) := \frac{a^{n-1}}{(n-1)(n-1)!}\big[(1-a)^{n-1} - (1-b)^{n-1}\big].$$

The minimal value of $M(a,b)$ corresponds to the maximal value of $R(a,b)$. The quantity $R(a,b)$ is an increasing function of b so its maximum is when $b=1$. Let

$$R(a) := \frac{\big(a(1-a)\big)^{n-1}}{(n-1)(n-1)!}.$$

The maximum of $R(a)$ occurs when $a=1/2$. Hence the minimal value of $M(a,b)$ is $M(1/2,1) = 2^{2n-2}(n-1)!(n-1)$.

We have the following numerical examples.

For $n=3, m=12, M(1/2,1)=64$, for $n=4, m=72, M(1/2,1)=1152$, and for $n=5, m=480, M(1/2,1)=24576$.

We could easily give examples for this case as we did for the first set of BCs.

5 The Third BC

The third BVP we study is

$$u^{(n)}(t) + \lambda g(t) f(t,u(t)) = 0,\ t \in (0,1), \tag{5.1}$$

$$u(0)=0,\ u^{(k)}(0)=0,\ 1 \le k \le n-2,\ \ u''(1) = \alpha[u]. \tag{5.2}$$

The Green's function for the local problem is

$$G(t,s) := \frac{t^{n-1}(1-s)^{n-3}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!} H(t-s). \tag{5.3}$$

The following properties hold.

Lemma 5.1 *For $n \ge 3$, The Green's function $G(t,s)$ satisfies the inequalities*

$$c_0(t)\Phi(s) \le G(t,s) \le \Phi(s),\ 0 \le s,t \le 1, \tag{5.4}$$

for $\Phi(s) = G(1,s) = \dfrac{(2s-s^2)(1-s)^{n-3}}{(n-1)!},\ \ c_0(t) = t^{n-1}.$

Remark 5.1 *In [2] it is shown that $c_0(t) \ge t^{n-1}/2$, half as large as possible.*

Using the method of Lemma 3.1 we obtain the stated upper bound and also find that

$$\frac{G_2(t,s)}{\Phi(s)} \ge \frac{t^{n-1}}{s(2-s)},\ \ t \le s \le 1,\ \ \frac{G_1(t,s)}{\Phi(s)} \ge \frac{2t-s}{2-s} t^{n-3},\ \ 0 \le s \le t,$$

hence $c_0(t) \le \min\Big\{t^{n-1}, \dfrac{t^{n-2}}{2-t}\Big\} = t^{n-1}$.

For $g \equiv 1$ we now compute the constants m and the optimal $M(a,b)$.

We have $\int_0^1 G(t,s)ds = \dfrac{t^{n-1}}{(n-2)(n-1)!} - \dfrac{t^n}{n!}$. This is an increasing function of t on $[0,1]$ so the maximum occurs when $t=1$ and hence $m = n!(n-2)/2$.

For $a < b$ we compute

$$R(a,b,t) := \int_a^b G(t,s)ds = \frac{t^{n-1}}{(n-2)(n-1)!}\big[(1-a)^{n-2} - (1-b)^{n-2}\big] - \frac{(t-a)^n}{n!}.$$

The minimum occurs at $t = a$ and $R(a,b) := R(a,b,a)$ has its maximum when $b = 1$. The maximum of $R(a,a)$ occurs when $a = (n-1)/(2n-3)$. Hence the minimal value of $M(a,b)$ is

$$M\Big(\frac{n-1}{2n-3}, 1\Big) = \frac{(n-1)!(n-2)(2n-3)^{2n-3}}{(n-1)^{n-1}(n-2)^{n-2}}.$$

Examples of the numbers obtained are:
for $n = 3, m = 3, M(2/3,1) = 27/2 = 13.5$, for $n = 4, m = 24, M(3/5,1) = 3125/9 \approx 347.222$, and for $n = 5,\ m = 180,\ M(4/7,1) = 823543/96 \approx 8578.573$.
We could similarly give examples for this case as we did for the first set of BCs.

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