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On fractional Cauchy-type problems containing Hilfer's derivative

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Abstract. In the paper we study fractional systems with generalized Riemann–Liouville derivatives. A theorem on the existence and uniqueness of a solution to a fractional nonlinear ordinary Cauchy problem is proved. Next a formula for the solution to a linear problem of such a type is presented.

Keywords: fractional integral, fractional derivatives in the Hilfer, Caputo and Riemann–Liouville sense, fractional ordinary Cauchy problem, integral equation.

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1 Introduction

In our paper we study the following fractional differential equation

$$(D^{\alpha,\beta}_{a+}y)(t) = g(t, y(t)), \qquad t \in [a, b] \ a.e. \tag{1.1}$$

with the initial condition

$$(I^{1-\gamma}_{a+}y)(a) = c, \tag{1.2}$$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, $c \in \mathbb{R}^n$, $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $D^{\alpha,\beta}_{a+}$ denotes the generalized Riemann–Liouville derivative operator introduced by Hilfer in [8]. It is easy to see that the $D^{\alpha,\beta}_{a+}$ derivative is considered as an interpolator between the Riemann–Liouville and Caputo derivative (cf. [7]).

In paper [7] the existence and uniqueness of a solution to such problem in a some weighted space of continuous functions has been investigated. The main idea of the proof relies on the change of such problem over to the equivalent integral equation and next, using the constructive method based on the Banach fixed point theorem, solving this equation.

We also investigate the question of the existence and uniqueness of a solution to problem (1.1)–(1.2) but in a different space of solutions, namely in the space so called "γ-absolutely continuous functions" denoted by $AC^{\gamma}_{a+}([a, b], \mathbb{R}^n)$ (generally this is the space of non-continuous

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functions). In our opinion such space of solutions is more useful in applications than the space of continuous functions (for example in the fields of control theory or calculus of variations). Similarly as in paper [7] we use the Banach contraction principle and additionally a notion of the Bielecki norm in the space of solutions. Such approach makes the proofs of our results not complicated and rather short.

Detailed description of our method is the following. First we consider a homogeneous problem (with zero initial condition). We prove that such problem is equivalent to integral equation (3.2). Next, in order to prove the existence of a solution to this integral equation, we use mentioned notion of the Bielecki norm in the space $I^{\alpha}_{a+}(L^1([a,b],\mathbb{R}^n))$ and the Banach fixed point theorem. The point of existence of a solution to nonhomogeneous problem reduces to point of existence of a solution to homogeneous problem.

In the second part of this work we consider the linear problem given by

$$\begin{cases} (D^{\alpha,\beta}_{a+}x)(t) = Ax(t) + w(t), \quad t \in [a,b] \text{ a.e.} \\ (I^{1-\gamma}_{a+}x)(a) = c, \end{cases} \tag{1.3}$$

where $A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$ and $w \in I^{\beta(1-\alpha)}_{a+}(L^1([a,b],\mathbb{R}^n))$.

Using a constructive method, provided by the Banach fixed point theorem, we obtain existence of a solution to such problem under different (less complicated) assumptions than in the case of the nonlinear problem. Moreover we give a formula for this solution. For the linear problem involving the Riemann–Liouville derivative such formula was derived in paper [10].

Problems of a type (1.1)–(1.2) involving the Riemann–Liouville and Caputo derivatives (the special cases $\beta = 0$ and $\beta = 1$, respectively) were investigated very well in many papers (cf. [10,12–14]). Generally fractional differential equations with such derivatives are a topic of research of many scientists (cf. [1,3–6,11,19]). The equations can be applied in various fields of science such as: physics, electronics, mechanics, calculus of variations, control theory, etc. (cf. [2,8,14–17]).

The paper is organized as follows. Section 2 contains some notions and facts concerning the fractional integrals and derivatives. In section 3, we prove theorems on the existence and uniqueness of a solution to problem (1.1) with zero and nonzero initial conditions (1.2). Results of a such type, for the linear problem (1.3), were obtained in Section 4. Moreover, a formula for the solution to such problem is given.

2 Preliminaries

In this section we recall some basic definitions and results concerning the fractional calculus, that we will use in the next sections (cf. [7,14,17]).

Let $\alpha > 0$ and $f \in L^1([a,b],\mathbb{R}^n)$. The functions

$$(I^{\alpha}_{a+}f)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau,$$
$$(I^{\alpha}_{b-}f)(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}}d\tau$$

defined for almost every $t \in [a,b]$ are called *the left-sided Riemann–Liouville integral* and *the right-sided Riemann–Liouville integral of the function f of order α*, respectively.

Remark 2.1. In view of convergence (cf. [17, Theorem 2.7])

$$\lim_{\alpha\to 0^+}(I^{\alpha}_{a+}f)(t) = f(t), \qquad t\in[a,b] \text{ a.e.}$$

it is natural to put

$$(I^{0}_{a+}f)(t) = f(t), \qquad t\in[a,b] \text{ a.e.}$$

Similarly, we put

$$(I^{0}_{b-}f)(t) = f(t), \qquad t\in[a,b] \text{ a.e.}$$

We have the following semigroup properties (cf. [17, formula 2.21])

Lemma 2.2. *If $\alpha_1>0$, $\alpha_2>0$ and $f\in L^1([a,b],\mathbb{R}^n)$ then*

$$(I^{\alpha_1}_{a+}I^{\alpha_2}_{a+}f)(t) = (I^{\alpha_1+\alpha_2}_{a+}f)(t), \qquad t\in[a,b] \text{ a.e.}$$
$$(I^{\alpha_1}_{b-}I^{\alpha_2}_{b-}f)(t) = (I^{\alpha_1+\alpha_2}_{b-}f)(t), \qquad t\in[a,b] \text{ a.e.}$$

The following rule of fractional integration by parts holds (cf. [17, formula 2.20]).

Theorem 2.3. *Let $\alpha>0$, $p\geq 1$, $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ (if $\frac{1}{p}+\frac{1}{q}=1+\alpha$ then $p\neq 1$ and $q\neq 1$). If $f\in L^p([a,b],\mathbb{R}^n)$ and $g\in L^q([a,b],\mathbb{R}^n)$ then*

$$\int_a^b f(t)(I^{\alpha}_{a+}g)(t)dt = \int_a^b g(t)(I^{\alpha}_{b-}f)(t)dt.$$

Now, let $\alpha\in(0,1)$ and $f\in L^1([a,b],\mathbb{R}^n)$. We say that the function f possesses *the left-sided Riemann–Liouville derivative $D^{\alpha}_{a+}f$ of order α*, if the function $I^{1-\alpha}_{a+}f$ is absolutely continuous on $[a,b]$ and

$$(D^{\alpha}_{a+}f)(t) := \frac{d}{dt}(I^{1-\alpha}_{a+}f)(t), \qquad t\in[a,b] \text{ a.e.}$$

In view of Remark 2.1, we put

$$(D^{1}_{a+}f)(t) := f'(t), \qquad t\in[a,b] \text{ a.e.}$$

By $I^{\alpha}_{a+}(L^1)$ we denote the set (cf. [14])

$$I^{\alpha}_{a+}(L^1) := \{f:[a,b]\to\mathbb{R}^n;\ f=I^{\alpha}_{a+}g \text{ a.e. on } [a,b],\ g\in L^1([a,b],\mathbb{R}^n)\}.$$

In [17, Theorem 2.3] the following characterization of the space $I^{\alpha}_{a+}(L^1)$ is proved.

Proposition 2.4. *Let $f\in L^1([a,b],\mathbb{R}^n)$ and $0<\alpha<1$. Then*

$$f\in I^{\alpha}_{a+}(L^1) \iff I^{1-\alpha}_{a+}f\in AC([a,b],\mathbb{R}^n) \quad \text{and} \quad (I^{1-\alpha}_{a+}f)(a)=0.$$

From the above proposition it follows that if $f\in I^{\alpha}_{a+}(L^1)$ then f possesses the left-sided Riemann–Liouville derivative $D^{\alpha}_{a+}f=g$, where g is the function from the definition of $I^{\alpha}_{a+}(L^1)$.

Let us introduce in the space $I^{\alpha}_{a+}(L^1)$ the norm given by

$$\|f\|_{I^{\alpha}_{a+}(L^1)} := \|D^{\alpha}_{a+}f\|_{L^1} \tag{2.1}$$

We have the following theorem.

Theorem 2.5. *The space $I^{\alpha}_{a+}(L^1)$ with the norm (2.1) is complete, i.e. it is a Banach space.*

Proof. Let $(u_k)_{k\in\mathbb{N}} \subset I^{\alpha}_{a+}(L^1)$ be a Cauchy sequence. So,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m,n > N \quad \|u_n - u_m\|_{I^{\alpha}_{a+}(L^1)} < \varepsilon.$$

From the definition of the norm in the space $I^{\alpha}_{a+}(L^1)$ and a linearity of the operator D^{α}_{a+} it follows that for $m,n > N$ we have

$$\|D^{\alpha}_{a+}u_n - D^{\alpha}_{a+}u_m\|_{L^1} = \|D^{\alpha}_{a+}(u_n - u_m)\|_{L^1} = \|u_n - u_m\|_{I^{\alpha}_{a+}(L^1)} < \varepsilon.$$

This means that the sequence $(D^{\alpha}_{a+}u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the space $L^1([a,b],\mathbb{R}^n)$. Consequently, since the space $L^1([a,b],\mathbb{R}^n)$ is complete, so there exists a function $x \in L^1([a,b],\mathbb{R}^n)$ such that

$$\|D^{\alpha}_{a+}u_k - x\|_{L^1} \underset{k\to\infty}{\longrightarrow} 0.$$

Let us put

$$u = I^{\alpha}_{a+}x.$$

Of course, $u \in I^{\alpha}_{a+}(L^1)$. Moreover, from Proposition 2.7 (a), we have

$$\begin{aligned}\|u_k - u\|_{I^{\alpha}_{a+}(L^1)} &= \|D^{\alpha}_{a+}(u_k - u)\|_{L^1} = \|D^{\alpha}_{a+}u_k - D^{\alpha}_{a+}u\|_{L^1} \\ &= \|D^{\alpha}_{a+}u_k - D^{\alpha}_{a+}I^{\alpha}_{a+}x\|_{L^1} = \|D^{\alpha}_{a+}u_k - x\|_{L^1} \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

This means that $I^{\alpha}_{a+}(L^1)$ is complete. □

We shall prove the following lemma.

Lemma 2.6. *Let* $0 < \alpha_1 < \alpha_2 < 1$. *Then*

$$I^{\alpha_2}_{a+}(L^1) \subset I^{\alpha_1}_{a+}(L^1).$$

Proof. Let $f \in I^{\alpha_2}_{a+}(L^1)$. Then there exists a function $\varphi \in L^1([a,b],\mathbb{R}^n)$ such that $f(t) = (I^{\alpha_2}_{a+}\varphi)(t)$ for a.e. $t \in [a,b]$. Let us put

$$\psi(t) = (I^{\alpha_2-\alpha_1}_{a+}\varphi)(t), \qquad t \in [a,b] \text{ a.e.}$$

From [14, Lemma 2.1(a)] it follows that $\psi \in L^1([a,b],\mathbb{R}^n)$. Moreover, from Lemma 2.2, we obtain

$$f(t) = (I^{\alpha_2}_{a+}\varphi)(t) = (I^{\alpha_1}_{a+}I^{\alpha_2-\alpha_1}_{a+}\varphi)(t) = (I^{\alpha_1}_{a+}\psi)(t), \qquad t \in [a,b] \text{ a.e.}$$

The proof is completed. □

We have the following composition properties.

Proposition 2.7 ([14, Lemmas 2.4, 2.5 (a)]). *Let* $0 < \alpha < 1$.

(a) If $f \in L^1([a,b],\mathbb{R}^n)$ *then*

$$(D^{\alpha}_{a+}I^{\alpha}_{a+}f)(t) = f(t), \qquad t \in [a,b] \text{ a.e.;}$$

(b) if $f \in I^{\alpha}_{a+}(L^1)$ *then*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(t) = f(t), \qquad t \in [a,b] \text{ a.e.}$$

Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $f \in L^1([a,b],\mathbb{R}^n)$. We say that the function f possesses *the left-sided generalized Riemann–Liouville derivative (so called Hilfer derivative) $D^{\alpha,\beta}_{a+}f$ of order α and type β*, if the function $I^{(1-\alpha)(1-\beta)}_{a+}f$ is absolutely continuous on $[a,b]$ and then

$$(D^{\alpha,\beta}_{a+}f)(t) := \left(I^{\beta(1-\alpha)}_{a+}\frac{d}{dt}I^{(1-\alpha)(1-\beta)}_{a+}f\right)(t), \qquad t \in [a,b] \text{ a.e.} \tag{2.2}$$

The operator $D^{\alpha,\beta}_{a+}f$, given by (2.2), was introduced by Hilfer in [8].

We have the following comments (cf. [7, Remark 19]).

Remark 2.8.

1. The Hilfer derivative $D^{\alpha,\beta}_{a+}f$ can be written as

$$(D^{\alpha,\beta}_{a+}f)(t) := \left(I^{\beta(1-\alpha)}_{a+}\frac{d}{dt}I^{1-\gamma}_{a+}f\right)(t) = (I^{\beta(1-\alpha)}_{a+}D^{\gamma}_{a+}f)(t) = (I^{\gamma-\alpha}_{a+}D^{\gamma}_{a+}f)(t)$$

   for a.e. $t \in [a,b]$, where $\gamma = \alpha + \beta - \alpha\beta$.

2. The $D^{\alpha,\beta}_{a+}f$ derivative is considered as an interpolator between the Riemann–Liouville and Caputo derivative since (cf. Remark 2.1)

$$D^{\alpha,\beta}_{a+}f = \begin{cases} D^{\alpha}_{a+}f, & \beta = 0 \\ {}^{C}D^{\alpha}_{a+}f, & \beta = 1. \end{cases}$$

3. The parameter γ satisfies

$$0 < \gamma \leqslant 1, \qquad \gamma \geqslant \alpha, \qquad \gamma > \beta, \qquad 1-\gamma < 1-\beta(1-\alpha).$$

Now, we shall prove the following composition properties for the Hilfer derivative.

Lemma 2.9. *Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$ and $f \in I^{\gamma}_{a+}(L^1)$. Then*

$$(I^{\alpha}_{a+}D^{\alpha,\beta}_{a+}f)(t) = f(t), \qquad t \in [a,b] \text{ a.e.}, \tag{2.3}$$

$$(D^{\alpha,\beta}_{a+}I^{\alpha}_{a+}f)(t) = f(t), \qquad t \in [a,b] \text{ a.e.} \tag{2.4}$$

Proof. First, let us note that since $f \in I^{\gamma}_{a+}(L^1)$, therefore $I^{1-\gamma}_{a+}f \in AC([a,b],\mathbb{R}^n)$, so the derivative $D^{\alpha,\beta}_{a+}f$ exists and belongs to $L^1([a,b],\mathbb{R}^n)$. From Proposition 2.7 (a) it follows that

$$(I^{\alpha}_{a+}D^{\alpha,\beta}_{a+}f)(t) = (I^{\gamma}_{a+}D^{\gamma}_{a+}f)(t) = f(t), \qquad t \in [a,b] \text{ a.e.}$$

Moreover, since $\gamma > \beta(1-\alpha)$, therefore, using Lemma 2.6, we assert that $I^{\gamma}_{a+}(L^1) \subset I^{\beta(1-\alpha)}_{a+}(L^1)$. Consequently, the derivative $D^{\beta(1-\alpha)}_{a+}f$, so also $D^{\alpha,\beta}_{a+}I^{\alpha}_{a+}f$, exist and belong to $L^1([a,b],\mathbb{R}^n)$. Using once again Proposition 2.7 (a) we conclude

$$(D^{\alpha,\beta}_{a+}I^{\alpha}_{a+}f)(t) = (I^{\beta(1-\alpha)}_{a+}D^{\beta(1-\alpha)}_{a+}f)(t) = f(t), \qquad t \in [a,b] \text{ a.e.}$$

The proof is completed. □

3 Cauchy problem

In this section we investigate the problem (1.1)–(1.2). First, we consider it with zero initial condition. We shall prove a theorem on the existence and uniqueness of a solution to such problem. Next, using obtained result, we shall prove the result of a such type for problem (1.1) with nonzero initial condition (1.2).

3.1 Homogenous Cauchy problem

Let us consider the following Cauchy problem

$$\begin{cases} (D^{\alpha,\beta}_{a+}x)(t) = h(t, x(t)), & t \in [a,b] \text{ a.e.} \\ (I^{1-\gamma}_{a+}x)(a) = 0, \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$ and $h : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

By a solution to this problem we shall mean a function $x \in I^{\gamma}_{a+}(L^1)$ satisfying the above equation almost everywhere on $[a,b]$ (from proposition 2.4 it follows that each function belonging to $I^{\gamma}_{a+}(L^1)$ satisfies the initial condition).

We have the following theorem.

Theorem 3.1. *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$ and $h(\cdot, x(\cdot)) \in L^1([a,b], \mathbb{R}^n)$ for any function $x \in I^{\alpha}_{a+}(L^1)$. If $x \in I^{\gamma}_{a+}(L^1)$ then x is a solution to problem (3.1) if and only if x satisfies the following integral equation*

$$x(t) = (I^{\alpha}_{a+}h(\cdot, x(\cdot))(t), \qquad t \in [a,b] \text{ a.e.} \tag{3.2}$$

Proof. Let $x \in I^{\gamma}_{a+}(L^1)$ be a solution to problem (3.1). Applying the operator I^{α}_{a+} to both sides of equation (3.1) and using equality (2.3) we assert that x is a solution to integral equation (3.2).

Now, let assume that $x \in I^{\gamma}_{a+}(L^1)$ satisfies (3.2). Then there exists the derivative $D^{\gamma}_{a+}x = \varphi$ almost everywhere on $[a,b]$, where $\varphi \in L^1([a,b], \mathbb{R}^n)$ is a function such that $x = I^{\gamma}_{a+}\varphi$. Consequently, there exists the derivative $D^{\alpha,\beta}_{a+}x$ and

$$(D^{\alpha,\beta}_{a+}x)(t) = (I^{\gamma-\alpha}_{a+}\varphi)(t), \qquad t \in [a,b] \text{ a.e.} \tag{3.3}$$

Moreover, from (3.2), it follows that

$$(I^{\gamma}_{a+}\varphi)(t) = x(t) = (I^{\alpha}_{a+}h(\cdot, x(\cdot)))(t), \qquad t \in [a,b] \text{ a.e.}$$

So

$$(I^{\alpha}_{a+}I^{\gamma-\alpha}_{a+}\varphi)(t) - (I^{\alpha}_{a+}h(\cdot, x(\cdot)))(t) = 0, \qquad t \in [a,b] \text{ a.e.}$$

Applying the operator D^{α}_{a+} to both sides of the last equality and using Proposition 2.7 (a) we obtain

$$(I^{\gamma-\alpha}_{a+}\varphi)(t) = h(t, x(t)), \qquad t \in [a,b] \text{ a.e.}$$

Hence and from equality (3.3) we conclude that x satisfies equation (3.1). Since $x \in I^{\gamma}_{a+}(L^1)$, therefore the inital condition is satisfied.

The proof is completed. □

Remark 3.2. It is easy to verify that the condition: $h(\cdot, x(\cdot)) \in L^1([a,b], \mathbb{R}^n)$ for any function $x \in I^{\alpha}_{a+}(L^1)$ is satisfied if h is measurable on $[a,b]$, satisfies the Lipschitz condition with respect to the second variable and the function $[a,b] \ni t \to h(t,0) \in \mathbb{R}^n$ is summable on $[a,b]$.

Now, we prove the following theorem.

Theorem 3.3. *Let* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If*

(1_h) $h(\cdot, x(\cdot)) \in I^{\beta(1-\alpha)}_{a+}(L^1)$ *for any function* $x \in I^{\alpha}_{a+}(L^1)$

(2_h) *there exists a constant* $N > 0$ *such that*

$$|h(t,x_1) - h(t,x_2)| \leq N|x_1 - x_2|, \qquad t \in [a,b] \text{ a.e., } x_1, x_2 \in \mathbb{R}^n,$$

then problem (3.1) *possesses a unique solution* $x \in I^{\gamma}_{a+}(L^1)$.

Proof. Let us consider the operator $S : I^{\alpha}_{a+}(L^1) \to I^{\alpha}_{a+}(L^1)$ given by

$$S(x)(t) = \left(I^{\alpha}_{a+} h(\cdot, x(\cdot))\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s,x(s))}{(t-s)^{1-\alpha}} ds, \qquad t \in [a,b] \text{ a.e.}$$

It is easy to check that S is well defined. Now, let us consider in $I^{\alpha}_{a+}(L^1)$ the Bielecki norm given by

$$\|x\|_k := \int_a^b e^{-kt} |D^{\alpha}_{a+} x(t)| dt,$$

where $k > 0$ is a fixed constant. We shall show that S is contractive.

Using Proposition 2.7 (b), assumption (2_h) and Theorem 2.3 we obtain

$$\begin{aligned}
\|S(x) - S(y)\|_k &= \int_a^b e^{-kt} |h(t,x(t)) - h(t,y(t))| dt \leq N \int_a^b e^{-kt} |x(t) - y(t)| dt \\
&= N \int_a^b e^{-kt} |I^{\alpha}_{a+} D^{\alpha}_{a+}(x(t) - y(t))| dt \leq N \int_a^b e^{-kt} (I^{\alpha}_{a+} |D^{\alpha}_{a+}(x-y)|)(t) dt \\
&= N \int_a^b |D^{\alpha}_{a+}(x-y)(t)| (I^{\alpha}_{b-} e^{-k\cdot})(t) dt \\
&= N \int_a^b |D^{\alpha}_{a+}(x-y)(t)| \left(\frac{1}{\Gamma(\alpha)} \int_t^b \frac{e^{-k\tau}}{(\tau - t)^{1-\alpha}} d\tau \right) dt.
\end{aligned}$$

Let us note that (cf. [10, proof of Theorem 3.1])

$$\int_t^b \frac{e^{-k\tau}}{(\tau - t)^{1-\alpha}} d\tau \leqslant \Gamma(\alpha) e^{-kt} k^{-\alpha}.$$

Consequently

$$\begin{aligned}
\|S(\varphi) - S(\psi)\|_k &\leq N \int_a^b |D^{\alpha}_{a+}(x-y)(t)| \left(\frac{1}{\Gamma(\alpha)} \Gamma(\alpha) e^{-kt} k^{-\alpha} \right) dt \\
&= N k^{-\alpha} \int_a^b e^{-kt} |D^{\alpha}_{a+}(x-y)(t)| dt = N k^{-\alpha} \|x - y\|_k.
\end{aligned}$$

Since $Nk^{-\alpha} \in (0,1)$ for sufficiently large k, therefore the operator S has a unique fixed point. It means that integral equation (3.2) possesses a unique solution $x_* \in I^{\alpha}_{a+}(L^1)$.

From assumption (1_h) it follows that there exists a function $\psi \in L^1([a,b], \mathbb{R}^n)$ such that for any $x \in I^{\alpha}_{a+}(L^1)$ $h(\cdot, x(\cdot)) = I^{\beta(1-\alpha)}_{a+} \psi(\cdot)$ almost everywhere on $[a,b]$. Thus

$$x_*(t) = \left(I^{\alpha}_{a+} h(\cdot, x_*(\cdot))\right)(t) = \left(I^{\alpha}_{a+} I^{\beta(1-\alpha)}_{a+} \psi\right)(t) = \left(I^{\gamma}_{a+} \psi\right)(t) \in I^{\gamma}_{a+}(L^1).$$

The proof is completed. □

3.2 Nonhomogenous Cauchy problem

Now, we consider the Cauchy problem (1.1)–(1.2) with $c \neq 0$.

By a solution to such problem we shall mean a function $y \in AC^{\gamma}_{a+}([a,b],\mathbb{R}^n)$, where (cf. [9])

$$AC^{\gamma}_{a+}([a,b],\mathbb{R}^n) = \Big\{ f : [a,b] \to \mathbb{R}^n : \quad f(t) = \frac{\tilde{c}}{\Gamma(\gamma)}(t-a)^{\gamma-1} + (I^{\gamma}_{a+}\varphi)(t), \\ t \in [a,b] \text{ a.e.},\ \varphi \in L^1([a,b],\mathbb{R}^n),\ \tilde{c} \in \mathbb{R}^n \Big\}.$$

It is easy to show that if $x_*(\cdot) \in I^{\gamma}_{a^+}(L^1)$ is a solution to problem (3.1) with the function h of the form

$$h(t,x) = g\left(t, x + \frac{c}{\Gamma(\gamma)} \frac{1}{(t-a)^{1-\gamma}}\right), \tag{3.4}$$

then the function

$$y_*(\cdot) = x_*(\cdot) + \frac{c}{\Gamma(\gamma)} \frac{1}{(\cdot - a)^{1-\gamma}} \tag{3.5}$$

is a solution to problem (1.1)–(1.2). Conversely, if $y_*(\cdot) \in AC^{\gamma}_{a^+}([a,b],\mathbb{R}^n)$ is a solution to problem (1.1)–(1.2) with the function g of the form

$$g(t,y) = h\left(t, y - \frac{\tilde{c}}{\Gamma(\gamma)} \frac{1}{(t-a)^{1-\gamma}}\right),$$

then $\tilde{c} = c$ and

$$x_*(\cdot) = y_*(\cdot) - \frac{c}{\Gamma(\gamma)} \frac{1}{(\cdot - a)^{1-\gamma}}$$

is a solution to problem (3.1).

So, using Theorem 3.3, we can prove the following

Theorem 3.4. *Let* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If*

(1_g) $g\left(\cdot, y(\cdot) + \frac{c}{\Gamma(\gamma)} \frac{1}{(\cdot - a)^{1-\gamma}}\right) \in I^{\beta(1-\alpha)}_{a+}(L^1)$ *for any function* $y \in I^{\alpha}_{a+}(L^1)$,

(2_g) *there exists a constant* $\tilde{N} > 0$ *such that*

$$|g(t,y_1) - g(t,y_2)| \leq \tilde{N}|y_1 - y_2|, \qquad t \in [a,b] \text{ a.e.}, y_1, y_2 \in \mathbb{R}^n,$$

then problem (1.1)–(1.2) *possesses a unique solution* $y \in AC^{\gamma}_{a+}([a,b],\mathbb{R}^n)$.

Proof. In order to prove the existence part of the above theorem it suffices to show that if g satisfies assumptions (1_g), (2_g), then the function h given by (3.4) satisfies conditions (1_h), (2_h) from Theorem 3.3. Indeed, the fact that h satisfies the Lipschitz condition with respect to the second variable is obvious. Moreover, for any $x \in I^{\alpha}_{a+}(L^1)$ we have

$$h(\cdot, x(\cdot)) = g\left(\cdot, x(\cdot) + \frac{c}{\Gamma(\gamma)} \frac{1}{(\cdot - a)^{1-\gamma}}\right) \in I^{\beta(1-\alpha)}_{a+}(L^1).$$

A uniqueness of the solution to problem (1.1)–(1.2) follows from the uniqueness of the solution to homogeneous problem.

The proof is completed. □

4 Linear Cauchy problem

In the previous section we obtained the existence of a unique solution to nonlinear Cauchy problem (1.1)–(1.2). Similarly as in paper [7] our method relies on the change of such problem over to the equivalent integral equation and next, using the Banach fixed point theorem, solving this equation. The obtained solution belongs to the space $AC^{\gamma}_{a+}([a,b],\mathbb{R}^n)$ (generally, in contrast to the paper [7], this is the space of non-continuous functions). An advantage of our paper is the fact that proofs of main results are not complicated and rather short. Unfortunately, the existence results were proved under the key assumption (1_h) $((1_g))$, which generally is difficult to check (except the case $\beta = 0$ – cf. Remark 3.2).

In this section we shall consider the linear Cauchy problem of a type (1.1)–(1.2). We shall show that in this case the mentioned assumption reduces to a condition, which is easier to verify. Moreover, we give the formula for a solution to such problem.

In our opinion the obtained results concerning the linear problem are useful in applications – for example in linear control systems involving the Hilfer derivative.

4.1 Homogenous problem

Let us consider the following linear Cauchy problem

$$\begin{cases} (D^{\alpha,\beta}_{a+}x)(t) = Ax(t) + v(t), & t \in [a,b] \text{ a.e.} \\ (I^{1-\gamma}_{a+}x)(a) = 0, \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$ and $A \in \mathbb{R}^{n\times n}$.

If $\beta(1-\alpha) < \alpha$ and $v \in I^{\beta(1-\alpha)}_{a+}(L^1)$, then Lemma 2.6 guarantees satisfying assumption (1_h) from Theorem 3.3. Consequently, there exists a unique solution $x \in I^{\gamma}_{a+}(L^1)$ to such problem.

Now, we shall show that the existence result can be obtained for any $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$. Indeed, from the proof of Theorem 3.3 it follows that the operator $S : I^{\alpha}_{a+}(L^1) \to I^{\alpha}_{a+}(L^1)$ given by

$$S(x)(t) = A(I^{\alpha}_{a+}x)(t) + (I^{\alpha}_{a+}v)(t), \qquad t \in [a,b] \text{ a.e.}$$

has a unique fixed point $x_* \in I^{\alpha}_{a+}(L^1)$. So there exists a function $\varphi_* \in L^1([a,b],\mathbb{R}^n)$ such that

$$\begin{aligned} x_*(t) &= S(x_*)(t) = A(I^{\alpha}_{a+}x_*)(t) + (I^{\alpha}_{a+}v)(t) \\ &= A^m(I^{m\alpha}_{a+}x_*)(t) + A^{m-1}(I^{m\alpha}_{a+}v)(t) + \cdots + A\left(I^{2\alpha}_{a+}v\right)(t) + (I^{\alpha}_{a+}v)(t) \\ &= A^m(I^{(m+1)\alpha}_{a+}\varphi_*)(t) + A^{m-1}(I^{m\alpha}_{a+}v)(t) + \cdots + A\left(I^{2\alpha}_{a+}v\right)(t) + (I^{\alpha}_{a+}v)(t) \end{aligned} \tag{4.2}$$

for all $m \in \mathbb{N}$ and $t \in [a,b]$ a.e.

Let us note that since $v \in I^{\beta(1-\alpha)}_{a+}(L^1)$, therefore there exists a function $\psi \in L^1([a,b],\mathbb{R}^n)$ such that

$$(I^{m\alpha}_{a+}v)(t) = (I^{m\alpha}_{a+}I^{\beta(1-\alpha)}_{a+}\psi)(t) = (I^{(m-1)\alpha}_{a+}I^{\gamma}_{a+}\psi)(t) = (I^{\gamma}_{a+}I^{(m-1)\alpha}_{a+}\psi)(t), \quad t \in [a,b] \text{ a.e.}, \ m \in \mathbb{N}.$$

From [17, Theorem 2.6] it follows that $I^{(m-1)\alpha}_{a+}\psi \in L^1([a,b],\mathbb{R}^n)$ for all $m \in \mathbb{N}$. It means that $A^{m-1}I^{m\alpha}_{a+}v \in I^{\gamma}_{a+}(L^1)$ for all $m \in \mathbb{N}$. Moreover, there exists $m \in \mathbb{N}$ such that $(m+1)\alpha \geqslant \gamma$ and $\delta := (m+1)\alpha - \gamma \in (0,1)$. Consequently

$$A^m(I^{(m+1)\alpha}_{a+}\varphi_*)(t) = A^m(I^{\gamma}_{a+}I^{\delta}_{a+}\varphi_*)(t), \qquad t \in [a,b] \text{ a.e.}$$

Using once again Theorem 2.6 from [17] we assert that $A^m I_{a+}^{(m+1)\alpha}\varphi_* \in I_{a+}^{\gamma}(L^1)$. So we showed that all terms of the equality (4.2) belong to the space $I_{a+}^{\gamma}(L^1)$. Thus and from Theorem 3.1 we conclude that there exists a unique solution x_* to problem (4.1) belonging to $I_{a+}^{\gamma}(L^1)$.

Using the Laplace transform one can prove that a formula for this solution is the following (cf. [18, Lemma 7]):

$$x_*(t) = \int_a^t \Phi_\alpha(t-s)v(s)ds, \qquad t \in [a,b] \ a.e., \tag{4.3}$$

where $\Phi_\alpha(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$.

4.2 Nonhomogeneous problem

Now, let us consider the following linear nonhomogeneous Cauchy problem

$$\begin{cases} (D_{a+}^{\alpha,\beta}y)(t) = Ay(t) + v(t), & t \in [a,b] \ a.e. \\ (I_{a+}^{1-\gamma}y)(a) = c, \end{cases} \tag{4.4}$$

where $c \in \mathbb{R}^n$ is a fixed point.

It is easy to check that if $x \in I_{a+}^{\gamma}(L^1)$ is a solution to homogeneous problem of the form

$$\begin{cases} (D_{a+}^{\alpha,\beta}x)(t) = Ax(t) + \frac{Ac}{\Gamma(\gamma)}\frac{1}{(t-a)^{1-\gamma}} + v(t), & t \in [a,b] \ a.e. \\ (I_{a+}^{1-\gamma}x)(a) = 0, \end{cases} \tag{4.5}$$

then

$$y(\cdot) = x(\cdot) + \frac{c}{\Gamma(\gamma)}\frac{1}{(\cdot - a)^{1-\gamma}} \in AC_{a+}^{\gamma}([a,b],\mathbb{R}^n) \tag{4.6}$$

is a solution to problem (4.4).

Since

$$\left(I_{a+}^{1-\beta(1-\alpha)}(\cdot - a)^{\gamma}\right)(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha+1)}(t-a)^{\alpha} \in AC([a,b],\mathbb{R}^n) \quad \text{for } t \in [a,b]$$

and $(I_{a+}^{1-\beta(1-\alpha)}(\cdot - a)^{\gamma})(a) = 0$, therefore $I_{a+}^{1-\beta(1-\alpha)}(\cdot - a)^{\gamma} \in I_{a+}^{\beta(1-\alpha)}(L^1)$ (cf. Proposition 2.4). Consequently if $v \in I_{a+}^{\beta(1-\alpha)}(L^1)$, then problem (4.5) has a unique solution $x \in I_{a+}^{\gamma}(L^1)$. Thus and from (4.6) it follows that there exists a unique solution $y \in AC_{a+}^{\gamma}([a,b],\mathbb{R}^n)$ to problem (4.4). Moreover, it is given by (cf. [18, Lemma 7])

$$y(t) = \Psi_{\alpha,\gamma}(t-a)c + \int_a^t \Phi_\alpha(t-s)v(s)ds, \qquad t \in [a,b] \ a.e., \tag{4.7}$$

where $\Phi_\alpha(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$ and $\Psi_{\alpha,\gamma}(t) = \sum_{k=0}^{\infty} \frac{A^k t^{\gamma+k\alpha-1}}{\Gamma(\gamma+k\alpha)}$.

Corollary 4.1.

1. *If $\beta = 0$, then $\gamma = \alpha$, $\Phi_\alpha = \Psi_{\alpha,\gamma}$ and*

$$y(t) = \Phi_\alpha(t-a)c + \int_a^t \Phi_\alpha(t-s)v(s)ds \in I_{a+}^{\alpha}(L^1), \qquad t \in [a,b] \ a.e. \tag{4.8}$$

is a solution to the following linear Cauchy problem involving the Riemann–Liouville derivative (cf. [10, Theorem 4.2])

$$\begin{cases} (D_{a+}^{\alpha}y)(t) = Ay(t) + v(t), & t \in [a,b] \ a.e. \\ (I_{a+}^{1-\alpha}y)(a) = c; \end{cases}$$

2. *if* $\beta = 1$, *then* $\gamma = 1$, $\Psi_{\alpha,\gamma} = E(At^\alpha)$, *where* $E(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z > 0$ *is the Mittag-Leffler function and*

$$y(t) = E(A(t-a)^\alpha)c + \int_a^t \Phi_\alpha(t-s)v(s)ds \in AC([a,b],\mathbb{R}^n), \qquad t \in [a,b] \tag{4.9}$$

is a solution to the following linear Cauchy problem involving the Caputo derivative

$$\begin{cases} ({}^C D^\alpha_{a+}y)(t) = Ay(t) + v(t), & t \in [a,b] \text{ a.e.} \\ y(a) = c; \end{cases}$$

3. *if we put* $I^0_{a+}y = y$, *then we can consider the following Cauchy problem of order* $\alpha = 1$

$$\begin{cases} y'(t) = Ay(t) + v(t), & t \in [a,b] \text{ a.e.} \\ y(a) = c. \end{cases}$$

Then all formulas for the solution: (4.7), (4.8) *and* (4.9) *reduce to the classical one*

$$y(t) = e^{A(t-a)}c + \int_a^t e^{A(t-s)}v(s)ds \in AC([a,b],\mathbb{R}^n), \qquad t \in [a,b].$$

5 Conclusion

In this work we have proved existence and uniqueness solution for fractional Cauchy problems involving Hilfer's derivative. Results of such a type in a some weighted space of continuous functions have been obtained by Furati at al. [7]. Here we consider a different space of solutions (generally the space of non-continuous functions), which is, in our opinion, more useful in applications. In proofs of our results, similarly as in paper [7], we apply the Banach contraction principle. Besides we use a notion of the Bielecki norm and due to such approach our argument is different than in [7] (we do not need to partition the interval $[a,b]$).

References

[1] M. Benchora, F. Ouaar, Existence results for nonlinear fractional differential equations with integral boundary conditions, *Bull. Math. Anal. Appl.* **2**(2010), No. 4, 7–15. MR2747882

[2] A. Carpinteri, F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Springer-Verlag, Wien, 1997. MR1611582; url

[3] K. Diethelm, N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265**(2002), 229–248. MR1876137; url

[4] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204**(1996), No. 2, 609–625. MR1421467; url

[5] K. M. Furati, N.-E. Tatar, An existence result for a nonlocal fractional differential problem, *J. Fract. Calc.* **26**(2004), 43–51. MR2096756

[6] K. M. Furati, N.-E. Tatar, Behavior of solutions for a weighted Cauchy-type fractional differential problem, *J. Fract. Calc.* **28**(2005), 23–42. MR2176059

[7] K. M. Furati, M. D. Kassim, N.-E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput. Math. Appl.* **64**(2012), 1616–1626. MR2176059; url

[8] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000. MR1890104; url

[9] D. Idczak, Fractional du Bois-Reymond lemma of order $\alpha \in (\frac{1}{2}, 1)$, in: *Differential equations and dynamical systems (Proc. 7th Internat. Workshop on Multidimensional (nD) Systems (nDs)*, Poitiers, France, 2011. url

[10] D. Idczak, R. Kamocki, On the existence and uniqueness and formula for the solution of R–L fractional Cauchy problem in $\mathbb{R}^n$, *Fract. Calc. Appl. Anal.* **14**(2011), No. 4, 538–553. MR2846375; url

[11] D. Idczak, A. Skowron, S. Walczak, Sensitivity of a fractional integrodifferential Cauchy problem of Volterra type, *Abstr. Appl. Anal.* **2013**, Art. ID 129478, 8 pp. MR3121404; url

[12] A. A. Kilbas, S. A. Marzan, Cauchy problem for differential equation with Caputo derivative, *Fract. Calc. Anal. Appl.* **7**(2004), No. 3, 297–320. MR2252568

[13] A. A. Kilbas, S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differ. Equ.* **41**(2005), No. 1, 84–89. MR2213269

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006. MR2218073

[15] A. B. Malinowska, D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imperial College Press, London, 2012. MR2984893

[16] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, Vol. 198, Academic Press, San Diego, 1999. MR1658022

[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993. MR1347689

[18] J. Sikora, W. Wójcik, *Modelling and optimization*, Lublin University of Technology, 2011.

[19] Y. Zhou, Existence and uniqueness of solutions for a system of fractional differential equations, *Fract. Calc. Anal. Appl.* **12**(2009), No. 2, 195–204. MR2498366