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Existence and multiplicity of periodic solutions to one-dimensional p-Laplacian

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Abstract. This paper deals with the existence and multiplicity of periodic solutions for the one-dimensional p-Laplacian. The minimization argument and extended Clark's theorem are applied to prove our results. The corresponding impulsive problem is considered as well.

Keywords: periodic solution, p-Laplacian, minimization theorem, Clark's theorem, weak solution, impulsive problem.

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1 Introduction

Let $s > 1$ be a real number and

$$\varphi_s(\tau) = \begin{cases} |\tau|^{s-2}\tau, & \tau \neq 0, \\ 0, & \tau = 0. \end{cases}$$

For $p > 1$, $q > r > 1$, and $a = a(t), b = b(t)$ positive continuous T-periodic functions on $[0, T]$, we consider the one-dimensional p-Laplacian periodic problem

$$\begin{cases} (\varphi_p(u'(t)))' - a(t)\varphi_q(u(t)) + b(t)\varphi_r(u(t)) = 0, & t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P$_\mathrm{T}$}$$

For $p = r = 2$ and $q = 4$, the equation in (P$_\mathrm{T}$) is known as the stationary Fisher–Kolmogorov equation and appears in biomathematical models (see, e.g., [1, 6]). Its periodic solutions have been studied in [10, 11, 16] using variational approach and critical point theorems. In [4], problem (P$_\mathrm{T}$) with $p = 2$ and $q > r > 1$ is studied using the minimization argument and Clark's theorem (see [5, 13, 15] for this assertion). The purpose of our paper is

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to treat the quasilinear case $p \neq 2$ variationally and to prove existence and multiplicity results for problem ($\mathrm{P_T}$) and associated impulsive problem.

We formulate our result for ($\mathrm{P_T}$) as follows.

Theorem 1.1. *Let $p > 1$, $q > r > 1$, and $a = a(t)$, $b = b(t)$ be positive continuous T-periodic functions on $[0, T]$. Then ($\mathrm{P_T}$) has at least one solution.*

If, in addition, we assume $p > r$ then ($\mathrm{P_T}$) has infinitely many pairs of solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\max_{t\in[0,T]} |u_m(t)| \to 0$ as $m \to \infty$.

Now, we extend our result to the following impulsive problem.

We denote $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = T$ and set $\mathcal{J} = \bigcup_{j=0}^{l} \mathcal{J}_j$, where $\mathcal{J}_j = (t_j, t_{j+1})$, $j = 0, \ldots, l$. We study the problem

$$\begin{cases} (\varphi_p(u'(t)))' - a(t)\varphi_q(u(t)) + b(t)\varphi_r(u(t)) = 0 & \text{for } t \in \mathcal{J}, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \\ \Delta(\varphi_p(u'(t_j))) = g_j\left(u(t_j)\right) & \text{for } j = 1, \ldots, l, \end{cases} \tag{$\mathrm{Q_T}$}$$

where $\Delta\big(\varphi_p(u'(t_j))\big) := \varphi_p\big(u'(t_j^+)\big) - \varphi_p\big(u'(t_j^-)\big)$, $u'(t_j^\pm) = \lim_{t\to t_j^\pm} u'(t)$, and $g_j : \mathbf{R} \to \mathbf{R}$ are given continuous functions.

Recently many authors applied variational methods to prove the existence results for similar impulsive problems (see [3,8,14,17]). Our impulsive conditions express the sudden changes in the "velocity" at given times $t_j \in (0, T)$. These changes depend on the "state" $u(t_j)$ via given continuous functions $g_j : \mathbf{R} \to \mathbf{R}$.

We formulate the result for impulsive problem ($\mathrm{Q_T}$) as follows.

Theorem 1.2. *Let $p > 1$, $q > r > 1$, $a = a(t)$, $b = b(t)$ be positive continuous T-periodic functions on $[0, T]$ and $g_j : \mathbf{R} \to \mathbf{R}$ $(j = 1, \ldots, l)$ be continuous functions satisfying for all $\tau \in \mathbf{R}$ and $j = 1, \ldots, l$,*

$$\int_0^\tau g_j(\sigma)\, \mathrm{d}\sigma \geq c \tag{1.1}$$

with a given constant $c \in \mathbf{R}$. Then ($\mathrm{Q_T}$) has at least one solution.

If, in addition, $p > r$ and for all $\tau \in \mathbf{R}$ and $j = 1, \ldots, l$,

$$\int_0^\tau g_j(\sigma)\, \mathrm{d}\sigma \leq 0, \tag{1.2}$$

and g_j are odd functions, then ($\mathrm{Q_T}$) has infinitely many pairs of solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\max_{t\in[0,T]} |u_m(t)| \to 0$ as $m \to \infty$.

Remark 1.3. Theorem 1.1 and 1.2 can be extended to equations with more general nonlinear terms as

$$(\varphi_p(u'(t)))' - f(t, u(t)) + h(t, u(t)) = 0.$$

Let

$$F(t, u) = \int_0^u f(t, \sigma)\, \mathrm{d}\sigma, \qquad H(t, u) = \int_0^u h(t, \sigma)\, \mathrm{d}\sigma.$$

Suppose that functions $f(t, \sigma)$ and $h(t, \sigma)$ are continuous in (t, σ) and there exist positive constants $a_1, a_2, b_1, b_2, q > r > 1$ such that for all $u \in \mathbf{R}$,

$$a_1|u|^q \leq F(t, u) \leq a_2|u|^q, \qquad b_1|u|^r \leq H(t, u) \leq b_2|u|^r.$$

With the same assumptions on p, q, r, the existence parts of Theorem 1.1 and 1.2 are valid. If, moreover, $f(t, \sigma)$ and $h(t, \sigma)$ are odd functions of σ, the multiplicity results are valid, too.

Remark 1.4. In order to illustrate an application of Theorem 1.2, we present two easy examples of impulsive functions.

Let $l = 1$ and $g_1(\sigma) = \frac{1}{1+\sigma^2}$. Then $\int_0^\tau g_1(\sigma)\,\mathrm{d}\sigma = \arctan\tau \geq -\frac{\pi}{2}$, i.e., (1.1) holds. However, g_1 is neither odd nor satisfies (1.2). Hence, only the existence part of Theorem 1.2 holds true.

On the other hand, for $g_1(\sigma) = \frac{-2\sigma}{(1+\sigma^2)^2}$, we have $-1 \leq \int_0^\tau g_1(\sigma)\,\mathrm{d}\sigma = \frac{-\tau^2}{1+\tau^2} \leq 0$, i.e., (1.1) and (1.2) hold. Since g_1 is odd, also the multiplicity result of Theorem 1.2 holds.

2 Preliminaries

Let $p > 1$ and

$$X_p := \left\{ u \in W^{1,p}(0,T) : u(0) = u(T) \right\}$$

be equipped with the Sobolev norm

$$\|u\| = \left(\int_0^T \left(|u'(t)|^p + |u(t)|^p \right) \mathrm{d}t \right)^{1/p}.$$

Then X_p is a uniformly convex (and hence reflexive) Banach space. Let X_p^* be the dual of X_p and $\langle \cdot, \cdot \rangle$ be the duality pairing between X_p^* and X_p.

In our estimates we use the following inequalities.

Lemma 2.1 (Wirtinger and Sobolev inequalities, see [7, 13])**.** *There exist constants $K_1 > 0$ and $K_2 > 0$ such that for*

$$u \in W := \left\{ W^{1,p}(0,T) : \int_0^T u(t)\,\mathrm{d}t = 0 \right\},$$

we have

$$\|u\|_{L^p}^p := \int_0^T |u(t)|^p\,\mathrm{d}t \leq K_1 \int_0^T |u'(t)|^p\,\mathrm{d}t,$$
$$\|u\|_{L^\infty} := \max_{t\in[0,T]} |u(t)| \leq K_2 \|u\|.$$

Remark 2.2. By Lemma 2.1, $\|u\|_W = \left(\int_0^T |u'(t)|^p\,\mathrm{d}t \right)^{1/p}$ defines the norm which is equivalent to $\|u\|$ on W.

We say that $u \in X_p$ is a *weak solution* of ($\mathrm{P_T}$) if the integral identity

$$\int_0^T \left[\varphi_p\left(u'(t)\right) v'(t) + a(t)\varphi_q\left(u(t)\right) v(t) - b(t)\varphi_r\left(u(t)\right) v(t) \right] \mathrm{d}t = 0$$

holds for any function $v \in X_p$.

Let $\Phi_s(\tau) = \frac{|\tau|^s}{s}$ be the antiderivative of $\varphi_s(\tau)$. We introduce the functional $I : X_p \to \mathbf{R}$ associated with ($\mathrm{P_T}$) as follows:

$$I(u) := \int_0^T \left[\Phi_p\left(u'(t)\right) + a(t)\Phi_q\left(u(t)\right) - b(t)\Phi_r\left(u(t)\right) \right] \mathrm{d}t.$$

Its Gâteaux derivative at $u \in X_p$ in the direction $v \in X_p$ is given by

$$\langle I'(u), v \rangle = \int_0^T \left[\varphi_p\left(u'(t)\right) v'(t) + a(t)\varphi_q\left(u(t)\right) v(t) - b(t)\varphi_r\left(u(t)\right) v(t) \right] \mathrm{d}t.$$

Hence, critical points of I are in one-to-one correspondence with weak solutions of ($\mathrm{P_T}$).

By a *classical solution* of ($\mathrm{P_T}$) we understand a function $u \in C^1[0,T]$ such that $\varphi_p(u'(\cdot)) \in C^1(0,T)$, the equation in ($\mathrm{P_T}$) holds pointwise in $(0,T)$ and $u(0) = u(T)$, $u'(0) = u'(T)$.

We say that $u \in X_p$ is a *weak solution* of impulsive problem ($\mathrm{Q_T}$) if the identity

$$\int_0^T \left[\varphi_p\left(u'(t)\right) v'(t) + a(t)\varphi_q\left(u(t)\right) v(t) - b(t)\varphi_r\left(u(t)\right) v(t)\right] \mathrm{d}t + \sum_{j=1}^{l} g_j\left(u(t_j)\right) v(t_j) = 0$$

holds for any $v \in X_p$. Let $G_j(\tau) = \int_0^\tau g_j(\sigma)\, \mathrm{d}\sigma$, $j = 1,\dots,l$. Then the functional $J : X_p \to \mathbf{R}$ associated with ($\mathrm{Q_T}$), defined by

$$J(u) := \int_0^T \left[\Phi_p\left(u'(t)\right) + a(t)\Phi_q\left(u(t)\right) - b(t)\Phi_r\left(u(t)\right)\right] \mathrm{d}t + \sum_{j=1}^{l} G_j\left(u(t_j)\right),$$

is Gâteaux differentiable at any $u \in X_p$ and its critical points are in one-to-one correspondence with weak solutions of ($\mathrm{Q_T}$).

By a *classical solution* of impulsive problem ($\mathrm{Q_T}$) we understand a function $u \in C[0,T]$ such that $u \in C^1(\mathcal{J}_j)$, $\varphi_p(u'(\cdot)) \in C^1(\mathcal{J}_j)$, $j = 0,\dots,l$, the equation in ($\mathrm{Q_T}$) holds pointwise in $\mathcal{J}$, $\Delta\left(\varphi_p\left(u'\left(t_j\right)\right)\right) = g_j\left(u\left(t_j\right)\right)$, $j = 1,\dots,l$, and $u(0) = u(T)$, $u'(0) = u'(T)$.

Note that by a standard regularity argument, every weak solution of ($\mathrm{P_T}$) and ($\mathrm{Q_T}$) is also a classical solution and vice versa (see, e.g., [8,16,17]).

Our approach is variational. The existence part of our result relies on the standard minimization argument (see, e.g., [2,9,13]) applied to I and J, respectively. We state it explicitly below for reader's convenience.

Theorem 2.3 (Minimization argument)**.** *Let $E : X \to \mathbf{R}$ be weakly sequentially lower semicontinuous functional on a reflexive Banach space X and let E have a bounded minimizing sequence. Then E has a minimum on X, i.e., there exists $u_0 \in X$ such that $E(u_0) = \inf_{u\in X} E(u)$. If E is differentiable then u_0 is a critical point of E.*

Our multiplicity result in Theorem 1.1 relies on the generalization of Clark's theorem. See [15, pp. 53–54] for the original version of Clark's theorem which has been applied by many authors (see, e.g., [4,11,16]). In our paper we use the extension of Clark's theorem proved recently by Liu and Wang [12]. For reader's convenience, we present this extended version.

Theorem 2.4 ([12, Theorem 1.1])**.** *Let X be a Banach space, $E \in C^1(X,\mathbf{R})$. Assume that E satisfies the (PS) condition, it is even and bounded from below, and $E(0) = 0$. If for any $k \in \mathbf{N}$, there exist a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} E < 0$, where $S_\rho = \{u \in X, \|u\|_X = \rho\}$, then at least one of the following conclusions holds.*

- (i) *There exists a sequence of critical points $\{u_k\}$ satisfying $E(u_k) < 0$ for all k and $\|u_k\|_X \to 0$ as $k \to \infty$.*
- (ii) *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_X = \alpha$ and $E(u) = 0$.*

In our approach, we use this assertion combined with the following remark.

Remark 2.5. It is already noted in [12], that Theorem 2.4 implies the existence of infinitely many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$, such that $E(u_m) \leq 0$, $E(u_m) \to 0$, and $\|u_m\|_X \to 0$ as $m \to \infty$.

3 Proofs of main results

We write the functional I as $I(u) = I_1(u) + I_2(u)$, where

$$I_1(u) = \int_0^T \Phi_p\left(u'(t)\right)\,\mathrm{d}t$$

and

$$I_2(u) = \int_0^T \left[a(t)\Phi_q\left(u(t)\right) - b(t)\Phi_r\left(u(t)\right)\right]\mathrm{d}t.$$

Clearly, the functional I_1 is continuous, convex and hence weakly sequentially lower semicontinuous on X_p. Due to the compact embedding $X_p \hookrightarrow\hookrightarrow C[0,T]$, I_2 is weakly sequentially continuous on X_p. Hence, I is weakly sequentially lower semicontinuous on X_p.

Similar arguments yield that the functional J is also weakly sequentially lower semicontinuous on X_p.

Since a and b are positive continuous functions on $[0,T]$, there exist constants a_i, b_i, $i = 1,2$, such that

$$0 < a_1 \le a(t) \le a_2, \qquad 0 < b_1 \le b(t) \le b_2. \tag{3.1}$$

We start with the proof of the *existence* of a solution of ($\mathrm{P_T}$). The plan is to apply Theorem 2.3 with $X = X_p$ and $E = I$. For this purpose we show that I is bounded from below on X_p and has a bounded minimizing sequence.

Consider the function $f(\tau) = \frac{1}{q}a_1\tau^q - \frac{1}{r}b_2\tau^r$, $\tau \ge 0$. Then

$$f(\tau) \ge \frac{r-q}{qr}\left(\frac{b_2^q}{a_1^r}\right)^{\frac{1}{q-r}} =: c_1.$$

Then we can estimate I from below on X_p as follows:

$$\begin{aligned} I(u) &\ge \int_0^T \Phi_p\left(u'(t)\right)\,\mathrm{d}t + \int_0^T \left(\frac{1}{q}a_1|u(t)|^q - \frac{1}{r}b_2|u(t)|^r\right)\mathrm{d}t \\ &\ge \frac{1}{p}\|u\|_W^p + Tc_1. \end{aligned}$$

Hence, $\inf_{u\in X_p} I(u) > -\infty$.

Let $(u_n) \subset X_p$ be a minimizing sequence, $I(u_n) \to \inf_{u\in X_p} I(u)$. Then there exists $c_2 \in \mathbf{R}$ such that

$$c_2 \ge I(u_n) \ge \frac{1}{p}\|\tilde{u}_n\|_W^p + Tc_1,$$

where $u_n = \bar{u}_n + \tilde{u}_n$, $\bar{u}_n \in \mathbf{R}$, $\tilde{u}_n \in W$. Hence, $(\tilde{u}_n)$ is a bounded sequence in W. Next we show that $(\bar{u}_n)$ is a bounded sequence in $\mathbf{R}$. We proceed via contradiction. Let $|\bar{u}_n| \to \infty$ as $n \to \infty$. Since $(\tilde{u}_n)$ is bounded in W, by Lemma 2.1 there exists $c_3 > 0$ such that $\|\tilde{u}_n\|_{L^\infty} \le c_3$. Thus, for $t \in [0,T]$, we have

$$|u_n(t)| \ge |\bar{u}_n| - |\tilde{u}_n(t)| \ge |\bar{u}_n| - c_3.$$

Therefore, $|u_n(t)| \to \infty$ uniformly in $[0,T]$. In other words, for any $R > 0$ there exists $N = N(R)$ such that for any $n > N$, we have

$$|u_n(t)| \ge R, \qquad t \in [0,T].$$

The function $f = f(\tau)$ is increasing for $\tau \geq \left(\frac{b_2^q}{a_1^r}\right)^{\frac{1}{q-r}} =: d$. Then, taking $R \geq d$ and $n > N(R)$, we have

$$\begin{aligned} c_2 \geq I(u_n) &\geq \int_0^T \left(\frac{1}{q} a_1 |u_n(t)|^q - \frac{1}{r} b_2 |u_n(t)|^r\right) \mathrm{d}t \\ &\geq \int_0^T \left(\frac{1}{q} a_1 R^q - \frac{1}{r} b_2 R^r\right) \mathrm{d}t = T\left(\frac{1}{q} a_1 R^q - \frac{1}{r} b_2 R^r\right). \end{aligned} \tag{3.2}$$

But $\left(\frac{1}{q} a_1 R^q - \frac{1}{r} b_2 R^r\right) \to \infty$ as $R \to \infty$ and this contradicts (3.2). Hence $(\bar{u}_n)$ is a bounded sequence in $\mathbf{R}$, i.e., (u_k) is bounded in X_p. Since I is weakly sequentially lower semicontinuous on X_p, Theorem 2.3 implies that I has a critical point in X_p. It follows from our discussions in Section 2 that this critical point is a solution of ($\mathrm{P_T}$). This concludes the proof of existence part of Theorem 1.1.

Similarly we prove the existence part of Theorem 1.2. Indeed, it follows from (1.1) that

$$J(u) \geq I(u) + cl \geq \frac{1}{p} \|u\|_W^p + Tc_1 + cl,$$

i.e., J is bounded from bellow on X_p. Due to (1.1), the boundedness of minimizing sequence is proved analogously as in the case of functional I. As mentioned above, J is weakly sequentially lower semicontinuous, and so the existence of a solution of ($\mathrm{Q_T}$) follows again from Theorem 2.3.

In order to prove the *multiplicity* result in Theorem 1.1, we need the following lemma.

Lemma 3.1. *The functional I satisfies the Palais–Smale condition on X_p.*

Proof of Lemma 3.1. Let (u_n) be a Palais–Smale sequence, i.e., $(I(u_n))$ is bounded in $\mathbf{R}$ and $I'(u_n) \to 0$ in X_p^*. From the boundedness of $(I(u_n))$, exactly as above, we deduce that (u_n) is bounded in X_p. Passing to a subsequence, if necessary, we may assume that there exists $u \in X_p$ such that $u_n \rightharpoonup u$ weakly in X_p and $u_n \to u$ strongly in $C[0, T]$. By $I'(u_n) \to 0$ in X_p^*, we have

$$\begin{aligned} 0 \leftarrow &\langle I'(u_n) - I'(u), u_n - u \rangle \\ = &\int_0^T \left[\varphi_p\left(u_n'(t)\right) - \varphi_p\left(u'(t)\right)\right]\left(u_n'(t) - u'(t)\right) \mathrm{d}t \\ &+ \int_0^T a(t)\left[\varphi_q\left(u_n(t)\right) - \varphi_q\left(u(t)\right)\right]\left(u_n(t) - u(t)\right) \mathrm{d}t \\ &- \int_0^T b(t)\left[\varphi_r\left(u_n(t)\right) - \varphi_r\left(u(t)\right)\right]\left(u_n(t) - u(t)\right) \mathrm{d}t. \end{aligned} \tag{3.3}$$

The last two terms in (3.3) tend to 0 due to the uniform convergence $u_n \to u$ in $C[0, T]$. Then, by (3.3) and Hölder's inequality, we obtain

$$\begin{aligned} 0 = &\lim_{n\to\infty} \int_0^T \left[\varphi_p\left(u_n'(t)\right) - \varphi_p\left(u'(t)\right)\right]\left(u_n'(t) - u'(t)\right) \mathrm{d}t \\ \geq &\lim_{n\to\infty} \left\{ \int_0^T |u_n'(t)|^p \,\mathrm{d}t - \int_0^T |u'(t)|^p \,\mathrm{d}t - \left(\int_0^T |u_n'|^p \,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_0^T |u'|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \right. \\ &\left. - \left(\int_0^T |u_n'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^T |u'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p'}} \right\} \end{aligned}$$

$$= \lim_{n\to\infty}\left[\left(\int_0^T |u_n'|^p \,\mathrm{d}t\right)^{\frac{1}{p}} - \left(\int_0^T |u'|^p \,\mathrm{d}t\right)^{\frac{1}{p}}\right]\left[\left(\int_0^T |u_n'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p'}} - \left(\int_0^T |u'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p'}}\right]$$
$$= \lim_{n\to\infty} (\|u_n\|_W - \|u\|_W)\left(\|u_n\|_W^{p-1} - \|u\|_W^{p-1}\right) \ge 0,$$

where $p' = \frac{p}{p-1}$ is the exponent conjugate to $p > 1$. This implies $\|u_n\|_W \to \|u\|_W$. Since also $\|u_n\|_{L^p} \to \|u\|_{L^p}$ by $u_n \to u$ in $C[0,T]$, we conclude $\|u_n\| \to \|u\|$. Hence, the weak convergence $u_n \rightharpoonup u$ in X_p and the uniform convexity of X_p yield $u_n \to u$ in X_p. □

Now we verify the "geometric" assumptions of Theorem 2.4. Recall that the functional I is bounded from below on X_p, even and $I(0) = 0$. Let $k \in \mathbf{N}$ be arbitrary and X^k be k-dimensional subspace of X_p spanned by the basis elements $\{\phi_1, \dots, \phi_m\} \subset W \subset X_p$. The separability of X_p allows for such construction. We use the fact that all norms $\|\cdot\|_W$, $\|\cdot\|_{L^q}$ and $\|\cdot\|_{L^r}$ are equivalent on X^k, i.e., there exist positive constants $c_4, \dots, c_7$ such that for all $u \in X^k$,

$$c_4 \|u\|_{L^q} \le \|u\|_W \le c_5 \|u\|_{L^q} \quad \text{and} \quad c_6 \|u\|_{L^r} \le \|u\|_W \le c_7 \|u\|_{L^r}. \tag{3.4}$$

Set

$$\mathcal{S}_\rho^k := \left\{ u = \alpha_1\phi_1 + \dots + \alpha_k\phi_k : \sum_{j=1}^k |\alpha_j|^p = \rho^p \right\} \subset X^k.$$

$\mathcal{S}_\rho^k$ is clearly homeomorphic to the unit sphere $\mathcal{S}^{k-1} \subset \mathbf{R}^k$. Then for $u = \sum_{j=1}^k \alpha_j\phi_j$, the expression $\|u\|_{X^k} = \left(\sum_{j=1}^k |\alpha_j|^p\right)^{1/p}$ defines also a norm on X^k equivalent to $\|\cdot\|_W$, i.e., there exist positive constants c_8 and c_9 such that for all $u \in X^k$,

$$c_8 \|u\|_{X^k} \le \|u\|_W \le c_9 \|u\|_{X^k}. \tag{3.5}$$

We show that there is (sufficiently small) $\rho > 0$ such that

$$\sup_{u\in\mathcal{S}_\rho^k} I(u) < 0. \tag{3.6}$$

Indeed, due to (3.1) and (3.4), for any $u \in \mathcal{S}_\rho^k$, we have

$$\begin{aligned} I(u) &= \int_0^T \left(\frac{1}{p}|u'(t)|^p + \frac{1}{q}a(t)|u(t)|^q - \frac{1}{r}b(t)|u(t)|^r\right) \mathrm{d}t \\ &\le \frac{1}{p}\|u\|_W^p + \frac{a_2}{q}\|u\|_{L^q}^q - \frac{b_1}{r}\|u\|_{L^r}^r \\ &\le \frac{1}{p}\|u\|_W^p + \frac{a_2}{qc_4^q}\|u\|_W^q - \frac{b_1}{rc_7^r}\|u\|_W^r \\ &= \|u\|_W^r\left[\frac{1}{p}\|u\|_W^{p-r} + \frac{a_2}{qc_4^q}\|u\|_W^{q-r} - \frac{b_1}{rc_7^r}\right]. \end{aligned} \tag{3.7}$$

Recall our assumptions $1 < r < p$ and $r < q$. Then (3.6) follows from (3.5) and (3.7). Due to Remark 2.2 and the fact $X^k \subset W$, there exists $\rho_k > 0$ such that $\sup_{X^k\cap S_{\rho_k}} I(u) < 0$, where $S_\rho = \{u \in X_p, \|u\| = \rho\}$.

We have verified all assumptions of Theorem 2.4. Taking into account Remark 2.5, the multiplicity result in Theorem 1.1 follows.

Similarly, we proceed to prove the multiplicity result in Theorem 1.2. Indeed, since every g_j ($j = 1, \dots, l$) is odd, J is even and the assumptions (1.1) and (1.2) guarantee that the assertion of Lemma 3.1 holds also for the functional J. The assumption (1.2) also guarantees that analogue of (3.7) holds also for J. Thus the multiplicity result for ($\mathrm{Q_T}$) follows again from Theorem 2.4.

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