Analysis of a stochastic delay competition system driven by Lévy noise under regime switching

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Abstract. This paper is concerned with a stochastic delay competition system driven by Lévy noise under regime switching. Both the existence and uniqueness of the global positive solution are examined. By comparison theorem, sufficient conditions for extinction and non-persistence in the mean are obtained. Some discussions are made to demonstrate that the different environment factors have significant impacts on extinction. Furthermore, we show that the global positive solution is stochastically ultimate boundedness under some conditions, and an important asymptotic property of system is given. In the end, numerical simulations are carried out to illustrate our main results.

Keywords: Lévy noise, regime switching, stochastically ultimate boundedness, non-persistence in the mean.

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1 Introduction

The stochastic Lotka–Volterra model has been an important topic in mathematical ecology and widely investigated (see e.g. [4, 5, 13, 27–29, 31, 37] and the references therein) by many scholars. Meng Liu and Ke Wang [16] investigated a stochastic two-species Lotka–Volterra model in the competition case, just as

$$
\begin{aligned}
dx_1(t) &= x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_1)]dt + \sigma_1 x_1(t)dB_1(t), \\
dx_2(t) &= x_2(t)[r_2 - a_{21}x_1(t - \tau_2) - a_{22}x_2(t)]dt + \sigma_2 x_2(t)dB_2(t),
\end{aligned}
\tag{1.1}
$$

with initial conditions

$$
x_i(s) = \varphi_i(s) > 0, \quad s \in [-\tau, 0]; \quad \varphi_i(0) > 0, \quad i = 1, 2,
$$

where $x_i(t)$ $(i = 1, 2)$ represent the ith population size at time t; r_i $(i = 1, 2)$ are positive constants which represent the intrinsic growth rate of the ith species; a_{11} and a_{22} denote the density-dependent coefficients of the 1th species and 2th species, respectively; a_{12} and a_{21}

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denote the interspecific competition coefficients between the 1th species and 2th species; σ_i^2 $(i = 1, 2)$ denote the intensity of white noise; $\tau = \max\{\tau_1 \geq 0, \tau_2 \geq 0\}$, and $\varphi_i(s)$ $(i = 1, 2)$ are continuous functions on $[-\tau, 0]$; $B(t) = (B_1(t), B_2(t))^T$ denotes a two-dimensional standard Brownian motion which is defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}\}_{t \in R}$ satisfying the usual conditions. Liu and Wang obtained the stability in time average and extinction of the system.

In the real world, however, the population systems may suffer sudden environmental shocks, such as earthquakes, epidemics, soaring, tsunamis, hurricanes and so on, see [1, 2, 6, 17, 18]. These natural calamities are so abrupt that they can change the population size greatly at short notice, and these phenomenon can't be accurately described by the white noise. So, introducing Lévy noise into the underlying population systems may be a reasonable way to explain these phenomena, see [18, 21, 23, 24, 36]. In [23], Qun Liu, Qingmei Chen and Zhenghai Liu considered the following stochastic delay Lotka–Volterra system driven by Lévy noise

$$
\begin{aligned}
dx_1(t) = {} & x_1(t^-)[r_1 - a_{11}x_1(t^-) - a_{12}x_2(t^- - \tau_1)]dt \\
& + \sigma_1 x_1(t^-)dB_1(t) + x_1(t^-)\int_Y \gamma_1(u)\tilde{N}(dt, du), \\
dx_2(t) = {} & x_2(t^-)[r_2 - a_{21}x_1(t^- - \tau_2) - a_{22}x_2(t^-)]dt \\
& + \sigma_2 x_2(t^-)dB_2(t) + x_2(t^-)\int_Y \gamma_2(u)\tilde{N}(dt, du),
\end{aligned}
\tag{1.2}
$$

with initial conditions

$$
x_i(s) = \varphi_i(s) > 0, \quad s \in [-\tau, 0]; \quad \varphi_i(0) > 0, \quad i = 1, 2.
$$

In the model, $x_i(t^-)$ $(i = 1, 2)$ represent the left limit of $x_i(t)$ $(i = 1, 2)$; N denotes a position counting measure with characteristic measure λ on a measurable subset Y of $(0, \infty)$ with $\lambda(Y) < \infty$; $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$ is the corresponding martingale measure. The pair (B, N) represents a Lévy noise.

The authors [23] studied the model in two types:

(i) competition system (1.2), that is $r_1 > 0$, $r_2 > 0$, $a_{12} > 0$, $a_{21} > 0$;

(ii) predator–prey system (1.2), that is $r_1 > 0$, $r_2 < 0$, $a_{12} > 0$, $a_{21} < 0$.

For each case, they obtained some sufficient and necessary criteria for stability in time average and extinction of each population, under some assumptions.

The above systems only consider that the intrinsic growth rate is perturbed by white noise. In practice, other system's parameters are also affected by white noise, such as the density-dependent coefficients and the interspecific competition coefficients. So far as our knowledge is concerned, another important type of environmental noise, the color noise, also called telegraph noise, has been widely studied by many famous scholars [25, 26, 34, 35]. The color noise can be regarded as a switching between two or more regimes of environment, which differ by factors such as rain falls or nutrition [3, 33], and the regime switching is always modelled by a right-continuous Markov chain $\gamma(t)$ with finite state space $S = \{1, 2, \ldots, N\}$. System (1.2) does not incorporate the effect of Markov chain.

Motivated by the above discussions and based on system (1.2), we consider the following

stochastic delay competition system driven by Lévy noise under regime switching

$$
\begin{aligned}
dx_1(t) &= x_1(t^-)[r_1(\gamma(t)) - a_{11}(\gamma(t))x_1(t^-) - a_{12}(\gamma(t))x_2(t^- - \tau_1(t))]dt \\
&\quad + \sigma_1(\gamma(t))x_1(t^-)dB_1(t) + \sigma_2(\gamma(t))x_1^2(t^-)dB_2(t) \\
&\quad + x_1(t^-)\int_Y \theta_1(\gamma(t),u)N(dt,du), \\
dx_2(t) &= x_2(t^-)[r_2(\gamma(t)) - a_{21}(\gamma(t))x_1(t^- - \tau_2(t)) - a_{22}(\gamma(t))x_2(t^-)]dt \\
&\quad + \sigma_3(\gamma(t))x_2(t^-)dB_3(t) + \sigma_4(\gamma(t))x_2^2(t^-)dB_4(t) \\
&\quad + x_2(t^-)\int_Y \theta_2(\gamma(t),u)N(dt,du).
\end{aligned}
\tag{1.3}
$$

Introduced by [15] and [35], we know that the mechanism of the ecosystem described by (1.3) can be explained by follows. If the initial state $\gamma(0) = i \in S$, then (1.3) obeys

$$
\begin{aligned}
dx_1(t) &= x_1(t^-)[r_1(i) - a_{11}(i)x_1(t^-) - a_{12}(i)x_2(t^- - \tau_1(t))]dt + \sigma_1(i)x_1(t^-)dB_1(t) \\
&\quad + \sigma_2(i)x_1^2(t^-)dB_2(t) + x_1(t^-)\int_Y \theta_1(i,u)N(dt,du), \\
dx_2(t) &= x_2(t^-)[r_2(i) - a_{21}(i)x_1(t^- - \tau_2(t)) - a_{22}(i)x_2(t^-)]dt + \sigma_3(i)x_2(t^-)dB_3(t) \\
&\quad + \sigma_4(i)x_2^2(t^-)dB_4(t) + x_2(t^-)\int_Y \theta_2(i,u)N(dt,du),
\end{aligned}
$$

until the Markov chain switches from state i to a new state j, then the system (1.3) obeys the following equation

$$
\begin{aligned}
dx_1(t) &= x_1(t^-)[r_1(j) - a_{11}(j)x_1(t^-) - a_{12}(j)x_2(t^- - \tau_1(t))]dt + \sigma_1(j)x_1(t^-)dB_1(t) \\
&\quad + \sigma_2(j)x_1^2(t^-)dB_2(t) + x_1(t^-)\int_Y \theta_1(j,u)N(dt,du), \\
dx_2(t) &= x_2(t^-)[r_2(j) - a_{21}(j)x_1(t^- - \tau_2(t)) - a_{22}(j)x_2(t^-)]dt + \sigma_3(j)x_2(t^-)dB_3(t) \\
&\quad + \sigma_4(j)x_2^2(t^-)dB_4(t) + x_2(t^-)\int_Y \theta_2(j,u)N(dt,du),
\end{aligned}
$$

until the next switching. The switch of the system (1.3) will be as long as the Markov chain switch. Meanwhile, the Markov chain has significant impacts on the system's analysis and many scholars (see [3, 15, 25, 26, 33–35]) have given many important results which reveal the effect of the environmental noise to the population system.

The distinguish between system (1.2) and system (1.3) is that system (1.3) not only considered the impacts of the white noise on the intrinsic growth rate, but also imposed the effect of the white noise on the density-dependent coefficients. In order to make our research more practical, we considered time-varying delay in system (1.3). In addition, the effects of color noise are also considered by system (1.3).

In this paper, we attempt to research how the different environmental factors affect the dynamical properties of system (1.3). So, the remaining part of this paper is organized as follows. The proof of the existence and the uniqueness for the global positive solution of system (1.3) for any initial value is given in Section 2. In Section 3, sufficient conditions for extinction and non-persistence in the mean of system (1.3) are established. The stochastically ultimate boundedness of the positive solution is examined in Section 4. An important asymptotic property of the system is obtained in section 5. Numerical simulations under certain parameters are presented to illustrate our main results in Section 6. Finally, a few comments will conclude the paper.

2   Global positive solution

Throughout this paper, let $\gamma(t)$ be a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $Q = (q_{ij})_{N \times N}$ given by

$$\mathbb{P} = \{\gamma(t + \Delta t) = j \mid \gamma(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & j \neq i, \\ 1 + q_{ii}\Delta t + o(\Delta t), & j = i. \end{cases}$$

where $\Delta t \geq 0$, $q_{ij} \geq 0$ is transition rate from i to j. If $i \neq j$, then $\sum_{j=1}^{N} q_{ij} = 0$. Furthermore, we should assume that Markov chain $\gamma(t)$ is irreducible which means that the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in R^{1 \times N}$ satisfying $\pi Q = 0$ and

$$\sum_{i=1}^{N} \pi_i = 1 \quad \text{and} \quad \pi_i > 0, \quad \text{for all } i \in S.$$

For simplicity and convenience, throughout this article the following assumption will be essential:

(A1) $\check{r}_i > 0$, $\check{a}_{ij} > 0$, $\check{\sigma}_i > 0$, where $\check{f} = \min_{i \in s} f(i)$, $\hat{f} = \max_{i \in s} f(i)$.

(A2) $\tau_i(t)(i = 1, 2)$ are nonnegative, bounded and continuous differential function on $[0, \infty]$; $\tau_i'(t)(i = 1, 2)$ are bounded function and $\tau^u = \sup_{t \in [0, +\infty]} \tau'(t) < 1$.

(A3) Let $\tau = \max_{i=1,2} \sup_{t \geq 0} \tau_i(t)$ and denotes by $C = C([-\tau, 0]; R_+)$ the family of continuous function defined on $[-\tau, 0]$. For any given $\varphi_i(s) \in C$, the initial condition of system (1.3) is

$$x_i(s) = \varphi_i(s) \geq 0, \quad s \in [-\tau, 0]; \quad \sup_{-\tau \leq s \leq 0} \varphi_i(s) < \infty, \quad i = 1, 2. \tag{2.1}$$

(A4) There exists a positive constant c such that $\int_Y [\ln(1 + \gamma(i, u))]^2 \lambda(du) < c$, for all $i \in S$.

(A5) For the sake of convenience and simplicity, we introduce the following notations:

$$\overline{f(t)} = \frac{1}{t} \int_0^t f(s)ds, \qquad f^* = \limsup_{t \to +\infty} f(t), \qquad f_* = \liminf_{t \to +\infty} f(t)$$

Before the properties of the solutions are considered, we should guarantee the existence of positive solutions, firstly. Then, the following result will be obtained.

Theorem 2.1. *Under assumptions (A1)–(A4), for any given initial value $\gamma(0) \in S$ and (2.1), system (1.3) admits a unique positive solution $X(t) = (x_1(t), x_2(t))$ on $t \in [-\tau, +\infty)$ and the solution remains in R_+^2 with probability* 1.

Proof. Our proof is inspired by [2] and [35]. Since the coefficients of system (1.3) are local Lipschitz continuous, then for any given initial state $\varphi_1(s) \geq 0, \varphi_2(s) \geq 0, -\tau \leq s \leq 0$, system (1.3) has a unique local positive solution $X(t)$ on $[0, \tau_e)$, where τ_e is the explosion time. To show this positive solution is global, we only need to show $\tau_e = \infty$. a.s. Let $k_0 > 0$ be sufficiently large for $\varphi_1(t)$, $\varphi_2(t)$ lying within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k > k_0$, we define a sequence of stopping time described by

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : x_i(t) \notin \left(\tfrac{1}{k}, k\right), \text{ for some } i = 1, 2 \right\}.$$

Clearly, τ_k increase as $k \to \infty$. If $\tau_\infty = \lim_{k \to \infty} \tau_k$, then $\tau_\infty \leq \tau_e$ a.s.

For any constant $p \in (0,1)$, we define a Lyapunov function $V : R_+^2 \longrightarrow R_+$ as

$$V(X) = x_1^p + x_2^p.$$

Now, in order to make the following writing more efficient and convenient, we omit t^- in $x(t^-)$. Let T be arbitrary positive constant, for any $0 \leq t \leq \tau_k \wedge T$, making use of general Itô formula with jumps to system (1.3) leads to

$$
\begin{aligned}
dV(X) = \; & x_1^p \Big\{ pr_1(\gamma(t)) - pa_{11}(\gamma(t))x_1 - pa_{12}(\gamma(t))x_2(t - \tau_1(t)) \\
& + \int_Y [(1 + \theta_1(\gamma(t), u))^p - 1]\lambda(du) \Big\} dt \\
& + x_2^p \Big\{ pr_2(\gamma(t)) - pa_{21}(\gamma(t))x_1(t - \tau_2(t)) - pa_{22}(\gamma(t))x_2 \\
& + \int_Y [(1 + \theta_2(\gamma(t), u))^p - 1]\lambda(du) \Big\} dt \\
& + \frac{1}{2}p(p-1)x_1^p(\sigma_1^2(\gamma(t)) + \sigma_2^2(\gamma(t))x_1^2)dt + px_1^p\sigma_1(\gamma(t))dB_1(t) \\
& + px_1^{p+1}\sigma_2(\gamma(t))dB_2(t) + \frac{1}{2}p(p-1)x_2^p(\sigma_3^2(\gamma(t)) + \sigma_4^2(\gamma(t))x_2^2)dt \\
& + px_2^p\sigma_3(\gamma(t))dB_3(t) + px_2^{p+1}\sigma_4(\gamma(t))dB_4(t) \\
& + x_1^p \int_Y [(1 + \theta_1(\gamma(t), u))^p - 1]\tilde{N}(dt, du) \\
& + x_2^p \int_Y [(1 + \theta_2(\gamma(t), u))^p - 1]\tilde{N}(dt, du) \\
= \; & LV(x_1, x_2)dt + px_1^p\sigma_1(\gamma(t))dB_1(t) + px_1^{p+1}\sigma_2(\gamma(t))dB_2(t) \\
& + px_2^p\sigma_3(\gamma(t))dB_3(t) + px_2^{p+1}\sigma_4(\gamma(t))dB_4(t) \\
& + x_1^p \int_Y [(1 + \theta_1(\gamma(t), u))^p - 1]\tilde{N}(dt, du) \\
& + x_2^p \int_Y [(1 + \theta_2(\gamma(t), u))^p - 1]\tilde{N}(dt, du),
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
& LV(x_1, x_2) \\
& = x_1^p \Big\{ pr_1(\gamma(t)) - pa_{11}(\gamma(t))x_1 - pa_{12}(\gamma(t))x_2(t - \tau_1(t)) \\
& \quad + \int_Y [(1 + \theta_1(\gamma(t), u))^p - 1]\lambda(du) + \frac{1}{2}p(p-1)(\sigma_1^2(\gamma(t)) + \sigma_2^2(\gamma(t))x_1^2) \Big\} \\
& \quad + x_2^p \Big\{ pr_2(\gamma(t)) - pa_{21}(\gamma(t))x_1(t - \tau_2(t)) - pa_{22}(\gamma(t))x_2 \\
& \quad + \int_Y [(1 + \theta_2(\gamma(t), u))^p - 1]\lambda(du) + \frac{1}{2}p(p-1)(\sigma_3^2(\gamma(t)) + \sigma_4^2(\gamma(t))x_2^2) \Big\} \\
& \leq x_1^p \Big\{ \frac{1}{2}p(p-1)\check{\sigma}_2^2 x_1^2 - p\check{a}_{11}x_1 + p\hat{r}_1 + \frac{1}{2}p(p-1)\check{\sigma}_1^2 + \int_Y [(1 + \hat{\theta}_1(u))^p - 1]\lambda(du) \Big\} \\
& \quad + x_2^p \Big\{ \frac{1}{2}p(p-1)\check{\sigma}_4^2 x_2^2 - p\check{a}_{22}x_2 + p\hat{r}_2 + \frac{1}{2}p(p-1)\check{\sigma}_3^2 + \int_Y [(1 + \hat{\theta}_2(u))^p - 1]\lambda(du) \Big\}.
\end{aligned}
\tag{2.3}
$$

As $p \in (0, 1)$, there exist two constants k_1 and k_2 such that

$$x_1^p \left\{ \frac{1}{2} p(p-1) \check{\sigma}_2^2 x_1^2 - p \check{a}_{11} x_1 + p \hat{r}_1 + \frac{1}{2} p(p-1) \check{\sigma}_1^2 + \int_Y [(1 + \hat{\theta}_1(u))^p - 1] \lambda(du) \right\} \le k_1,$$

and

$$x_2^p \left\{ \frac{1}{2} p(p-1) \check{\sigma}_4^2 x_2^2 - p \check{a}_{22} x_2 + p \hat{r}_2 + \frac{1}{2} p(p-1) \check{\sigma}_3^2 + \int_Y [(1 + \hat{\theta}_2(u))^p - 1] \lambda(du) \right\} \le k_2.$$

Thus, we can get that

$$LV(x_1, x_2) \le k_1 + k_2. \tag{2.4}$$

Applying inequality (2.4) to equation (2.2), and integrating from 0 to $\tau_k \wedge T$, yields

$$\int_0^{\tau_k \wedge T} dV(x_1, x_2) \le \int_0^{\tau_k \wedge T} (k_1 + k_2) dt + \int_0^{\tau_k \wedge T} p \hat{\sigma}_1 x_1^p dB_1(t) + \int_0^{\tau_k \wedge T} p \hat{\sigma}_2 x_1^{p+1} dB_1(t)$$
$$+ \int_0^{\tau_k \wedge T} p \hat{\sigma}_3 x_2^p dB_3(t) + \int_0^{\tau_k \wedge T} x_1^p \int_Y [(1 + \hat{\theta}_1(u))^p - 1] \tilde{N}(dt, du)$$
$$+ \int_0^{\tau_k \wedge T} p \hat{\sigma}_4 x_2^{p+1} dB_4(t) + \int_0^{\tau_k \wedge T} x_2^p \int_Y [(1 + \hat{\theta}_2(u))^p - 1] \tilde{N}(dt, du),$$

Taking expectations, the above inequality changes into

$$\mathbb{E} V(X(\tau_k \wedge T)) - V(X(0)) \le \mathbb{E}[(k_1 + k_2)(\tau_k \wedge T)],$$

that is to say

$$\mathbb{E} V(x_1(\tau_k \wedge T), x_2(\tau_k \wedge T)) \le V(x_1(0), x_2(0)) + (k_1 + k_2) \mathbb{E}(\tau_k \wedge T)$$
$$\le V(x_1(0), x_2(0)) + (k_1 + k_2) T. \tag{2.5}$$

For each $u \ge 0$, we define $\mu(u) = \inf\{V(X), |x_i| \ge u, i = 1, 2\}$. Clearly, if $u \to \infty$, then $\mu(u) \to \infty$. Let us set $\Omega_k = \tau_k \le T$ and $\mathbb{P}(\Omega_k) \ge \varepsilon$, for any $\omega \in \Omega_k$, then it is easy to see that

$$\mu(k) \mathbb{P}(\tau_k \le T) \le \mathbb{E}(V(X(\tau_k)) I_{\Omega_k}) \le V(X(0)) + (k_1 + k_2) T.$$

When $k \to \infty$, we can get that $\mathbb{P}(\tau_\infty \le T) = 0$. Due to the arbitrariness of T, then $\mathbb{P}(\tau_\infty = \infty) = 1$. So, this completes the proof. □

Basing the view of biomathematics, the positivity and nonexplosion property of the solutions are often not good enough in the population dynamical system. Then, the critical value between extinction and persistence of the system (1.3) will be investigated in the next.

3   Critical value between extinction and persistence

Now, in order to obtain our main results, several lemmas and definitions which play an important role in our article will be given.

Lemma 3.1 ([12]). *Under assumption (A4) and $x(t) \in C(\Omega \times [0, +\infty), R_+)$, then the following statements hold.*

(i) If there exist two positive constants T and δ_0 such that

$$\ln x(t) \leq \delta t - \delta_0 \int_0^t x(s)ds + \alpha B(t) + \sum_{i=1}^{2} \delta_i \int_0^t \int_Y \ln(1 + \gamma_i(u))\tilde{N}(ds, du), \quad a.s.$$

for all $t \geq T$, where α, δ_1 and δ_2 are constants, then

$$\begin{cases} \bar{x}^* \leq \frac{\delta}{\delta_0}, \ a.s. & \delta \geq 0, \\ \lim_{t \to +\infty} x(t) = 0, \ a.s. & \delta \leq 0. \end{cases}$$

(ii) If there exist three positive constants T, δ and δ_0 such that

$$\ln x(t) \geq \delta t - \delta_0 \int_0^t x(s)ds + \alpha B(t) + \sum_{i=1}^{2} \delta_i \int_0^t \int_Y \ln(1 + \gamma_i(u))\tilde{N}(ds, du), \quad a.s.$$

for all $t \geq T$, then $\bar{x}_* \geq \frac{\delta}{\delta_0}$ a.s.

Lemma 3.2 ([35]). *Suppose that $M(t)$, $t \geq 0$, is a local martingale vanishing at zero, then*

$$\lim_{t \to +\infty} \rho M(t) < \infty \Rightarrow \lim_{t \to +\infty} \frac{M(t)}{t} = 0, \ a.s.$$

where

$$\rho M(t) = \int_0^t \frac{d\langle M \rangle(s)}{(1+s)^2}, \quad t \geq 0,$$

and $\langle M \rangle(t)$ is Meyer's angle bracket process.

Definition 3.3.

(i) Population $x(t)$ is said to go to extinction, if $\lim_{t \to +\infty} x(t) = 0$.

(ii) Population $x(t)$ is said to be non-persistence in the mean, if $\lim_{t \to +\infty} \overline{x(t)} = 0$.

In order to obtain the above results, we will consider the following stochastic competition system driven by Lévy noise under regime switching

$$\begin{aligned} dy_1(t) &= y_1(t^-)[r_1(\gamma(t)) - a_{11}(\gamma(t))y_1(t^-)]dt + \sigma_1(\gamma(t))y_1(t^-)dB_1(t) \\ &\quad + \sigma_2(\gamma(t))y_1^2(t^-)dB_2(t) + y_1(t^-)\int_Y \theta_1(\gamma(t), u)N(dt, du), \\ dy_2(t) &= y_2(t^-)[r_2(\gamma(t)) - a_{22}(\gamma(t))y_2(t^-)]dt + \sigma_3(\gamma(t))y_2(t^-)dB_3(t) \\ &\quad + \sigma_4(\gamma(t))y_2^2(t^-)dB_4(t) + y_2(t^-)\int_Y \theta_2(\gamma(t), u)N(dt, du), \end{aligned} \tag{3.1}$$

with initial condition $y_1(0) > 0$, $y_2(0) > 0$ and $\gamma(0) \in S$.

Lemma 3.4. *Let assumption (A4) hold, then for the initial value $y_1(0) > 0$, $y_2(0) > 0$ and $\gamma(0) \in S$, the solution $(y_1(t), y_2(t))$ of system (3.1) satisfies*

$$\limsup_{t \to +\infty} \frac{\ln y_1(t)}{t} \leq \sum_{i=1}^{N} h_1(i)\pi_i \quad \text{and} \quad \limsup_{t \to +\infty} \frac{\ln y_2(t)}{t} \leq \sum_{i=1}^{N} h_2(i)\pi_i,$$

where

$$h_1(i) = r_1(i) - \frac{1}{2}\sigma_1^2(i) + \int_Y \ln(1 + \theta_1(i,u))\lambda(du),$$

and

$$h_2(i) = r_2(i) - \frac{1}{2}\sigma_3^2(i) + \int_Y \ln(1 + \theta_2(i,u))\lambda(du).$$

Proof. Our proof is motivated by [14] and [19]. For system (2.5), making use of generalized Itô's formula with jumps [20–22] to $\ln y_1$ and $\ln y_2$, then

$$
\begin{aligned}
d\ln y_1(t) = {} & \left[r_1(\gamma(t)) - \frac{1}{2}\sigma_1^2(\gamma(t)) + \int_Y \ln(1 + \theta_1(\gamma(t),u))\lambda(du)\right]dt - a_{11}(\gamma(t))y_1(t)dt \\
& + \sigma_1(\gamma(t))dB_1(t) + \sigma_2(\gamma(t))y_1(t)dB_2(t) - \frac{1}{2}\sigma_2^2(\gamma(t))y_1^2(t)dt \\
& + \int_Y \ln[1 + \theta_1(\gamma(t),u)]\tilde{N}(dt,du), \\
d\ln y_2(t) = {} & \left[r_2(\gamma(t)) - \frac{1}{2}\sigma_3^2(\gamma(t)) + \int_Y \ln(1 + \theta_2(\gamma(t),u))\lambda(du)\right]dt - a_{22}(\gamma(t))y_2(t)dt \\
& + \sigma_3(\gamma(t))dB_3(t) + \sigma_4(\gamma(t))y_2(t)dB_4(t) - \frac{1}{2}\sigma_4^2(\gamma(t))y_2^2(t)dt \\
& + \int_Y \ln[1 + \theta_2(\gamma(t),u)]\tilde{N}(dt,du).
\end{aligned}
$$

Integrating from 0 to t, leads to

$$
\begin{aligned}
\ln y_1(t) - \ln y_1(0) = {} & \int_0^t \left[r_1(\gamma(s)) - \frac{1}{2}\sigma_1^2(\gamma(s)) + \int_Y \ln(1 + \theta_1(\gamma(s),u))\lambda(du)\right]ds \\
& - \int_0^t a_{11}(\gamma(s))y_1(s)ds + \int_0^t \sigma_1(\gamma(s))dB_1(s) \\
& + \int_0^t \sigma_2(\gamma(s))y_1(s)dB_2(s) - \int_0^t \frac{1}{2}\sigma_2^2(\gamma(s))y_1^2(s)ds + M_1,
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\ln y_2(t) - \ln y_2(0) = {} & \int_0^t [r_2(\gamma(s)) - \frac{1}{2}\sigma_3^2(\gamma(s)) + \int_Y \ln(1 + \theta_2(\gamma(s),u))\lambda(du)]ds \\
& - \int_0^t a_{22}(\gamma(s))y_2(s)ds + \int_0^t \sigma_3(\gamma(s))dB_3(s) \\
& + \int_0^t \sigma_4(\gamma(s))y_2(s)dB_4(s) - \int_0^t \frac{1}{2}\sigma_4^2(\gamma(s))y_2^2(s)ds + M_2,
\end{aligned}
\tag{3.3}
$$

where

$$M_1 = \int_0^t \int_Y \ln[1 + \theta_1(\gamma(s),u)]\tilde{N}(ds,du) \quad \text{and} \quad M_2 = \int_0^t \int_Y \ln[1 + \theta_2(\gamma(s),u)]\tilde{N}(ds,du).$$

According to assumption (A4) and Lemma 3.2, we can get

$$\lim_{t \to +\infty} \frac{M_1(t)}{t} = 0 \text{ a.s.} \quad \text{and} \quad \lim_{t \to +\infty} \frac{M_2(t)}{t} = 0 \text{ a.s.} \tag{3.4}$$

Let $p_1(t) = \int_0^t \sigma_2(\gamma(s))y_1(s)dB_2(s)$, $p_2(t) = \int_0^t \sigma_4(\gamma(s))y_2(s)dB_4(s)$, then the quadratic variations of $p_1(t)$ and $p_2(t)$ are

$$\langle p_1(t), p_1(t) \rangle = \int_0^t \sigma_2^2(\gamma(s))y_1^2(s)ds \quad \text{and} \quad \langle p_2(t), p_2(t) \rangle = \int_0^t \sigma_4^2(\gamma(s))y_2^2(s)ds.$$

An application of exponential martingale inequality [12, 35] gives

$$\mathbb{P}\left\{\sup_{0\leq t\leq k}\left[p_1(t)-\frac{1}{2}\langle p_1(t),p_1(t)\rangle\right]>2\ln k\right\}\leq\frac{1}{k^2},$$

and

$$\mathbb{P}\left\{\sup_{0\leq t\leq k}\left[p_2(t)-\frac{1}{2}\langle p_2(t),p_2(t)\rangle\right]>2\ln k\right\}\leq\frac{1}{k^2},$$

making use of the classical Borel–Cantelli Lemma, we have that for almost all $\omega\in\Omega$, there is a random integer $k_0=k_0(\omega)$ such that for $k\geq k_0$

$$\sup_{0\leq t\leq k}\left[p_1(t)-\frac{1}{2}\langle p_1(t),p_1(t)\rangle\right]\leq 2\ln k,$$

and

$$\sup_{0\leq t\leq k}\left[p_2(t)-\frac{1}{2}\langle p_2(t),p_2(t)\rangle\right]\leq 2\ln k.$$

They are equivalent to

$$p_1(t)\leq 2\ln k+\frac{1}{2}\langle p_1(t),p_1(t)\rangle=2\ln k+\frac{1}{2}\int_0^t\sigma_2^2(\gamma(s))y_1^2(s)d(s), \tag{3.5}$$

and

$$p_2(t)\leq 2\ln k+\frac{1}{2}\langle p_2(t),p_2(t)\rangle=2\ln k+\frac{1}{2}\int_0^t\sigma_4^2(\gamma(s))y_2^2(s)d(s), \tag{3.6}$$

for all $0\leq t\leq k$, $k\geq k_0$. According to (3.5), (3.6) and assumption (A1), equations (3.2) and (3.3) change into

$$\begin{aligned}\ln y_1(t)-\ln y_1(0)\leq&\int_0^t h_1(\gamma(s))ds-\int_0^t a_{11}(\gamma(s))y_1(s)dt\\&+\int_0^t\sigma_1(\gamma(s))dB_1(s)+2\ln k+M_1,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\ln y_2(t)-\ln y_2(0)\leq&\int_0^t h_2(\gamma(s))ds-\int_0^t a_{22}(\gamma(s))y_2(s)dt\\&+\int_0^t\sigma_3(\gamma(s))dB_3(s)+2\ln k+M_2,\end{aligned} \tag{3.8}$$

where

$$h_1(\gamma(s))=r_1(\gamma(s))-\frac{1}{2}\sigma_1^2(\gamma(s))+\int_Y\ln(1+\theta_1((\gamma(s)),u))\lambda(du),$$

and

$$h_2(\gamma(s))=r_2(\gamma(s))-\frac{1}{2}\sigma_3^2(\gamma(s))+\int_Y\ln(1+\theta_2((\gamma(s)),u))\lambda(du).$$

Dividing (3.7) and (3.8) by t, for $k-1\leq t\leq k, k\geq k_0$, we obtain

$$\begin{aligned}t^{-1}[\ln y_1(t)-\ln y_1(0)]\leq&\frac{1}{t}\int_0^t h_1(\gamma(s))ds-\frac{1}{t}\int_0^t a_{11}(\gamma(s))y_1(s)dt\\&+\frac{1}{t}\int_0^t\sigma_1(\gamma(s))dB_1(s)+\frac{2\ln k}{t}+\frac{M_1}{t},\end{aligned}$$

$$\begin{aligned}t^{-1}[\ln y_2(t)-\ln y_2(0)]\leq&\frac{1}{t}\int_0^t h_2(\gamma(s))ds-\frac{1}{t}\int_0^t a_{22}(\gamma(s))y_2(s)dt\\&+\frac{1}{t}\int_0^t\sigma_3(\gamma(s))dB_3(s)+\frac{2\ln k}{t}+\frac{M_2}{t}.\end{aligned}$$

In the case of using the property of the Markov chain and (3.4), taking the superior limit, we have

$$\limsup_{t \to +\infty} \frac{\ln y_1(t)}{t} \le \sum_{i=1}^{N} h_1(i)\pi_i,$$

$$\limsup_{t \to +\infty} \frac{\ln y_2(t)}{t} \le \sum_{i=1}^{N} h_2(i)\pi_i.$$

Then, this completes the proof. □

Theorem 3.5. *Let assumption (A4) hold, then for any given initial value $\gamma(0) \in S$ and (2.1), the solution $X(t) = (x_1(t), x_2(t))$ of system (1.3) satisfies*

$$\limsup_{t \to +\infty} \frac{\ln x_1(t)}{t} \le \sum_{i=1}^{N} h_1(i)\pi_i \quad \text{and} \quad \limsup_{t \to +\infty} \frac{\ln x_2(t)}{t} \le \sum_{i=1}^{N} h_2(i)\pi_i,$$

where

$$h_1(i) = r_1(i) - \frac{1}{2}\sigma_1^2(i) + \int_Y \ln(1 + \theta_1(i, u))\lambda(du),$$

and

$$h_2(i) = r_2(i) - \frac{1}{2}\sigma_3^2(i) + \int_Y \ln(1 + \theta_2(i, u))\lambda(du).$$

Proof. By the comparison theorem for stochastic differential equation with jumps [30], we have

$$x_1(t) \le y_1(t) \quad \text{and} \quad x_2(t) \le y_2(t).$$

Applying Lemma 3.4, we can obtain

$$\limsup_{t \to +\infty} \frac{\ln x_1(t)}{t} \le \sum_{i=1}^{N} h_1(i)\pi_i \quad \text{and} \quad \limsup_{t \to +\infty} \frac{\ln x_2(t)}{t} \le \sum_{i=1}^{N} h_2(i)\pi_i. \qquad □$$

Theorem 3.6. *If $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then the species $x_1(t)$ and $x_2(t)$ will go to extinction a.s.*

Proof. Basing the result of Theorem 3.5, if $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then $\limsup_{t \to +\infty} \frac{\ln x_1(t)}{t} < 0$ and $\limsup_{t \to +\infty} \frac{\ln x_2(t)}{t} < 0$. It is easy to find that $\lim_{t \to +\infty} x_1(t) = 0$, a.s. and $\lim_{t \to +\infty} x_2(t) = 0$ a.s. So, species $x_1(t)$ and $x_2(t)$ go to extinction a.s. □

Theorem 3.7. *If $\sum_{i=1}^{N} h_1(i)\pi_i = 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then the species $x_1(t)$ and $x_2(t)$ will be non-persistence in the mean a.s.*

Proof. For system (1.3), making use of generalized Itô's formula with jumps to $\ln x_1(t)$ and $\ln x_2(t)$, and integrating from 0 to t, we have

$$\ln x_1(t) - \ln x_1(0) = \int_0^t [r_1(\gamma(s)) - \frac{1}{2}\sigma_1^2(\gamma(s)) + \int_Y \ln(1 + \theta_1(\gamma(s), u))\lambda(du)]ds$$

$$- \int_0^t a_{11}(\gamma(s))x_1(s)ds - \int_0^t a_{12}(\gamma(s))x_2(s - \tau_1(s))ds + \int_0^t \sigma_1(\gamma(s))dB_1(s)$$

$$+ \int_0^t \sigma_2(\gamma(s))x_1(s)dB_2(s) - \int_0^t \frac{1}{2}\sigma_2^2(\gamma(s))x_1^2(s)ds + N_1,$$

$$\ln x_2(t) - \ln x_2(0) = \int_0^t [r_2(\gamma(s)) - \frac{1}{2}\sigma_3^2(\gamma(s)) + \int_Y \ln(1 + \theta_2(\gamma(s), u))\lambda(du)]ds$$

$$- \int_0^t a_{22}(\gamma(s))x_2(s)ds - \int_0^t a_{21}(\gamma(s))x_1(s - \tau_2(s))ds + \int_0^t \sigma_3(\gamma(s))dB_3(s)$$

$$+ \int_0^t \sigma_4(\gamma(s))x_2(s)dB_4(s) - \int_0^t \frac{1}{2}\sigma_4^2(\gamma(s))x_2^2(s)ds + N_2,$$

where

$$N_1 = \int_0^t \int_Y \ln[1 + \theta_1(\gamma(s), u)]\tilde{N}(dt, du),$$

and

$$N_2 = \int_0^t \int_Y \ln[1 + \theta_2(\gamma(s), u)]\tilde{N}(dt, du).$$

According to assumption (A4) and Lemma 3.2, we can get

$$\lim_{t \to +\infty} \frac{N_1(t)}{t} = 0 \ a.s. \quad \text{and} \quad \lim_{t \to +\infty} \frac{N_2(t)}{t} = 0 \ a.s.$$

Let $p_1'(t) = \int_0^t \sigma_2(\gamma(s))x_1(s)dB_2(s)$, $p_2'(t) = \int_0^t \sigma_4(\gamma(s))x_2(s)dB_4(s)$, making use of exponential martingale inequality, we have

$$\mathbb{P}\left\{ \sup_{0 \le t \le k} \left[p_1'(t) - \frac{1}{2}\langle p_1'(t), p_1'(t)\rangle \right] > 2\ln k \right\} \le \frac{1}{k^2},$$

and

$$\mathbb{P}\left\{ \sup_{0 \le t \le k} \left[p_2'(t) - \frac{1}{2}\langle p_2'(t), p_2'(t)\rangle \right] > 2\ln k \right\} \le \frac{1}{k^2},$$

by the classical Borel–Cantelli lemma, we have that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that for $k \ge k_0$

$$\sup_{0 \le t \le k} \left[p_1'(t) - \frac{1}{2}\langle p_1'(t), p_1'(t)\rangle \right] \le 2\ln k,$$

and

$$\sup_{0 \le t \le k} \left[p_2'(t) - \frac{1}{2}\langle p_2'(t), p_2'(t)\rangle \right] \le 2\ln k.$$

Obviously,

$$p_1'(t) \le 2\ln k + \frac{1}{2}\langle p_1'(t), p_1'(t)\rangle = 2\ln k + \frac{1}{2}\int_0^t \sigma_2^2(\gamma(s))x_1^2(s)ds,$$

and

$$p_2'(t) \le 2\ln k + \frac{1}{2}\langle p_2'(t), p_2'(t)\rangle = 2\ln k + \frac{1}{2}\int_0^t \sigma_2^2(\gamma(s))x_2^2(s)ds,$$

for all $0 \le t \le k, \ k \ge k_0$.

According to the above discussion, we obtain

$$\ln x_1(t) - \ln x_1(0) \le \int_0^t h_1(\gamma(s))ds - \int_0^t a_{11}(\gamma(s))x_1(s)ds$$

$$+ \int_0^t \sigma_1(\gamma(s))dB_1(s) + 2\ln k + N_1, \tag{3.9}$$

and

$$\ln x_2(t) - \ln x_2(0) \le \int_0^t h_2(\gamma(s))ds - \int_0^t a_{22}(\gamma(s))x_2(s)ds$$
$$+ \int_0^t \sigma_3(\gamma(s))dB_3(s) + 2\ln k + N_2. \tag{3.10}$$

Dividing (3.9) and (3.10) by t, for $k - 1 \le t \le k$, $k \ge k_0$, we obtain

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \le \frac{1}{t}\int_0^t h_1(\gamma(s))ds - \frac{1}{t}\int_0^t a_{11}(\gamma(s))x_1(s)dt$$
$$+ \frac{1}{t}\int_0^t \sigma_1(\gamma(s))dB_1(s) + \frac{2\ln k}{t} + \frac{N_1}{t},$$
$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \le \frac{1}{t}\int_0^t h_2(\gamma(s))ds - \frac{1}{t}\int_0^t a_{22}(\gamma(s))x_2(s)dt$$
$$+ \frac{1}{t}\int_0^t \sigma_3(\gamma(s))dB_3(s) + \frac{2\ln k}{t} + \frac{N_2}{t}.$$

Based on the fact that $\lim_{t\to+\infty} t^{-1}\int_0^t h_1(\gamma(s))ds = \sum_{i=1}^N h_1(i)\pi_i$ and $\lim_{t\to+\infty} t^{-1}\int_0^t h_2(\gamma(s))ds = \sum_{i=1}^N h_2(i)\pi_i$, for arbitrary $\varepsilon \ge 0$, there exists a constant $T_1 > 0$ such that

$$t^{-1}\int_0^t h_1(\gamma(s))ds \le \sum_{i=1}^N h_1(i)\pi_i + \frac{\varepsilon}{4} = \frac{\varepsilon}{4}, \quad t > T_1,$$
$$t^{-1}\int_0^t h_2(\gamma(s))ds \le \sum_{i=1}^N h_2(i)\pi_i + \frac{\varepsilon}{4} = \frac{\varepsilon}{4}, \quad t > T_1,$$

and

$$\frac{N_1}{t} \le \frac{\varepsilon}{4} \quad \text{and} \quad \frac{N_2}{t} \le \frac{\varepsilon}{4}.$$

By the strong laws of large numbers, we have

$$t^{-1}\int_0^t \sigma_1(\gamma(s))dB_1(s) \le \frac{\varepsilon}{4} \quad \text{and} \quad t^{-1}\int_0^t \sigma_3(\gamma(s))dB_3(s) \le \frac{\varepsilon}{4}.$$

Then, for $T_1 < t \le k, k \ge k_0$, (3.9) and (3.10) change into

$$\ln x_1(t) - \ln x_1(0) \le \frac{\varepsilon t}{4} - \check{a}_{11}\int_0^t x_1(s)dt + \frac{\varepsilon t}{4} + 2\ln k + \frac{\varepsilon t}{4},$$
$$\ln x_2(t) - \ln x_2(0) \le \frac{\varepsilon t}{4} - \check{a}_{22}\int_0^t x_2(s)dt + \frac{\varepsilon t}{4} + 2\ln k + \frac{\varepsilon t}{4}.$$

Note that for sufficiently large t with $T_1 < k - 1 \le t \le k$, $k \ge k_0$, we have $t^{-1}\ln k \le \frac{\varepsilon}{8}$. Due to the above results, we obtain

$$\ln x_1(t) - \ln x_1(0) \le \varepsilon t - \check{a}_{11}\int_0^t x_1(s)dt,$$
$$\ln x_2(t) - \ln x_2(0) \le \varepsilon t - \check{a}_{22}\int_0^t x_2(s)dt.$$

Making use of Lemma 3.1, we have $\bar{x}_1^* \le \frac{\varepsilon}{\check{a}_{11}}$ and $\bar{x}_2^* \le \frac{\varepsilon}{\check{a}_{22}}$, by the arbitrariness of ε, we obtain our result. $\qquad \square$

Remark 3.8. Theorem 3.6 and Theorem 3.7 have an obvious biological interpretation. It is that the extinction and non-persistence in the mean of the system (1.3) only depend on the values of $\sum_{i=1}^{N} h_1(i)\pi_i$ and $\sum_{i=1}^{N} h_2(i)\pi_i$, where $h_1(i) = r_1(i) - \frac{1}{2}\sigma_1^2(i) + \int_Y \ln(1 + \theta_1(i, u))\lambda(du)$ and $h_2(i) = r_2(i) - \frac{1}{2}\sigma_3^2(i) + \int_Y \ln(1 + \theta_2(i, u))\lambda(du)$. We can see that the white noise σ_2 and σ_4 imposed on the interspecific competition coefficients have no impact on the extinction and non-persistence in the mean of system (1.3).

Remark 3.9. Let us consider the effect of time delay on the extinction and non-persistence in the mean of system (1.3). It is clearly that time-delay has no impact on the values of $\sum_{i=1}^{N} h_1(i)\pi_i$ and $\sum_{i=1}^{N} h_2(i)\pi_i$, so the same to extinction and non-persistence in the mean of system (1.3).

Remark 3.10. Let us consider the effect of jump-diffusion coefficients $\theta_1(i, u)$ and $\theta_2(i, u)$ on the extinction of system (1.3). If $\theta_1(i, u) < 0$ and $\theta_2(i, u) < 0$, which mean that the jump noise are always disadvantageous for the ecosystem, such as earthquakes and tsunamis, so the jump noise can make the system extinctive. If $\theta_1(i, u) > 0$ and $\theta_2(i, u) > 0$, which imply that the jump noise are always advantageous for the ecosystem, for example, ocean red tide, this case is very complex, so we will study it in the future.

Remark 3.11. When $\gamma(t) = i$, $i \in S$, we can see the different subsystems of system (1.3). Similarly, we can obtain the same results as Theorem 3.6 and Theorem 3.7.

Remark 3.12. By Remark 3.11, we can consider the impact of Markovian switching on system (1.3) easily. If every subsystem of system (1.3) is extinctive, then as a result of Markovian switching, the overall behavior of system (1.3) remains extinctive. But, if only some subsystems of (1.3) are extinctive, then the values of $\sum_{i=1}^{N} h_1(i)\pi_i$ and $\sum_{i=1}^{N} h_2(i)\pi_i$ compared with zero may be not clear, so the overall behavior of system (1.3) is uncertain.

Theorem 3.13. *If $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then $\bar{x}_1^* \leq \frac{\sum_{i=1}^{N} h_1(i)\pi_i}{\check{a}_{11}}$ and $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\check{a}_{22}}$.*

Proof. Basing Theorem 3.7, we have

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \frac{1}{t}\int_0^t h_1(\gamma(s))ds - \frac{1}{t}\int_0^t a_{11}(\gamma(s))x_1(s)dt$$
$$+ \frac{1}{t}\int_0^t \sigma_1(\gamma(s))dB_1(s) + \frac{2\ln k}{t} + \frac{N_1}{t},$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \frac{1}{t}\int_0^t h_2(\gamma(s))ds - \frac{1}{t}\int_0^t a_{22}(\gamma(s))x_2(s)dt$$
$$+ \frac{1}{t}\int_0^t \sigma_3(\gamma(s))dB_3(s) + \frac{2\ln k}{t} + \frac{N_2}{t}.$$

By assumption (A1) and the fact that

$$\lim_{t \to +\infty} t^{-1}\int_0^t h_1(\gamma(s))ds = \sum_{i=1}^{N} h_1(i)\pi_i$$

and

$$\lim_{t \to +\infty} t^{-1}\int_0^t h_2(\gamma(s))ds = \sum_{i=1}^{N} h_2(i)\pi_i,$$

for arbitrary $\varepsilon \geq 0$, there exists a constant $T_1 > 0$ such that

$$t^{-1} \int_0^t h_1(\gamma(s))ds \leq \sum_{i=1}^N h_1(i)\pi_i + \frac{\varepsilon}{4}, \qquad t > T_1,$$

$$t^{-1} \int_0^t h_2(\gamma(s))ds \leq \sum_{i=1}^N h_2(i)\pi_i + \frac{\varepsilon}{4}, \qquad t > T_1,$$

and

$$\frac{N_1}{t} \leq \frac{\varepsilon}{4} \quad \text{and} \quad \frac{N_2}{t} \leq \frac{\varepsilon}{4}.$$

Then the inequalities will be changed into

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \sum_{i=1}^N h_1(i)\pi_i + \frac{\varepsilon}{4} - \frac{\check{a}_{11}}{t} \int_0^t x_1(s)dt + \frac{\hat{\sigma}_1}{t} B_1(t) + \frac{2\ln k}{t} + \frac{\varepsilon}{4},$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^N h_2(i)\pi_i + \frac{\varepsilon}{4} - \frac{\check{a}_{22}}{t} \int_0^t x_2(s)dt + \frac{\hat{\sigma}_3}{t} B_3(t) + \frac{2\ln k}{t} + \frac{\varepsilon}{4}.$$

Note that for sufficiently large t with $T_1 < k - 1 \leq t \leq k$, $k \geq k_0$, we have $t^{-1} \ln k \leq \frac{\varepsilon}{4}$. Then

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \sum_{i=1}^N h_1(i)\pi_i + \varepsilon - \frac{\check{a}_{11}}{t} \int_0^t x_1(s)dt + \frac{\hat{\sigma}_1}{t} B_1(t),$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^N h_2(i)\pi_i + \varepsilon - \frac{\check{a}_{22}}{t} \int_0^t x_2(s)dt + \frac{\hat{\sigma}_3}{t} B_3(t).$$

Let ε be sufficiently small, if $\sum_{i=1}^N h_1(i)\pi_i > 0$ and $\sum_{i=1}^N h_2(i)\pi_i > 0$, making use of Lemma 3.1, we can derive that $\bar{x}_1^* \leq \frac{\sum_{i=1}^N h_1(i)\pi_i}{\check{a}_{11}}$ and $\bar{x}_2^* \leq \frac{\sum_{i=1}^N h_2(i)\pi_i}{\check{a}_{22}}$.   □

Theorem 3.14.

i) *If $\sum_{i=1}^N h_1(i)\pi_i < 0$ and $\sum_{i=1}^N h_2(i)\pi_i > 0$, then species $x_1(t)$ will go to extinction and species $x_2(t)$ will satisfy $\bar{x}_2^* \leq \frac{\sum_{i=1}^N h_2(i)\pi_i}{\check{a}_{22}}$.*

ii) *If $\sum_{i=1}^N h_1(i)\pi_i > 0$ and $\sum_{i=1}^N h_2(i)\pi_i < 0$, then species $x_2(t)$ will go to extinction and species $x_1(t)$ will satisfy $\bar{x}_1^* \leq \frac{\sum_{i=1}^N h_1(i)\pi_i}{\check{a}_{11}}$.*

Proof. According to Theorem 3.7, we obtain the following inequalities

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \frac{1}{t} \int_0^t h_1(\gamma(s))ds - \frac{1}{t} \int_0^t a_{11}(\gamma(s))x_1(s)ds$$

$$- \frac{1}{t} \int_0^t a_{12}(\gamma(s))x_2(s - \tau_1(s))ds$$

$$+ \frac{1}{t} \int_0^t \sigma_1(\gamma(s))dB_1(s) + \frac{2\ln k}{t} + \frac{N_1}{t},$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \frac{1}{t} \int_0^t h_2(\gamma(s))ds - \frac{1}{t} \int_0^t a_{21}(\gamma(s))x_1(s - \tau_2(s))ds$$

$$- \frac{1}{t} \int_0^t a_{22}(\gamma(s))x_2(s)ds + \frac{1}{t} \int_0^t \sigma_3(\gamma(s))dB_3(s) + \frac{2\ln k}{t} + \frac{N_2}{t}.$$

By assumption (A1) and the fact that

$$\lim_{t \to +\infty} t^{-1} \int_0^t h_1(\gamma(s))ds = \sum_{i=1}^N h_1(i)\pi_i$$

and

$$\lim_{t \to +\infty} t^{-1} \int_0^t h_2(\gamma(s))ds = \sum_{i=1}^N h_2(i)\pi_i,$$

for arbitrary $\varepsilon \geq 0$, there exists a constant $T_1 > 0$ such that

$$t^{-1} \int_0^t h_1(\gamma(s))ds \leq \sum_{i=1}^N h_1(i)\pi_i + \frac{\varepsilon}{4}, \qquad t > T_1,$$

$$t^{-1} \int_0^t h_2(\gamma(s))ds \leq \sum_{i=1}^N h_2(i)\pi_i + \frac{\varepsilon}{4}, \qquad t > T_1,$$

and

$$\frac{N_1}{t} \leq \frac{\varepsilon}{4} \quad \text{and} \quad \frac{N_2}{t} \leq \frac{\varepsilon}{4}.$$

We can obtain

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \sum_{i=1}^N h_1(i)\pi_i + \frac{\varepsilon}{4} - \frac{\check{a}_{11}}{t} \int_0^t x_1(s)ds - \frac{\check{a}_{12}}{t} \int_0^t x_2(s - \tau_1(s))ds$$
$$+ \frac{\hat{\sigma}_1}{t} B_1(t) + \frac{2\ln k}{t} + \frac{\varepsilon}{4},$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^N h_2(i)\pi_i + \frac{\varepsilon}{4} - \frac{\check{a}_{21}}{t} \int_0^t x_1(s - \tau_2(s))ds - \frac{\check{a}_{22}}{t} \int_0^t x_2(s)ds$$
$$+ \frac{\hat{\sigma}_3}{t} B_3(t) + \frac{2\ln k}{t} + \frac{\varepsilon}{4}.$$

Note that for sufficiently large t with $T_1 < k - 1 \leq t \leq k$, $k \geq k_0$, we have $t^{-1}\ln k \leq \frac{\varepsilon}{4}$. Then

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \sum_{i=1}^N h_1(i)\pi_i + \varepsilon - \frac{\check{a}_{11}}{t} \int_0^t x_1(s)ds - \frac{\check{a}_{12}}{t} \int_0^t x_2(s - \tau_1(s))ds + \frac{\hat{\sigma}_1}{t} B_1(t),$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^N h_2(i)\pi_i + \varepsilon - \frac{\check{a}_{21}}{t} \int_0^t x_1(s - \tau_2(s))ds - \frac{\check{a}_{22}}{t} \int_0^t x_2(s)ds + \frac{\hat{\sigma}_3}{t} B_3(t).$$

i) Let ε be sufficiently small, and $\sum_{i=1}^N h_1(i)\pi_i < 0$, one can derive that $\lim_{t \to +\infty} x_1(t) = 0$. Then, for arbitrary $\varepsilon > 0$, there exists a constant $T_2 > T_1$ such that $-\varepsilon \leq x_1(t) \leq \varepsilon$, for $t > T_2$. So, we can have

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^N h_2(i)\pi_i + \varepsilon + \check{a}_{21}\varepsilon - \frac{\check{a}_{22}}{t} \int_0^t x_2(s)ds + \frac{\hat{\sigma}_3}{t} B_3(t).$$

According to Lemma 3.1 and $\sum_{i=1}^N h_2(i)\pi_i > 0$, one can derive that $\bar{x}_2^* \leq \frac{\sum_{i=1}^N h_2(i)\pi_i + (1 + \check{a}_{21})\varepsilon}{\check{a}_{22}}$. Let ε be sufficiently small, then $\bar{x}_2^* \leq \frac{\sum_{i=1}^N h_2(i)\pi_i}{\check{a}_{22}}$.

ii) The proof is similar to that of i), so we omit it. $\qquad\square$

Remark 3.15. According to Theorem 3.13, we know that if $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then $\bar{x}_1^* \leq \frac{\sum_{i=1}^{N} h_1(i)\pi_i}{\breve{a}_{11}}$ and $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\breve{a}_{22}}$.

That is to say, species x_1 and species x_2 are no longer extinct. Based on Theorem 3.14, if $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then species x_1 will go to extinction, a.s. and species x_2 may not be extinct; if $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then species x_1 may not be extinct and species x_2 will go to extinction a.s.

Theorem 3.16.

    i) If $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then species $x_1(t)$ will go to extinction a.s. and species $x_2(t)$ will be non-persistence in the mean a.s.

    ii) If $\sum_{i=1}^{N} h_1(i)\pi_i = 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then species $x_1(t)$ will be non-persistence in the mean a.s. and species $x_2(t)$ will go to extinction a.s.

Proof. By the proof of Theorem 3.14, we obtain

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \sum_{i=1}^{N} h_1(i)\pi_i + \varepsilon - \frac{\breve{a}_{11}}{t}\int_0^t x_1(s)ds - \frac{\breve{a}_{12}}{t}\int_0^t x_2(s-\tau_1(s))ds + \frac{\hat{\sigma}_1}{t}B_1(t),$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^{N} h_2(i)\pi_i + \varepsilon - \frac{\breve{a}_{21}}{t}\int_0^t x_1(s-\tau_2(s))ds - \frac{\breve{a}_{22}}{t}\int_0^t x_2(s)ds + \frac{\hat{\sigma}_3}{t}B_3(t).$$

i) The proof is similar to that of Theorem 3.14 i), if $\sum_{i=1}^{N} h_1(i)\pi_i < 0$, we can derive that $\lim_{t\to+\infty} x_1(t) = 0$. Then, for arbitrary $\varepsilon > 0$, there exists a constant $T_2 > T_1$ such that $-\varepsilon \leq x_1(t) \leq \varepsilon$, for $t > T_2$. So, we can have

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^{N} h_2(i)\pi_i + \varepsilon + \breve{a}_{21}\varepsilon - \frac{\breve{a}_{22}}{t}\int_0^t x_2(s)ds + \frac{\hat{\sigma}_3}{t}B_3(t).$$

Making use of Lemma 3.1 and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then $\bar{x}_2^* \leq \frac{(1+\breve{a}_{21})\varepsilon}{\breve{a}_{22}}$, by the arbitrariness of ε, we know the species $x_2(t)$ will be be non-persistence in the mean a.s.

ii) The proof is similar to that of i), so we omit it. □

Theorem 3.17.

    i) If $\sum_{i=1}^{N} h_1(i)\pi_i = 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then species $x_1(t)$ will be non-persistence in the mean a.s. and species $x_2(t)$ will satisfy $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\breve{a}_{22}}$.

    ii) If $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then species $x_1(t)$ will satisfy $\bar{x}_1^* \leq \frac{\sum_{i=1}^{N} h_1(i)\pi_i}{\breve{a}_{11}}$ and species $x_2(t)$ will be non-persistence in the mean a.s.

Proof. By the above Theorem 3.13, we have

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \sum_{i=1}^{N} h_1(i)\pi_i + \varepsilon - \frac{\breve{a}_{11}}{t}\int_0^t x_1(s)ds + \frac{\hat{\sigma}_1}{t}B_1(t),$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \sum_{i=1}^{N} h_2(i)\pi_i + \varepsilon - \frac{\breve{a}_{22}}{t}\int_0^t x_2(s)ds + \frac{\hat{\sigma}_3}{t}B_3(t).$$

i) Basing on Lemma 3.1 and $\sum_{i=1}^{N} h_1(i)\pi_i = 0$, we have $\bar{x}_1^* \leq \frac{\varepsilon}{\breve{a}_{11}}$, by the arbitrariness of ε, we know the species $x_1(t)$ will be non-persistence in the mean a.s. If $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, by Lemma 3.1 we have $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i + \varepsilon}{\breve{a}_{22}}$. Let ε be sufficiently small, we get $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\breve{a}_{22}}$. So, this proof is completed.

ii) The proof is similar to i), so we omit it. □

4 Stochastically ultimate boundedness

In this section, we continue to examine another important asymptotic property: stochastically ultimate boundedness which means that the solution is ultimately bounded with the large probability. Firstly, its definition will be given.

Definition 4.1. The solution of system (1.3) is said to stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there is a positive constant $H = H_\varepsilon$, such that for any initial state $x_i(s) = \varphi_i(s) \geq 0$, $-\tau \leq s \leq 0$ and $\gamma(0) \in S$, the solution $X(t) = (x_1(t), x_2(t))$ of system (1.3) satisfies $\lim_{t \to +\infty} \mathbb{P}\{|X(t)| > H\} < \varepsilon$.

We provide a useful lemma from which the stochastically ultimate boundedness follows directly.

Lemma 4.2. *For any $p \in (0,1]$, there exists a constant C such that the solution of system (1.3) has the property*

$$\limsup_{t \to +\infty} \mathbb{E}|X(t)|^p \leq C,$$

where C is independent of the initial state.

Proof. Let V be defined by Theorem 2.1. For any $|x_i(0)| < k$ ($i = 1, 2.$), we define a stopping time

$$\sigma_k = \inf\{t \geq 0, |x_1(t)| > k \ \text{ or } \ |x_2(t)| > k\}.$$

Then $\sigma_k \to \infty$ a.s. as $k \to \infty$.
Applying the Itô formula to $e^t V(X(t))$, where $V(X(t)) = x_1^p(t) + x_2^p(t)$, yields

$$
\begin{aligned}
&d(e^t V(X(t))) \\
&\quad = e^t(x_1^p + x_2^p)dt + e^t dV(X(t)) \\
&\quad = e^t x_1^p[1 + pr_1(\gamma(t)) - pa_{11}(\gamma(t))x_1 - pa_{12}(\gamma(t))x_2(t - \tau_1(t)) + \tfrac{1}{2}p(p-1)\sigma_1^2(\gamma(t)) \\
&\qquad + \tfrac{1}{2}p(p-1)\sigma_2^2(\gamma(t))x_1^2]dt + e^t x_2^p[1 + pr_2(\gamma(t)) - pa_{21}(\gamma(t))x_1(t - \tau_2(t)) \\
&\qquad - pa_{22}(\gamma(t))x_2 + \tfrac{1}{2}p(p-1)\sigma_3^2(\gamma(t)) + \tfrac{1}{2}p(p-1)\sigma_4^2(\gamma(t))x_2^2]dt \\
&\qquad + pe^t x_1^p \sigma_1(\gamma(t))dB_1(t) + pe^t x_1^{p+1}\sigma_2(\gamma(t))dB_2(t) + pe^t x_2^p\sigma_3(\gamma(t))dB_3(t) \\
&\qquad + pe^t x_2^{p+1}\sigma_4(\gamma(t))dB_4(t) + pe^t x_1^p \int_Y [(1 + \theta_1(\gamma(t), u))^p - 1]N(dt, du) \\
&\qquad + pe^t x_2^p \int_Y [(1 + \theta_2(\gamma(t), u))^p - 1]N(dt, du),
\end{aligned}
$$

making use of assumption (A1), we obtain

$$
\begin{aligned}
&d(e^t V(X(t))) \\
&\quad \leq e^t x_1^p \Big[\tfrac{1}{2}p(p-1)\check{\sigma}_2^2(\gamma(t))x_1^2 - p\check{a}_{11}(\gamma(t))x_1 + 1 + p\hat{r}_1(\gamma(t)) + \tfrac{1}{2}p(p-1)\check{\sigma}_1^2(\gamma(t)) \\
&\qquad + \int_Y [(1 + \hat{\theta}_1(\gamma(t), u))^p - 1]\lambda(du) \Big]dt + e^t x_2^p \Big\{ \tfrac{1}{2}p(p-1)\check{\sigma}_4^2(\gamma(t))x_2^2 - p\check{a}_{22}(\gamma(t))x_2 \\
&\qquad + 1 + p\hat{r}_2(\gamma(t)) + \tfrac{1}{2}p(p-1)\check{\sigma}_3^2(\gamma(t)) + \int_Y [(1 + \hat{\theta}_2(\gamma(t), u))^p - 1]\lambda(du) \Big\}dt
\end{aligned}
$$

$$+ pe^t \hat{\sigma}_1(\gamma(t)) x_1^p dB_1(t) + pe^t \hat{\sigma}_2(\gamma(t)) x_1^{p+1} dB_2(t)$$

$$+ pe^t \hat{\sigma}_3(\gamma(t)) x_2^p dB_3(t) + pe^t x_1^p \int_Y [(1 + \hat{\theta}_1(\gamma(t), u))^p - 1] \tilde{N}(dt, du)$$

$$+ pe^t \hat{\sigma}_4(\gamma(t)) x_2^{p+1} dB_4(t) + pe^t x_2^p \int_Y [(1 + \hat{\theta}_2(\gamma(t), u))^p - 1] \tilde{N}(dt, du). \tag{4.1}$$

By the value $p \in (0, 1]$, there exist two constants k_1' and k_2' such that

$$e^t x_1^p \left[\frac{1}{2} p(p-1) \check{\sigma}_2^2(\gamma(t)) x_1^2 - p \check{a}_{11}(\gamma(t)) x_1 + 1 + p \hat{r}_1(\gamma(t)) \right.$$

$$\left. + \frac{1}{2} p(p-1) \check{\sigma}_1^2(\gamma(t)) + \int_Y [(1 + \hat{\theta}_1(\gamma(t), u))^p - 1] \lambda(du) \right] dt \le k_1',$$

and

$$e^t x_2^p \left[\frac{1}{2} p(p-1) \check{\sigma}_4^2(\gamma(t)) x_2^2 - p \check{a}_{22}(\gamma(t)) x_2 + 1 + p \hat{r}_2(\gamma(t)) \right.$$

$$\left. + \frac{1}{2} p(p-1) \check{\sigma}_3^2(\gamma(t)) + \int_Y [(1 + \hat{\theta}_2(\gamma(t), u))^p - 1] \lambda(du) \right] dt \le k_2'.$$

By the above two inequalities and integrating (4.1) from 0 to $t \wedge \sigma_k$, we have

$$\int_0^{t \wedge \sigma_k} d(e^s V(X(s)))$$

$$\le \int_0^{t \wedge \sigma_k} (k_1' + k_2') dt + \int_0^{t \wedge \sigma_k} pe^t \hat{\sigma}_1(\gamma(s)) x_1^p dB_1(s) + \int_0^{t \wedge \sigma_k} pe^s \hat{\sigma}_2(\gamma(s)) x_1^{p+1} dB_2(s)$$

$$+ \int_0^{t \wedge \sigma_k} pe^s \hat{\sigma}_3(\gamma(s)) x_2^p dB_3(s) + \int_0^{t \wedge \sigma_k} pe^s \hat{\sigma}_4(\gamma(s)) x_2^{p+1} dB_4(s)$$

$$+ \int_0^{t \wedge \sigma_k} pe^s x_1^p \int_Y [(1 + \hat{\theta}_1(\gamma(s), u))^p - 1] \tilde{N}(ds, du)$$

$$+ \int_0^{t \wedge \sigma_k} pe^s x_2^p \int_Y [(1 + \hat{\theta}_2(\gamma(s), u))^p - 1] \tilde{N}(ds, du).$$

Taking expectation, yields

$$\mathbb{E}[e^{t \wedge \sigma_k} V(X(t \wedge \sigma_k))] \le V(X(0)) + \mathbb{E} \int_0^{t \wedge \sigma_k} (k_1' + k_2') e^s dt.$$

Hence,

$$\mathbb{E}[e^t V(X(t))] \le V(X(0)) + (k_1' + k_2')(e^t - 1).$$

Clearly,

$$\mathbb{E}[V(X(t))] \le e^{-t} V(X(0)) + (k_1' + k_2')(1 - e^{-t}).$$

For $X \in R_+^2$ and $p > 0$, we have inequality $2^{(1 - \frac{p}{2}) \wedge 0} |X|^p \le x_1^p + x_2^p$. Taking the superior limit for both sides, we obtain

$$\limsup_{t \to +\infty} \mathbb{E}|X(t)|^p \le (\frac{p}{2})^{(1 - \frac{1}{2}) \wedge 0} \limsup_{t \to +\infty} \mathbb{E}[x_1^p + x_2^p].$$

That is to say

$$\limsup_{t \to +\infty} \mathbb{E}|X(t)|^p \le (\frac{p}{2})^{(1 - \frac{1}{2}) \wedge 0} (k_1' + k_2') \doteq C.$$

Then, this completes the proof.   □

According to Chebyshev's inequality and the application of Lemma 4.2, we have the following result.

Theorem 4.3. *Under assumption (A1), system (1.3) is stochastically ultimate bounded.*

Proof. By Lemma 4.2, we see that

$$\limsup_{t \to +\infty} \mathbb{E}\{|X(t)|^p\} \leq C.$$

For any $\varepsilon \in (0,1)$, let $H = (\frac{C}{\varepsilon})^{\frac{1}{p}}$. Then according to Chebyshev's inequality

$$\mathbb{P}\{|X(t)| > H\} \leq \frac{\mathbb{E}|X(t)|^p}{H^p},$$

obviously

$$\lim_{t \to +\infty} \sup \mathbb{P}\{|X(t)| > H\} \leq \varepsilon$$

will be obtained. This proof is completed. □

5 Asymptotic property

Before we investigate this asymptotic property of system (1.3), we need to provide some useful conditions, firstly.

(A6) Let assumption (A4) hold, assume further that for any $t \geq 0$ and $i \in S$,

$$\sup_{t \geq 0} \int_0^t \int_Y e^{s-t}(\theta(i,u) - \ln(1 + \theta(i,u)))\lambda(du)ds < \infty.$$

Lemma 5.1 ([22]). *Assume that $g : [0,\infty) \to R$ and $h : [0,\infty) \times Y \to R$ are both predictable $\mathcal{F}_t$-adapted processes such that for any $T > 0$,*

$$\int_0^T |g(t)|^2 dt < \infty \ a.s. \quad \text{and} \quad \int_0^T \int_Y |h(t,u)|^2 \lambda(du)dt < \infty \ a.s.$$

Then for any constants $\alpha, \beta > 0$,

$$\mathbb{P}\left\{ \sup_{t \geq 0} \left[\int_0^t g(s)dB(s) - \frac{\alpha}{2} \int_0^t |g(s)|^2 ds + \int_0^t \int_Y h(s,u)\tilde{N}(ds,du) \right.\right.$$
$$\left.\left. - \frac{1}{\alpha} \int_0^t \int_Y [e^{\alpha h(s,u)} - 1 - \alpha h(s,u)]\lambda(du)ds \right] > \beta \right\} \leq e^{-\alpha\beta}.$$

Theorem 5.2. *Let assumptions (A4) and (A6) hold, then for any given initial value $\gamma(0) \in S$ and (2.1), the solution $X(t) = (x_1(t), x_2(t))$ of system (1.3) has the property*

$$\limsup_{t \to +\infty} \frac{\ln x_1(t) + \ln x_2(t)}{\ln t} \leq 6 \ a.s.$$

Proof. For any $t \geq 0$, applying Itô's formula to $e^t \ln(x_1(t) + x_2(t))$ yields

$$
\begin{aligned}
e^t(\ln & x_1(t) + \ln x_2(t)) \\
= & \ln x_1(0) + \ln x_2(0) + \int_0^t e^s[\ln x_1(s) + r_1(\gamma(s)) - a_{11}(\gamma(s))x_1(s) \\
& - a_{12}(\gamma(s))x_2(s - \tau_1(s)) + \ln x_2(s) + r_2(\gamma(s)) - a_{21}(\gamma(s))x_1(s - \tau_2(s)) \\
& - a_{22}(\gamma(s))x_2(s) + \int_Y \ln(1 + \theta_1(\gamma(s), u)))\lambda(du) + \int_Y \ln(1 + \theta_2(\gamma(s), u)))\lambda(du)]ds \\
& + \int_0^t e^s \sigma_1(\gamma(s))dB_1(s) + \int_0^t \int_Y e^s \ln(1 + \theta_1(\gamma(s), u))\tilde{N}(ds, du) - \int_0^t e^s \sigma_1^2(\gamma(s))ds \\
& + \int_0^t e^s \sigma_2(\gamma(s))x_1(s)dB_2(s) - \int_0^t e^s \sigma_2^2(\gamma(s))x_1^2(s)ds + \int_0^t e^s \sigma_3(\gamma(s))dB_3(s) \\
& + \int_0^t \int_Y e^s \ln(1 + \theta_2(\gamma(s), u))\tilde{N}(ds, du) - \int_0^t e^s \sigma_3^2(\gamma(s))ds \\
& + \int_0^t e^s \sigma_3(\gamma(s))x_2(s)dB_4(s) - \int_0^t e^s \sigma_4^2(\gamma(s))x_2^2(s)ds.
\end{aligned}
\tag{5.1}
$$

It is easy for us to see that there exists a constant $\tilde{C}$ such that

$$
\begin{aligned}
\ln x_1(s) & + r_1(\gamma(s)) - a_{11}(\gamma(s))x_1(s) - a_{12}(\gamma(s))x_2(s - \tau_1(s)) + \ln x_2(s) + r_2(\gamma(s)) \\
& - a_{21}(\gamma(s))x_1(s - \tau_2(s)) - a_{22}(\gamma(s))x_2(s) + \int_Y \ln(1 + \theta_1(\gamma(s), u)))\lambda(du) \\
& + \int_Y \ln(1 + \theta_2(\gamma(s), u)))\lambda(du) \leq \tilde{C}.
\end{aligned}
\tag{5.2}
$$

By virtue of Lemma 5.1, for any $\alpha, \beta, T > 0$, we have

$$
\begin{aligned}
\mathbb{P}\Big\{ \limsup_{0 \leq t \leq T} \Big[& \int_0^t e^s \sigma_1(\gamma(s))dB_1(s) - \frac{\alpha}{2} \int_0^t e^{2s} \sigma_1^2(\gamma(s))ds + \int_0^t \int_Y e^s \ln(1 + \theta_1(\gamma(s), u))\tilde{N}(ds, du) \\
& - \frac{1}{\alpha} \int_0^t \int_Y [e^{\alpha e^s \ln(1 + \theta_1(\gamma(s), u))} - 1 - \alpha e^s \ln(1 + \theta_1(\gamma(s), u))]\lambda(du)ds \Big] > \beta \Big\} \leq e^{-\alpha\beta}.
\end{aligned}
$$

Choose $T = k\eta$, $\alpha = \epsilon e^{-k\eta}$ and $\beta = \frac{\theta e^{k\eta} \ln k}{\epsilon}$, where $k \in \mathbb{N}$, $0 < \epsilon < 1$, $\eta > 0$ and $\theta > 1$ in the above equation. Since $\sum_{k=1}^\infty \frac{1}{k^\theta} < \infty$, then by the classical Borel–Cantelli lemma, one can conclude that there is an $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for any $\epsilon \in \Omega_1$, an integer $k_1 = k_1(\omega, \epsilon)$ can be found such that

$$
\begin{aligned}
\int_0^t & e^s \sigma_1(\gamma(s))dB_1(s) + \int_0^t \int_Y e^s \ln(1 + \theta_1(\gamma(s), u))\tilde{N}(ds, du) \\
& \leq \frac{1}{\epsilon e^{-k\eta}} \int_0^t \int_Y \Big[\ln(1 + \theta_1(\gamma(s), u))^{\epsilon e^{s-k\eta}} - 1 - \epsilon e^{s-k\eta} \ln(1 + \theta_1(\gamma(s), u)) \Big] \lambda(du)ds \\
& \quad + \frac{\theta e^{k\eta} \ln k}{\epsilon} + \frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s} \sigma_1^2(\gamma(s))ds,
\end{aligned}
\tag{5.3}
$$

whenever $k \geq k_1$, $0 \leq t \leq k\eta$. Next, for $x \geq 0$, $0 \leq r \leq 1$, there exists an inequality $x^r \leq 1 + r(x - 1)$ such that

$$
\begin{aligned}
\frac{1}{\epsilon e^{t-k\eta}} & \int_0^t \int_Y [\ln(1 + \theta_1(\gamma(s), u))^{\epsilon e^{s-k\eta}} - 1 - \epsilon e^{s-k\eta} \ln(1 + \theta_1(\gamma(s), u))]\lambda(du)ds \\
& \leq \int_0^t \int_Y e^{s-t}(\theta_1(\gamma(s), u) - \ln(1 + \theta_1(\gamma(s), u)))\lambda(du)ds.
\end{aligned}
\tag{5.4}
$$

According to the exact calculation, one can derive that

$$\frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s}\sigma_1^2(\gamma(s))ds - \frac{1}{2}\int_0^t e^s\sigma_1^2(\gamma(s))ds = \frac{1}{2}\int_0^t (\epsilon e^{2s-k\eta} - e^s)\sigma_1^2(\gamma(s))ds \leq 0. \tag{5.5}$$

Combining the inequality (5.3) and (5.5) results in

$$\int_0^t e^s\sigma_1(\gamma(s))dB_1(s) + \int_0^t \int_Y e^s \ln(1 + \theta_1(\gamma(s), u))\tilde{N}(ds, du) - \frac{1}{2}\int_0^t e^s\sigma_1^2(\gamma(s))ds$$
$$\leq \frac{\theta e^{k\eta} \ln k}{\epsilon} \tag{5.6}$$
$$+ \frac{1}{\epsilon e^{-k\eta}}\int_0^t \int_Y [\ln(1 + \theta_1(\gamma(s), u))^{\epsilon e^{s-k\eta}} - 1 - \epsilon e^{s-k\eta}\ln(1 + \theta_1(\gamma(s), u))]\lambda(du)ds.$$

Similarly, making use of Lemma 5.1, we have

$$\int_0^t e^s\sigma_3(\gamma(s))dB_3(s) + \int_0^t \int_Y e^s \ln(1 + \theta_2(\gamma(s), u))\tilde{N}(ds, du)$$
$$\leq \frac{1}{\epsilon e^{-k\eta}}\int_0^t \int_Y [\ln(1 + \theta_2(\gamma(s), u))^{\epsilon e^{s-k\eta}} - 1 - \epsilon e^{s-k\eta}\ln(1 + \theta_2(\gamma(s), u))]\lambda(du)ds \tag{5.7}$$
$$+ \frac{\theta e^{k\eta} \ln k}{\epsilon} + \frac{\epsilon e^{-k\eta}}{2}\int_0^t e^{2s}\sigma_3^2(\gamma(s))ds,$$

and then

$$\frac{1}{\epsilon e^{t-k\eta}}\int_0^t \int_Y [\ln(1 + \theta_2(\gamma(s), u))^{\epsilon e^{s-k\eta}} - 1 - \epsilon e^{s-k\eta}\ln(1 + \theta_2(\gamma(s), u))]\lambda(du)ds$$
$$\leq \int_0^t \int_Y e^{s-t}(\theta_2(\gamma(s), u) - \ln(1 + \theta_2(\gamma(s), u)))\lambda(du)ds, \tag{5.8}$$

obviously

$$\frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s}\sigma_3^2(\gamma(s))ds - \frac{1}{2}\int_0^t e^s\sigma_3^2(\gamma(s))ds = \frac{1}{2}\int_0^t (\epsilon e^{2s-k\eta} - e^s)\sigma_3^2(\gamma(s))ds \leq 0. \tag{5.9}$$

Combining the inequality (5.7) and (5.9) leads to

$$\int_0^t e^s\sigma_3(\gamma(s))dB_3(s) + \int_0^t \int_Y e^s \ln(1 + \theta_2(\gamma(s), u))\tilde{N}(ds, du) - \frac{1}{2}\int_0^t e^s\sigma_3^2(\gamma(s))ds$$
$$\leq \frac{\theta e^{k\eta} \ln k}{\epsilon} \tag{5.10}$$
$$+ \frac{1}{\epsilon e^{-k\eta}}\int_0^t \int_Y [\ln(1 + \theta_2(\gamma(s), u))^{\epsilon e^{s-k\eta}} - 1 - \epsilon e^{s-k\eta}\ln(1 + \theta_2(\gamma(s), u))]\lambda(du)ds.$$

Let $D_1(t) = \int_0^t e^s\sigma_2(\gamma(s))x_1(s)dB_2(s)$ and $D_2(t) = \int_0^t e^s\sigma_4(\gamma(s))x_2(s)dB_4(s)$, applying the exponential martingale inequality [12, 35], for any positive constants T, α and β, we have

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left[D_1(t) - \frac{\alpha}{2}\langle D_1(t), D_1(t)\rangle\right] > \beta\right) \leq e^{-\alpha\beta},$$

and

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left[D_2(t) - \frac{\alpha}{2}\langle D_2(t), D_2(t)\rangle\right] > \beta\right) \leq e^{-\alpha\beta}.$$

To maintain the value of T, α and β, we have

$$D_1(t) \leq \frac{\theta e^{k\eta} \ln k}{\epsilon} + \frac{\epsilon e^{-k\eta}}{2} \langle D_1(t), D_1(t) \rangle$$

and

$$D_1(t) \leq \frac{\theta e^{k\eta} \ln k}{\epsilon} + \frac{\epsilon e^{-k\eta}}{2} \langle D_1(t), D_1(t) \rangle,$$

that is to say

$$\int_0^t e^s \sigma_2(\gamma(s)) x_1(s) dB_2(s) \leq \frac{\theta e^{k\eta} \ln k}{\epsilon} + \frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s} \sigma_2^2(\gamma(s)) x_1^2(s) ds, \tag{5.11}$$

and

$$\int_0^t e^s \sigma_4(\gamma(s)) x_2(s) dB_4(s) \leq \frac{\theta e^{k\eta} \ln k}{\epsilon} + \frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s} \sigma_4^2(\gamma(s)) x_2^2(s) ds. \tag{5.12}$$

According to the precise calculation, we can obtain

$$\frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s} \sigma_2^2(\gamma(s)) x_1^2(s) ds - \frac{1}{2} \int_0^t e^s \sigma_2^2(\gamma(s)) x_1^2(s) ds$$
$$= \frac{1}{2} \int_0^t (\epsilon e^{2s-k\eta} - e^s) \sigma_2^2(\gamma(s)) x_1^2(s) ds \leq 0, \tag{5.13}$$

$$\frac{\epsilon e^{-k\eta}}{2} \int_0^t e^{2s} \sigma_4^2(\gamma(s)) x_2^2(s) ds - \frac{1}{2} \int_0^t e^s \sigma_4^2(\gamma(s)) x_2^2(s) ds$$
$$= \frac{1}{2} \int_0^t (\epsilon e^{2s-k\eta} - e^s) \sigma_4^2(\gamma(s)) x_2^2(s) ds \leq 0. \tag{5.14}$$

Combining the inequality (5.11) and (5.13) gives

$$\int_0^t e^s \sigma_2(\gamma(s)) x_1(s) dB_2(s) - \frac{1}{2} \int_0^t e^s \sigma_2^2(\gamma(s)) x_1^2(s) ds \leq \frac{\theta e^{k\eta} \ln k}{\epsilon}. \tag{5.15}$$

Then combining the inequality (5.12) and (5.14) gives

$$\int_0^t e^s \sigma_4(\gamma(s)) x_2(s) dB_4(s) - \frac{1}{2} \int_0^t e^s \sigma_4^2(\gamma(s)) x_2^2(s) ds \leq \frac{\theta e^{k\eta} \ln k}{\epsilon}. \tag{5.16}$$

According to the inequalities (5.2), (5.4), (5.6), (5.8), (5.10), (5.15), (5.16) and dividing equation (5.1) by $e^t \ln t$, for any $\omega \in \Omega_1$ and $(k-1)\eta \leq t \leq k\eta$ with $k \geq k_1 + 1$, one can derive that

$$\frac{\ln x_1(t) + \ln x_2(t)}{\ln t} \leq \frac{\ln x_1(0) + \ln x_2(0)}{e^t \ln t} + \frac{\tilde{C}(1 - e^{-t})}{\ln t} + \frac{6\theta e^{k\eta} \ln k}{\epsilon e^{(k-1)\eta} \ln((k-1)\eta)}$$
$$+ \frac{1}{\ln t} \int_0^t \int_Y e^{s-t} (\theta_1(\gamma(s), u) - \ln(1 + \theta_1(\gamma(s), u))) \lambda(du) ds$$
$$+ \frac{1}{\ln t} \int_0^t \int_Y e^{s-t} (\theta_2(\gamma(s), u) - \ln(1 + \theta_2(\gamma(s), u))) \lambda(du) ds.$$

Letting $k \to +\infty$, then combining with assumption (A5) results in

$$\limsup_{t \to +\infty} \frac{\ln x_1(t) + \ln x_2(t)}{\ln t} \leq \frac{6\theta e^{\eta}}{\epsilon}.$$

Letting $\eta \to 0$, $\epsilon \to 1$ and $\theta \to 1$ gives

$$\limsup_{t \to +\infty} \frac{\ln x_1(t) + \ln x_2(t)}{\ln t} \le 6 \quad a.s.$$

So, this proof is completed. □

6 Discussion and numerical simulations

In this section, some numerical simulations is given to support our main results. We assume the Markov chain $\gamma(t)$ takes values in the state space $S = \{1, 2\}$, and the generator Q be expressed by $Q = \begin{pmatrix} -7 & 7 \\ 5 & -5 \end{pmatrix}$. According to equation $\pi Q = 0$ and exact calculation, the unique stationary distribution $\pi = (\pi_1, \pi_2) = (\frac{5}{12}, \frac{7}{12})$ will be obtained. In Figure 1–2, we choose $a_{11}(\gamma) \equiv 0.02$, $a_{12}(\gamma) \equiv 0.01$, $a_{21}(\gamma) \equiv 0.03$, $a_{22}(\gamma) \equiv 0.03$, $\sigma_1(\gamma) \equiv 2.0$, $\sigma_3(\gamma) \equiv 2.2$ $\sigma_2(\gamma) \equiv 0.04$, $\sigma_4(\gamma) \equiv 0.03$. The initial values are $x_1(0) = 0.8$, $x_2(0) = 10$ and $\lambda(Y) = 1$. The only difference between conditions of Figure 1 and Figure 2 is in the values of θ_1 and θ_2.

Comparing Figure 1 with Figure 2, we can observe when the jumping function is positive, the population size of species $x_1(t)$ and $x_2(t)$ will increase rapidly in a short period of time. In addition, whether the jumping function identically equal to zero or is positive, the species $x_1(t)$ always rapidly go to extinction. But, when the jumping function is positive, the species $x_2(t)$ will go to extinction more slowly. That is to say, the species $x_2(t)$ is more likely to be affected by the Lévy noise.

In Figure 3, the parameters of system (1.3) are given as follows $a_{11}(\gamma) \equiv 0.02$, $a_{12}(\gamma) \equiv 0.01$, $a_{21}(\gamma) \equiv 0.3$, $a_{22}(\gamma) \equiv 0.5$, $\sigma_1(\gamma) \equiv 2.0$, $\sigma_3(\gamma) \equiv 1.6$ $\sigma_2(\gamma) \equiv 0.02$, $\sigma_4(\gamma) \equiv 0.4$. The initial values are $x_1(0) = 0.3$, $x_2(0) = 5$ and $\lambda(Y) = 1$.

According to Figure 3, we can find that the species $x_2(t)$ go to extinction more quickly than the species $x_1(t)$. However, the results of Figure 1 and Figure 2 shows that species $x_1(t)$ will be extinct more rapidly, no matter the value of jumping function is positive or identically equal to zero. That is to say, the Lévy noise can cause both favorable and unfavorable influence on ecosystem. If the jumping function is positive, it indicates that the Lévy noise is advantage for a ecosystem. Inversely, if the jumping function is negative, it indicates that the Lévy noise is disadvantage for a ecosystem.

In Figure 4, the parameters of system (1.3) are given as follows $a_{11}(\gamma) \equiv 0.02$, $a_{12}(\gamma) \equiv 0.04$, $a_{21}(\gamma) \equiv 0.03$, $a_{22}(\gamma) \equiv 0.01$, $\sigma_1(\gamma) \equiv 1.0$, $\sigma_3(\gamma) \equiv 1.2$ $\sigma_2(\gamma) \equiv 0.04$, $\sigma_4(\gamma) \equiv 0.03$. The initial values are $x_1(0) = 0.8$, $x_2(0) = 0.9$ and $\lambda(Y) = 1$.

Comparing Figure 1 with Figure 4, the jumping function both identically equal to zero, and the changes between species $x_1(t)$ and $x_2(t)$ in Figure 1 and Figure 4 are perfectly clear. But, the important difference between conditions of Figure 1 and Figure 4 is that the values of σ_1 and σ_3. In Figure 4 we choose $\sigma_1(\gamma) \equiv 1.0$ and $\sigma_3(\gamma) \equiv 1.2$, and in Figure 1 we give $\sigma_1(\gamma) \equiv 2.0$ and $\sigma_3(\gamma) \equiv 2.2$. That is to say, when the jumping function identically equal to zero, the values of σ_1 and σ_3 have great significance in the population size of species $x_1(t)$ and species $x_2(t)$.

Figures 1–4 all describe the extinction of species $x_1(t)$ and species $x_2(t)$. Now, we will give some simulations to substantiate that it is possible for species $x_1(t)$ and species $x_2(t)$ are no longer extinct.

In Figure 5 the parameters of system (1.3) are given as follows $a_{11}(\gamma) \equiv 0.02$, $a_{12}(\gamma) \equiv 0.01$, $a_{21}(\gamma) \equiv 0.3$, $a_{22}(\gamma) \equiv 0.5$, $\sigma_1(\gamma) \equiv 1.0$, $\sigma_3(\gamma) \equiv 0.4$ $\sigma_2(\gamma) \equiv 0.52$, $\sigma_4(\gamma) \equiv 0.4$. The initial values are $x_1(0) = 0.8$, $x_2(0) = 1.2$ and $\lambda(Y) = 1$.

In Figure 6.6 the parameters of system (1.3) are chosen as follows $a_{11}(\gamma) \equiv 0.2$, $a_{12}(\gamma) \equiv$ 0.01, $a_{21}(\gamma) \equiv 0.3$, $a_{22}(\gamma) \equiv 0.4$, $\sigma_1(\gamma) \equiv 0.8$, $\sigma_3(\gamma) \equiv 1.2$ $\sigma_2(\gamma) \equiv 0.01$, $\sigma_4(\gamma) \equiv 0.25$. The initial values are $x_1(0) = 0.8$, $x_2(0) = 5$ and $\lambda(Y) = 1$.

In Figure 7 the parameters of system (1.3) are given as follows $a_{11}(\gamma) \equiv 0.2$, $a_{12}(\gamma) \equiv 0.01$, $a_{21}(\gamma) \equiv 0.1$, $a_{22}(\gamma) \equiv 0.4$, $\sigma_1(\gamma) \equiv 0.8$, $\sigma_3(\gamma) \equiv 0.8$ $\sigma_2(\gamma) \equiv 0.01$, $\sigma_4(\gamma) \equiv 0.05$. The initial values are $x_1(0) = 0.8$, $x_2(0) = 1.2$ and $\lambda(Y) = 1$.

In Figure 5, we choose the values of the parameters such that $\sum_{i=1}^{2} h_1(i)\pi_i = -0.01 < 0$ and $\sum_{i=1}^{2} h_2(i)\pi_i = 1.37 > 0$. In view of Theorem 3.14 i), we obtain that $x_1(t)$ is extinct and $x_2(t)$ is no longer extinct. Similarly, in Figure 6, we choose the values of the parameters such that $\sum_{i=1}^{2} h_1(i)\pi_i = 0.4 > 0$ and $\sum_{i=1}^{2} h_2(i)\pi_i = -0.1 < 0$. An application of Theorem 3.14 ii), species $x_2(t)$ is extinct and $x_1(t)$ is no longer extinct. In Figure 6.7, by simply calculation about the values of parameters, $\sum_{i=1}^{2} h_1(i)\pi_i = 0.88 > 0$ and $\sum_{i=1}^{2} h_2(i)\pi_i = 1.08 > 0$ will be obtained. It therefore from Theorem 3.13. That is to say, population $x_1(t)$ and $x_2(t)$ are no longer extinct. Figure 7 just confirms these.

7 Conclusion and future directions

This paper studies a stochastic delay competition system driven by Lévy noise under regime switching. The main results are as follows.

(1) For any initial value, the system exists an unique global positive solution.

(2) If $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then both the species x_1 and x_2 go to extinction a.s.

(3) If $\sum_{i=1}^{N} h_1(i)\pi_i = 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then both the species x_1 and x_2 are non-persistence in the mean a.s.

(4) If $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then $\bar{x}_1^* \leq \frac{\sum_{i=1}^{N} h_1(i)\pi_i}{\check{a}_{11}}$ and $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\check{a}_{22}}$.

(5) i) If $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then species $x_1(t)$ goes to extinction and species $x_2(t)$ satisfies $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\check{a}_{22}}$.

ii) If $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then species $x_2(t)$ goes to extinction and species $x_1(t)$ satisfies $\bar{x}_1^* \leq \frac{\sum_{i=1}^{N} h_1(i)\pi_i}{\check{a}_{11}}$.

(6) i) If $\sum_{i=1}^{N} h_1(i)\pi_i < 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then species $x_1(t)$ goes to extinction and species $x_2(t)$ is non-persistence in the mean a.s.

ii) If $\sum_{i=1}^{N} h_1(i)\pi_i = 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i < 0$, then species $x_1(t)$ is non-persistence in the mean a.s. and species $x_2(t)$ goes to extinction.

(7) i) If $\sum_{i=1}^{N} h_1(i)\pi_i = 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i > 0$, then species $x_1(t)$ is non-persistence in the mean a.s. and species $x_2(t)$ satisfies $\bar{x}_2^* \leq \frac{\sum_{i=1}^{N} h_2(i)\pi_i}{\check{a}_{22}}$.

ii) If $\sum_{i=1}^{N} h_1(i)\pi_i > 0$ and $\sum_{i=1}^{N} h_2(i)\pi_i = 0$, then species $x_1(t)$ satisfies $\bar{x}_1^* \leq \frac{\sum_{i=1}^{N} h_1(i)\pi_i}{\check{a}_{11}}$ and species $x_2(t)$ is non-persistence in the mean a.s.

(8) Some asymptotic properties of system (1.3) have been given.

For stochastic population models, the persistence in the mean and weak persistence are not good definitions of permanence. In recent years, some authors have introduced a more

appropriate definition of permanence for stochastic population models, that is stochastically persistent in probability (see [10,11,32]), which is a more appropriate definition of persistence. Moreover, for system (1.3), we also consider its stochastically persistent in probability. Because of the existence of time-varying delay in our model, it makes the task more complicated to deal with. So far, we have looked up a lot of relevant known references, but we still can't find a suitable Lyapunov function to solve the problem of the system (1.3) with variable time delay. It is a pity that we have to take it as our research work in the future.

Furthermore, the stability of the positive equilibrium state is a very interesting study for population models. For models with noise, the stochastic models do not keep the positive equilibrium state of the corresponding deterministic systems. Recently, the stability in distribution of stochastic population models has been one of the most interest topics, and many authors have studied the stability in distribution of various stochastic population models (see [7–9]). Then, in order to increase the interest of our articles, we will study the stability in distribution of stochastic population models in our future investigation.

Acknowledgements

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Figures

(a)

(b)
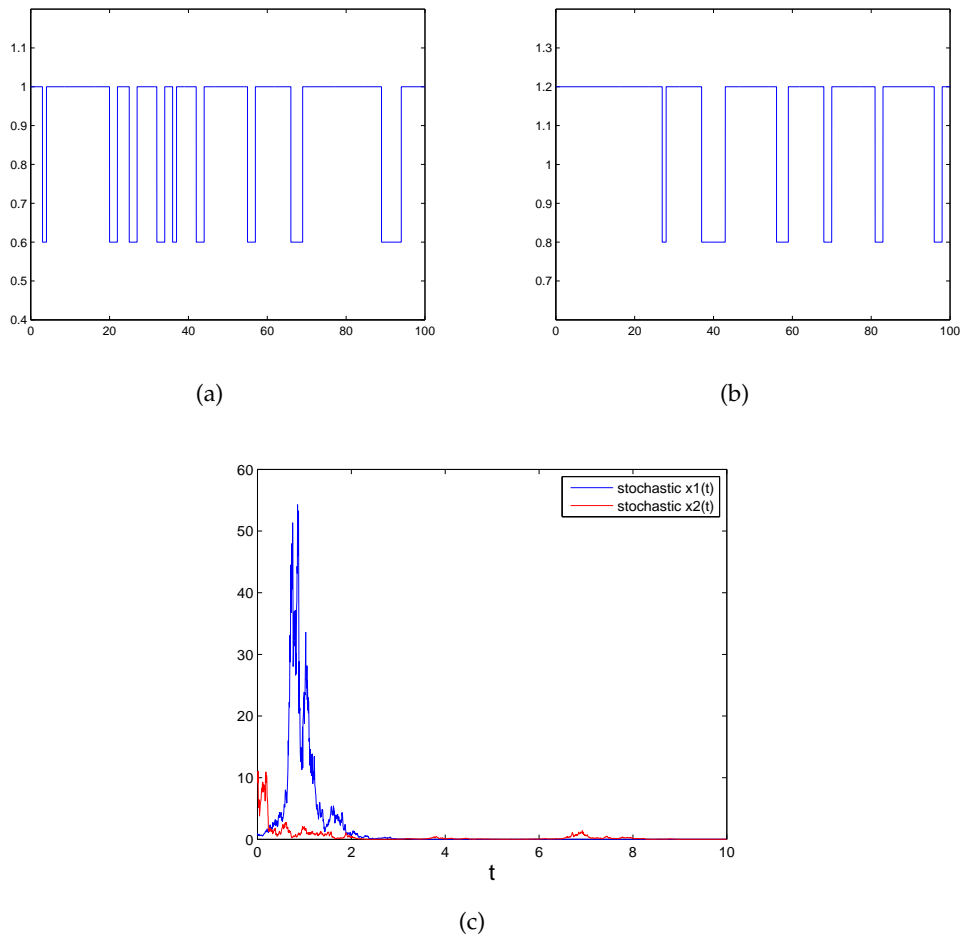
(c)

Figure 1: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv 0$ and $\theta_2(\gamma, u) \equiv 0$.
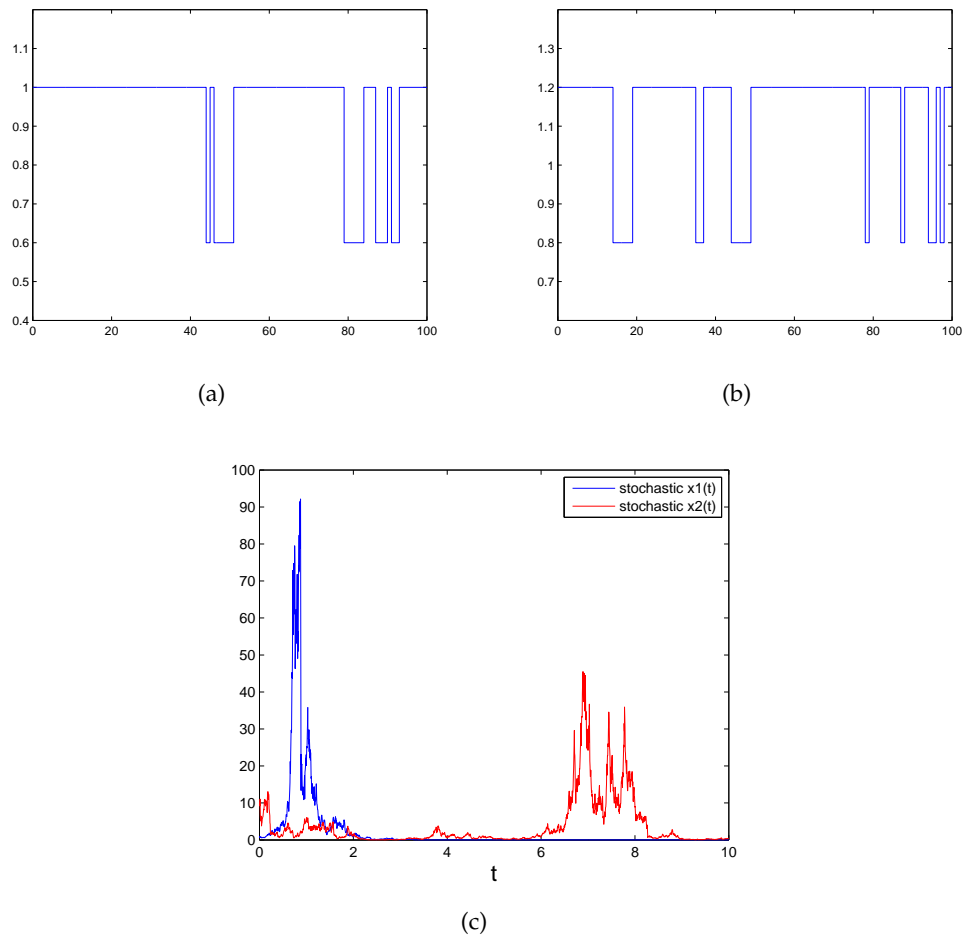
(a)

(b)

(c)

Figure 2: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv 0.2$ and $\theta_2(\gamma, u) \equiv 0.2$.
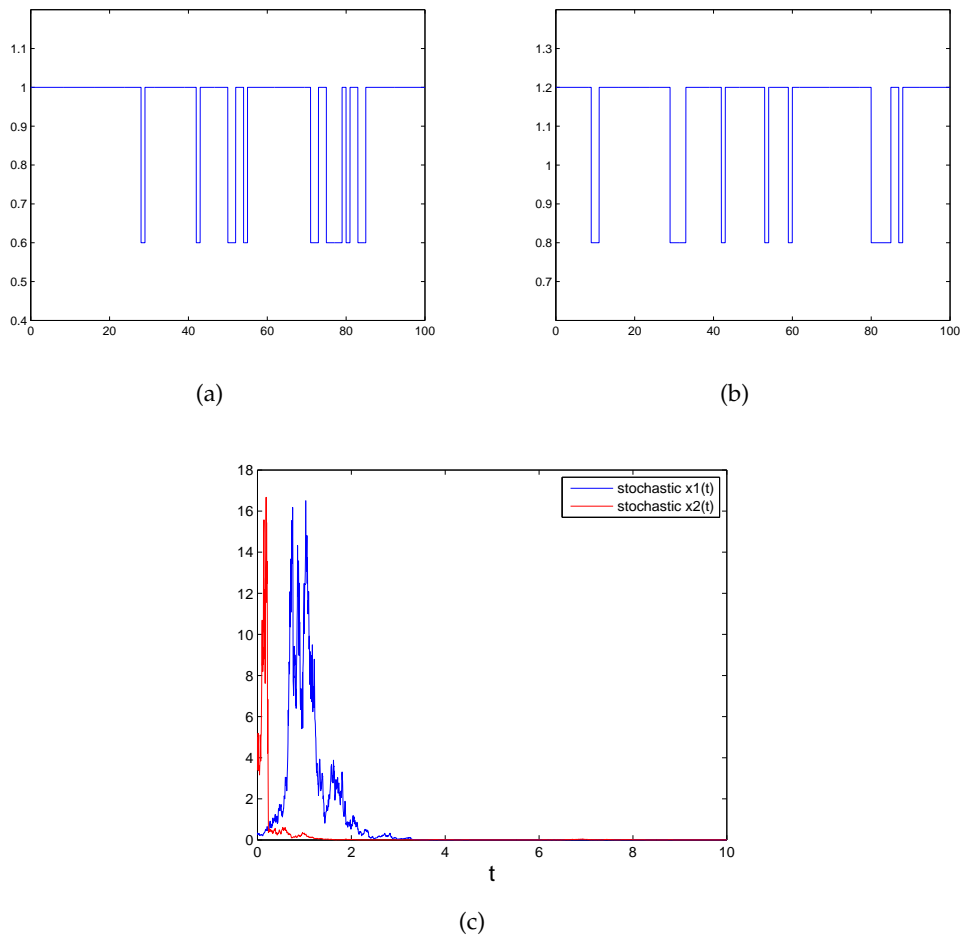
(a)

(b)

(c)

Figure 3: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv 0.01$ and $\theta_2(\gamma, u) \equiv -0.15$.
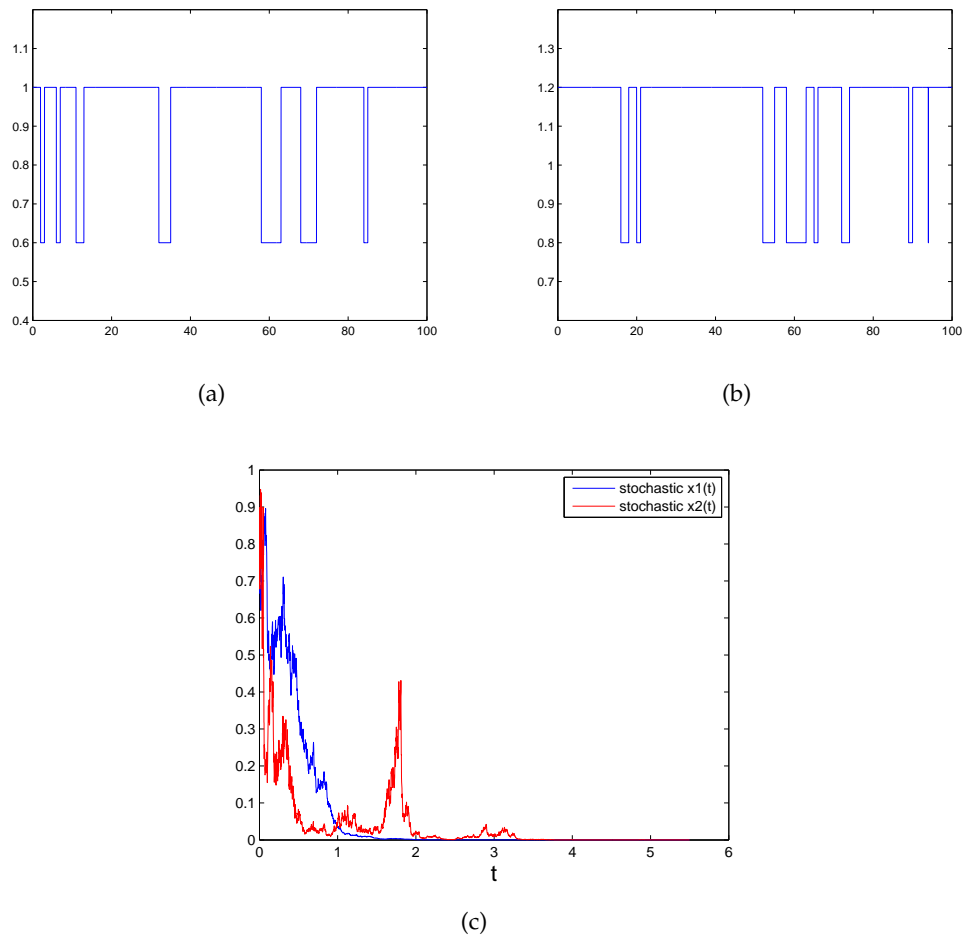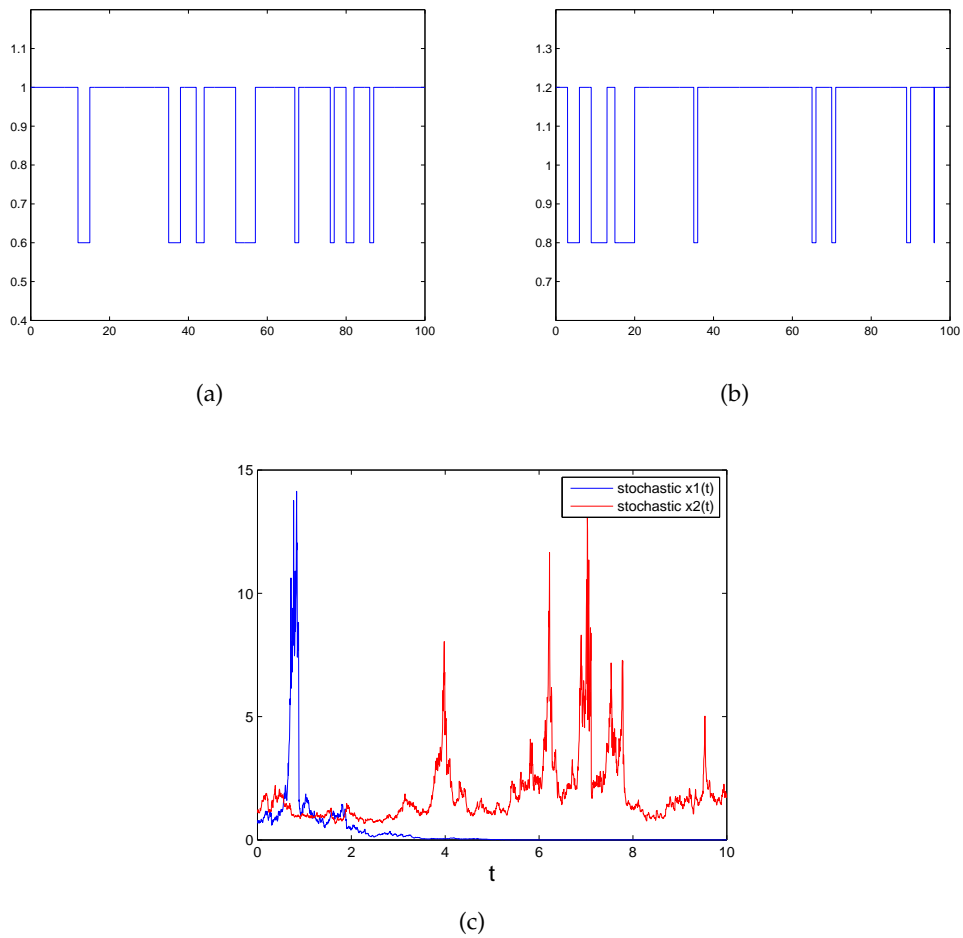
(a)

(b)

(c)

Figure 4: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv 0$ and $\theta_2(\gamma, u) \equiv 0$.

(a)

(b)

(c)

Figure 5: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv -0.51$ and $\theta_2(\gamma, u) \equiv 0.25$.
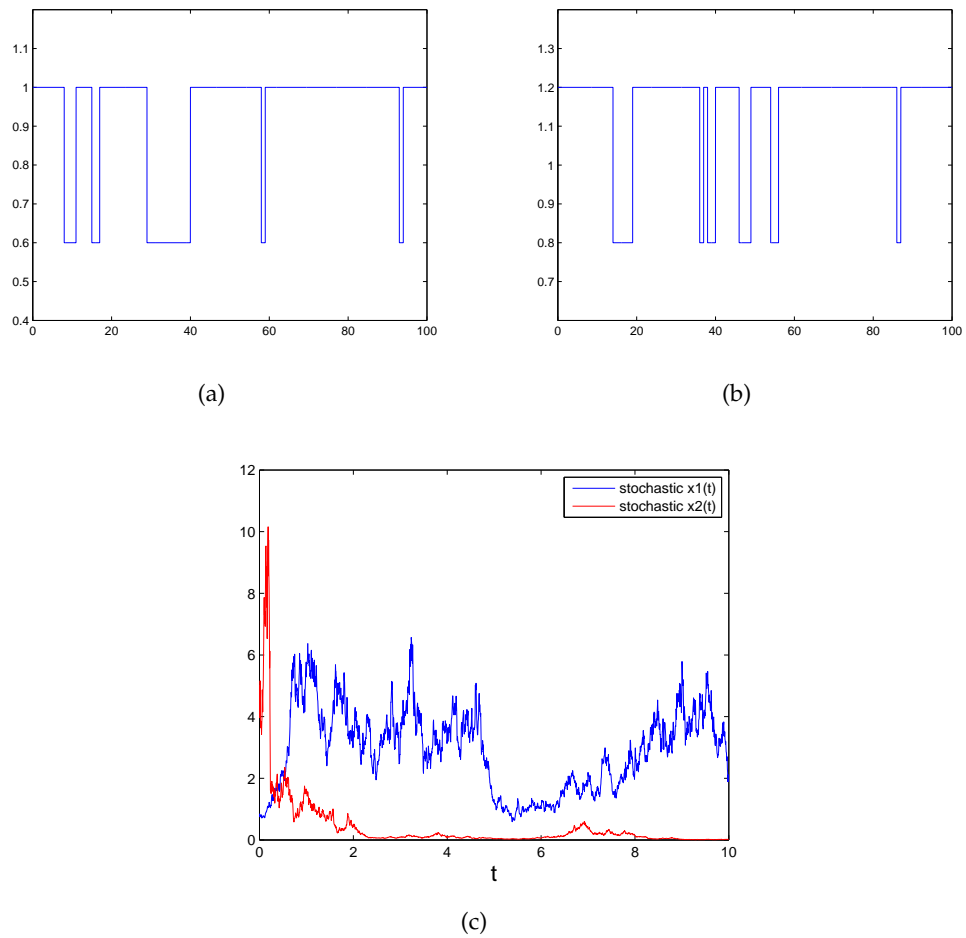
(a)

(b)

(c)

Figure 6: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv 0.2$ and $\theta_2(\gamma, u) \equiv -0.1$.
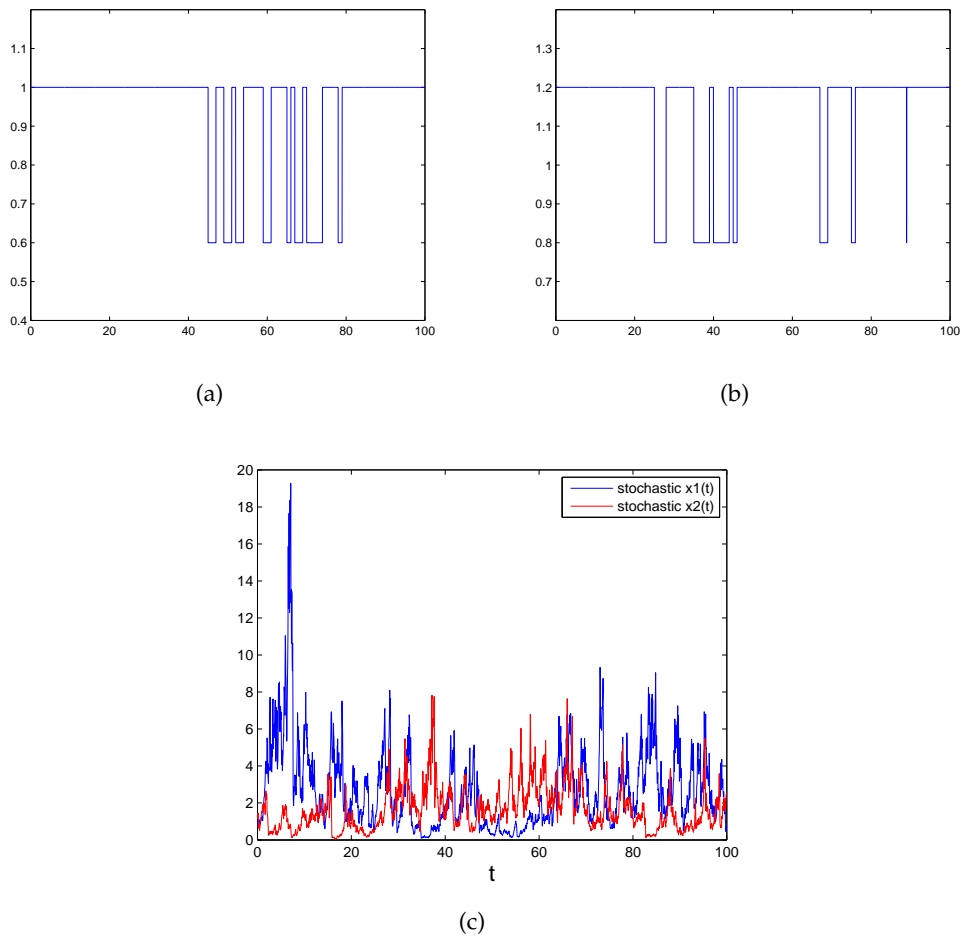
(a)

(b)

(c)

Figure 7: Numerical simulations for system (1.3) with $\theta_1(\gamma, u) \equiv 0.2$ and $\theta_2(\gamma, u) \equiv 0.2$.