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Multiple positive solutions for a nonlinear 2n-th order m-point boundary value problems *†

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Abstract In this paper, we consider the existence of multiple positive solutions for the 2n-th order m-point boundary value problems:

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \cdots, x^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ x^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{ij} x^{(2i+1)}(\xi_j), \quad x^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), \quad 0 \le i \le n-1, \end{cases}$$

where α_{ij}, β_{ij} $(0 \le i \le n-1, 1 \le j \le m-2) \in [0, \infty)$, $\sum_{j=1}^{m-2} \alpha_{ij}$, $\sum_{j=1}^{m-2} \beta_{ij} \in (0, 1)$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$. Using Leggett-Williams fixed point theorem, we provide sufficient conditions for the existence of at least three positive solutions to the above boundary value problem.

Keywords Higher order m-point boundary value problem, Leggett-Williams fixed point theorem, Green's function, Positive solution.

1. Introduction

The multi-point boundary value problems for ordinary differential equations arises in a variety of different areas of applied mathematics and physics. Linear and nonlinear second order multi-point boundary value problems have also been studied by several authors. We refer the reader to

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[2-8] and references therein. Davis et al. [9,10] studied the following 2n-th Lidstone BVP

$$\begin{cases} x^{(2n)} = f(x(t), x^{''}(t), \cdots, x^{(2(n-1))}(t)), & t \in [0,1], \\ x^{(2i)}(0) = x^{(2i)}(1) = 0, & 0 \le i \le n-1, \end{cases} \tag{1}$$

where $(-1)^n f : R^n \to [0, \infty)$ is continuous. They obtained the existence of three symmetric positive solutions of the BVP (1).

Y. Guo et al. [11] studied the following 2n-th BVP

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x^{''}(t), \cdots, x^{(2(n-1))}(t)), & 0 \le t \le 1, \\ x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0, \quad x^{(2i)}(1) = \sum\limits_{j=1}^{m-2} k_{ij} y^{(2i)}(\xi_j), & 0 \le i \le n-1. \end{cases} \tag{2}$$

They obtained the existence of at least two positive solution for the above BVP.

Recently, Y. Guo et al. [13] studied the following 2n-th BVP

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x^{''}(t), \cdots, x^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ x^{(2i)}(0) = 0, \quad x^{(2i)}(1) = \sum\limits_{j=1}^{m-2} k_{ij} y^{(2i)}(\xi_j), \quad 0 \le i \le n-1. \end{cases} \tag{3}$$

By using Leggett-Williams fixed point theorem, they got at least three positive solutions for the BVP(3).

The authors [14,15] investigated the following two BVPs

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x^{''}(t), \cdots, x^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ x^{(2i)}(0) = \sum\limits_{j=1}^{m-2} \alpha_{ij} x^{(2i)}(\xi_j), \quad x^{(2i)}(1) = \sum\limits_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), \quad 0 \le i \le n-1, \end{cases} \tag{4}$$

and

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x^{''}(t), \cdots, x^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ x^{(2i)}(0) - a_i x^{(2i+1)}(0) = \sum\limits_{j=1}^{m-2} \alpha_{ij} x^{(2i)}(\xi_j), \\ x^{(2i)}(1) + b_i x^{(2i+1)}(1) = \sum\limits_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), \quad 0 \le i \le n-1, \end{cases} \tag{5}$$

Motivated by the above results, in this paper, we study the existence of multiple positive solutions for the following 2n-th order m-point boundary value problem

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x^{''}(t), \cdots, x^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ x^{(2i+1)}(0) = \sum\limits_{j=1}^{m-2} \alpha_{ij} x^{(2i+1)}(\xi_j), \quad x^{(2i)}(1) = \sum\limits_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), \quad 0 \le i \le n-1, \end{cases} \tag{6}$$

To the best of our knowledge, existence results for positive solutions of above boundary value problems have not been studied previously.

Throughout the paper, we assume the following conditions satisfied:

(H_1) α_{ij}, β_{ij} $(0 \leq i \leq n-1, 1 \leq j \leq m-2) \in [0,\infty)$, $\sum_{j=1}^{m-2} \alpha_{ij}, \sum_{j=1}^{m-2} \beta_{ij} \in (0,1)$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$;

(H_2) $(-1)^n f : [0,1] \times R^n \to [0,\infty)$ is continuous;

2. Preliminaries

Our main results will depend on the Leggett-Williams fixed point theorem. For convenience, we present here the necessary definitions from the theory of cones in Banach spaces.

Definition 2.1 Let E be a real Banach space . A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$ for all $u \in P$ and all $a \geq 0$ and

(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

Definition 2.2 The map α is said to be a nonnegative continuous **concave** functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Similarly, we say the map β is a nonnegative continuous **convex** functional on a cone P of a real Banach space E provided that $\beta : P \to [0,\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 2.3 An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

For positive real numbers a, b, we define the following convex sets:

$$P_r = \{x \in P |\ \|x\| < r\},$$

$$P(\alpha, a, b) = \{x \in P |\ a \leq \alpha(x), \|x\| \leq b\},$$

Theorem 2.1 [1] (Leggett-Williams Fixed Point Theorem) Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous operators and let α be a nonnegative continuous concave function on P such that $\alpha(x) \le \|x\|$ for all $x \in \overline{P}_c$. Suppose there exists $0 < a < b < d \le c$ such that

(C1) $\{x \in P(\alpha, b, d)|\ \alpha(x) > b\} \neq \emptyset$ and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$,

(C2) $\|Ax\| < a$ for $\|x\| \le a$, and

(C3) $\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2 and x_3 such that $\|x_1\| < a$, $b < \alpha(x_2)$, and $\|x_3\| > a$ with $\alpha(x_3) < b$.

3. Multiple positive solutions of (6)

In order to apply Theorem 2.1, we must define an appropriate operator on a Banach space. We first consider the the unique solution of the following second order boundary value problem:

Lemma 3.1[12] Let $(1 - \sum\limits_{i=1}^{m-2} \alpha_i)(1 - \sum\limits_{i=1}^{m-2} \beta_i) \neq 0$. Then for $f(t) \in C[0, 1]$, the problem

$$\begin{cases} x''(t) + f(t) = 0, \quad 0 \le t \le 1 \\ x'(0) = \sum\limits_{i=1}^{m-2} \alpha_i x'(\xi_i), \quad x(1) = \sum\limits_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{7}$$

has a unique solution

$$x(t) = -\int_0^t (t - s) f(s) ds + At + B,$$

where

$$\begin{aligned} A &= -\frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} f(s) ds \right), \\ B &= \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i} \left[\int_0^1 (1 - s) f(s) ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi_i - s) f(s) ds \right. \\ &\quad \left. + \frac{1 - \sum_{i=1}^{m-2} \beta_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} f(s) ds \right) \right]. \end{aligned}$$

Lemma 3.2[12] Suppose $\alpha_i, \beta_i > 0$ $(i = 1, 2, \cdots, m-2)$, $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1, 0 < \sum\limits_{i=1}^{m-2} \beta_i < 1$. If $f(t) \in C[0, 1]$ and $f \ge 0$, then the unique solution of (7) satisfies

$$\inf_{t \in [0,1]} x(t) \ge \gamma \|x\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} \beta_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \beta_i \xi_i}.$$

Lemma 3.3 Suppose $\alpha_i, \beta_i > 0$ $(i = 1, 2, \cdots, m-2)$, $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1, 0 < \sum\limits_{i=1}^{m-2} \beta_i < 1$, and let $M = (1 - \sum\limits_{i=1}^{m-2} \alpha_i)(1 - \sum\limits_{i=1}^{m-2} \beta_i)$. Then the Green's function for the boundary value problem

$$\begin{cases} -x''(t) = 0, \quad 0 \le t \le 1, \\ x'(0) = \sum\limits_{i=1}^{m-2} \alpha_i x'(\xi_i), \quad x(1) = \sum\limits_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

is given by

$$G^*(t,s) = \frac{1}{M} \begin{cases} (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j), \\ \qquad 0 \le t \le 1, \quad 0 \le s \le \xi_1, \quad s \le t; \\ \sum_{j=1}^{m-2} \alpha_j \left[(1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum\limits_{j=1}^{m-2} \beta_j) \right] \\ +(1 - \sum\limits_{j=1}^{m-2} \alpha_j) \left[(1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=1}^{m-2} \beta_j) \right], \\ \qquad 0 \le t \le 1, \quad 0 \le s \le \xi_1, \quad t \le s; \\ \sum_{j=i}^{m-2} \alpha_j \left[(1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum\limits_{j=1}^{m-2} \beta_j) \right] \\ +(1 - \sum\limits_{j=1}^{m-2} \alpha_j) \left[(1 - \sum_{j=i}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=i}^{m-2} \beta_j) \right], \\ \qquad \xi_{i-1} \le s \le \xi_i, \quad 2 \le i \le m-2, \quad t \le s; \\ -M(t-s) + \sum_{j=i}^{m-2} \alpha_j \left[(1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j) \right] \\ +(1 - \sum_{j=1}^{m-2} \alpha_j) \left[(1 - \sum_{j=i}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=i}^{m-2} \beta_j) \right], \\ \qquad \xi_{i-1} \le s \le \xi_i, \quad 2 \le i \le m-2, \quad s \le t; \\ (1 - \sum_{j=1}^{m-2} \alpha_j) \left[(1-t) + \sum_{j=1}^{m-2} \beta_j (t-s) \right], \\ \qquad \xi_{m-2} \le s \le 1, \quad s \le t; \\ (1 - \sum_{j=1}^{m-2} \alpha_j)(1-s), \\ \qquad 0 \le t \le 1, \quad \xi_{m-2} \le s \le 1, \quad t \le s. \end{cases}$$

Lemma 3.4 Suppose $\alpha_i, \beta_i > 0$ $(i = 1, 2, \cdots, m-2)$, $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1, 0 < \sum\limits_{i=1}^{m-2} \beta_i < 1$. Then

$$G^*(t,s) \ge 0 \quad \text{for} \quad (t,s) \in [0,1] \times [0,1].$$

Proof. We only check that if $s \le t$, then

$$\begin{aligned} Q &= -M(t-s) + \sum_{j=i}^{m-2} \alpha_j \left[(1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j) \right] \\ &\quad +(1 - \sum_{j=1}^{m-2} \alpha_j) \left[(1 - \sum_{j=i}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=i}^{m-2} \beta_j) \right] \ge 0. \end{aligned}$$

In fact

$$Q = \sum_{j=i}^{m-2} \alpha_j \left(1 - \sum_{j=1}^{m-2} \beta_j \right) (1-t) + \sum_{j=i}^{m-2} \alpha_j \left(\sum_{j=1}^{m-2} \beta_j - \sum_{j=1}^{m-2} \beta_j \xi_j \right)$$

$$
\begin{aligned}
&+\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(1-\sum_{j=i}^{m-2}\beta_j\right)(1-s)+\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(\sum_{j=i}^{m-2}\beta_j-\sum_{j=i}^{m-2}\beta_j\xi_j\right)\\
&-\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(1-\sum_{j=1}^{m-2}\beta_j\right)(t-s)\\
\geq\ &\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(1-\sum_{j=i}^{m-2}\beta_j\right)(1-s)-\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(1-\sum_{j=1}^{m-2}\beta_j\right)(t-s)\\
\geq\ &\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(1-\sum_{j=i}^{m-2}\beta_j\right)(t-s)-\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\left(1-\sum_{j=1}^{m-2}\beta_j\right)(t-s)\\
=\ &\left(1-\sum_{j=1}^{m-2}\alpha_j\right)\sum_{j=1}^{i-1}\beta_j(t-s)\\
\geq\ &0.
\end{aligned}
$$

Lemma 3.5 Suppose (H_1) holds. Then $g_i(t,s)\leq 0$ $(0\leq i\leq n-1)$, where $g_i(t,s)$ is the Green's function for the BVP

$$
\begin{cases}
x''(t)=0, \quad 0\leq t\leq 1,\\
x'(0)=\sum_{j=1}^{m-2}\alpha_{ij}x'(\xi_j), \quad x(1)=\sum_{j=1}^{m-2}\beta_{ij}x(\xi_j).
\end{cases}
$$

Proof. It is easy to see that $g_i(t,s)\leq 0$ by using Lemma 3.4.

Let $G_1(t,s)=g_{n-2}(t,s)$, then for $2\leq j\leq n-1$ we recursively define

$$
G_j(t,s)=\int_0^1 g_{n-j-1}(t,r)G_{j-1}(r,s)dr.
$$

Lemma 3.6 Suppose (H_1) holds. If $f(t)\in C[0,1]$, then the boundary value problem

$$
\begin{cases}
u^{(2l)}(t)=f(t), & 0\leq t\leq 1,\\
u^{(2i+1)}(0)=\sum_{j=1}^{m-2}\alpha_{n-l+i-1,j}u^{(2i+1)}(\xi_j), & \\
u^{(2i)}(1)=\sum_{j=1}^{m-2}\beta_{n-l+i-1,j}u^{(2i)}(\xi_j), & 0\leq i\leq l-1,
\end{cases}
\tag{8}
$$

has a unique solution for each $1\leq l\leq n-1$, $G_l(t,s)$ is the associated Green's function for the boundary value problem (8).

Proof. We prove the result by using induction. Obviously, the result holds by using Lemma 3.3 for $l=1$.

We assume that the result holds for $l-1$. Now we consider the case for l. Let $u''(t)=v(t)$, then (8) is equivalent to

$$\begin{cases} u^{''}(t) = v(t), \qquad 0 \le t \le 1, \\ u'(0) = \sum\limits_{j=1}^{m-2} \alpha_{n-l-1,j} u'(\xi_j), \\ u(1) = \sum\limits_{j=1}^{m-2} \beta_{n-l-1,j} u(\xi_j), \end{cases} \tag{9}$$

and

$$\begin{cases} v^{(2(l-1))}(t) = f(t), \qquad 0 \le t \le 1, \\ v^{(2i+1)}(0) = \sum\limits_{j=1}^{m-2} \alpha_{n-l+i,j} v^{(2i+1)}(\xi_j), \\ v^{(2i)}(1) = \sum\limits_{j=1}^{m-2} \beta_{n-l+i,j} v^{(2i)}(\xi_j), \quad 0 \le i \le l-2. \end{cases} \tag{10}$$

Lemma 3.3 implies that (9) has a unique solution $u(t) = \int_0^1 g_{n-l-1}(t,r)v(r)dr$, and (10) has also a unique solution $v(t) = \int_0^1 G_{l-1}(t,s)f(s)ds$ by the inductive hypothesis. Thus, (8) has a unique solution

$$\begin{aligned} u(t) &= \int_0^1 g_{n-l-1}(t,r) \int_0^1 G_{l-1}(r,s)f(s)dsdr \\ &= \int_0^1 \left(\int_0^1 g_{n-l-1}(t,r)G_{l-1}(r,s)dr \right) f(s)ds \\ &= \int_0^1 G_l(t,s)f(s)ds \end{aligned}$$

Therefore, the result hold for l. Lemma 3.6 is now completed.

For each $1 \le l \le n-1$, we define $A_l : C[0,1] \to C[0,1]$ by

$$A_l v(t) = \int_0^1 G_l(t,\tau)v(\tau)d\tau.$$

With the use of Lemma 3.6, for each $1 \le l \le n-1$, we have

$$\begin{cases} (A_l v)^{(2l)}(t) = v(t), & 0 \le t \le 1, \\ (A_l v)^{(2i+1)}(0) = \sum\limits_{j=1}^{m-2} \alpha_{n-l+i-1,j} (A_l v)^{(2i+1)}(\xi_j), \\ (A_l v)^{(2i)}(1) = \sum\limits_{j=1}^{m-2} \beta_{n-l+i-1,j} (A_l v)^{(2i)}(\xi_j), & 0 \le i \le l-1. \end{cases}$$

Therefore (6) has a solution if and only if the boundary value problem

$$\begin{cases} v^{''}(t) = f(t, A_{n-1}v(t), A_{n-2}v(t), \cdots, A_1 v(t), v(t)), 0 \le t \le 1, \\ v'(0) = \sum\limits_{j=1}^{m-2} \alpha_{n-1,j} v'(\xi_j), \quad v(1) = \sum\limits_{j=1}^{m-2} \beta_{n-1,j} v(\xi_j), \end{cases} \tag{11}$$

has a solution. If x is a solution of (6), then $v = x^{(2(n-1))}$ is a solution of (11). Conversely, if v is a solution of (11), then $x = A_{n-1}v$ is a solution of (6).

Define $A : C[0,1] \to C[0,1]$ by

$$Av(t) = \int_0^1 g_{n-1}(t,s)f(s, A_{n-1}v(s), A_{n-2}v(s), \cdots, A_1v(s), v(s))ds.$$

It now follows that there exists a solution of BVP (6) if, and only if , there exists a continuous fixed point of A. Moreover, the relationship between a solution of BVP (6) and a fixed point of A is given by $x = A_{n-1}v(t)$, or equivalently, $x^{(2(n-1))} = v$.

Note that x is a positive solution of (6) if, and only if, $(-1)^{n-1}x^{(2(n-1))} = (-1)^{n-1}v$ is positive, where v is the corresponding continuous fixed point of A.

For each $0 \le t \le 1, 0 \le i \le n-1$, there are only finitely many points s such that $g_i(t,s) = 0$. Let

$$M_i = \max_{0\le t\le 1} \int_0^1 |g_i(t,s)|ds, \quad m_i = \min_{0\le t\le 1} \int_0^1 |g_i(t,s)|ds,$$

obviously, $M_i > m_i > 0$.

Let $X = C[0,1]$ with the maximum norm $\|x\| = \max_{0\le t\le 1} |x(t)|$ and define the cone $P \subset X$ by

$$P = \left\{ x \in X : (-1)^{n-1}x(t) \ge 0, (-1)^{n-1}x \text{ is concave on } [0,1], \text{and } \min_{t\in[0,1]} (-1)^{n-1}x(t) \ge \gamma\|x\| \right\}.$$

Let $\alpha : P \to [0,\infty)$ be the nonnegative continuous concave functional

$$\alpha(x) = \min_{t\in[0,1]} (-1)^{n-1}x(t) \quad \text{for} \quad x \in P.$$

We now present our main result.

Theorem 3.1. Suppose $(H_1) - (H_2)$ hold. In addition there exist nonnegative numbers a, b, and c such that $0 < a < b \le \min\{\gamma, m_{n-1}/M_{n-1}\}c$ and $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0)$ satisfies the following growth conditions:

(H_3) $(-1)^n f(t, u_{n-1}, \cdots, u_0) < a/M_{n-1}$ for $(t, |u_{n-1}|, |u_{n-2}|, \cdots, |u_0|) \in [0,1]\times \prod_{j=n-1}^{1}[0, \prod_{i=2}^{j+1} M_{n-i}a] \times [0,a]$;

(H_4) $(-1)^n f(t, u_{n-1}, \cdots, u_0) < c/M_{n-1}$ for $(t, |u_{n-1}|, |u_{n-2}|, \cdots, |u_0|) \in [0,1]\times \prod_{j=n-1}^{1}[0, \prod_{i=2}^{j+1} M_{n-i}c] \times [0,c]$;

(H_5) $(-1)^n f(t, u_{n-1}, \cdots, u_0) \ge b/m_{n-1}$ for $(t, |u_{n-1}|, |u_{n-2}|, \cdots, |u_0|) \in [0,1]\times \prod_{j=n-1}^{1} [\prod_{i=2}^{j+1} m_{n-i}b, \prod_{i=2}^{j+1} M_{n-i}b/\gamma] \times [b, b/\gamma]$.

Then the boundary value problem (6) has at least three positive solutions x_1, x_2 and x_3 such that

$$\|x_1^{(2(n-1))}\| < a, \quad b < \min_{0\le t\le 1}(-1)^{n-1}x_2^{(2(n-1))}(t),$$

and

$$\|x_3^{(2(n-1))}\| > a \quad \text{with} \quad \min_{0\le t\le 1}(-1)^{n-1}x_3^{(2(n-1))}(t) < b.$$

Proof. At first we show that $A : P \to P$. Let $x \in P$ then $(-1)^{n-1}Ax(t) \ge 0$. Moreover,

$$(-1)^{n-1}(Ax)''(t) = (-1)^{n-1}f(t, A_{n-1}x(t), A_{n-2}x(t), \cdots, A_1x(t), x(t)) < 0.$$

By lemma 3.2, $\min_{t\in[0,1]}(-1)^{n-1}Ax(t) \ge \gamma\|Ax\|$, this implies that $A : P \to P$. Also, it is easy to see that the operator A is completely continuous.

Choose $x \in \overline{P}_c$, then $\|x\| \le c$. Note that

$$\|A_jx\| = \max_{t\in[0,1]}\left|\int_0^1 G_j(t,s)x(s)ds\right| \le \prod_{i=2}^{j+1}M_{n-i}\|x\| \le \prod_{i=2}^{j+1}M_{n-i}c.$$

Thus, according to assumption (H_4) we have

$$\begin{aligned}
\|Ax\| &= \max_{0\le t\le 1}|Ax(t)| \\
&= \max_{0\le t\le 1}\left\{\int_0^1 |g_{n-1}(t,s)f(s, A_{n-1}x(s), A_{n-2}x(s), \cdots, A_1x(s), x(s))|ds\right\} \\
&\le \frac{c}{M_{n-1}}\max_{0\le t\le 1}\left\{\int_0^1 |g_{n-1}(t,s)|ds\right\} \\
&= c.
\end{aligned}$$

Therefore, $A : \overline{P}_c \to \overline{P}_c$.

In a completely analogous argument, assumption (H_3) implies that Condition (C2) of the Leggett-Williams Fixed Point Theorem is satisfied.

We now show that condition (C1) is satisfied. Note that for $0 \le t \le 1$.

$$x(t) = (-1)^{n-1}\frac{b}{\gamma} \in P\left(\alpha, b, \frac{b}{\gamma}\right) \quad \text{and} \quad \alpha(x) = \frac{b}{\gamma} > b.$$

Thus,

$$\{x \in P(\alpha, b, \frac{b}{\gamma})|\ \alpha(x) > b\} \ne \emptyset.$$

Also, if $x \in P(\alpha, b, \frac{b}{\gamma})$, then $\alpha(x) = \min\limits_{t\in[0,1]}(-1)^{n-1}x(t) \geq b$ for each $0 \leq t \leq 1$, so $(-1)^{n-1}x(t) \geq b$, $0 \leq t \leq 1$, this implies

$$\begin{aligned}(-1)^{n-2}A_1x(t) &= \int_0^1 -G_1(t,s)(-1)^{n-1}x(s)ds \\ &\geq b\int_0^1 |G_1(t,s)|ds \geq bm_{n-2}.\end{aligned}$$

Inductively, we have

$$(-1)^{n-1-j}A_jx(t) \geq \prod_{i=2}^{j+1} m_{n-j}b, \quad 0 \leq t \leq 1,\ 1 \leq j \leq n-1.$$

and it is easy to see that

$$|A_jx(t)| \leq \prod_{i=2}^{j+1} M_{n-j}\frac{b}{\gamma}.$$

Applying condition (H_5) we get

$$(-1)^n f(t, A_{n-1}x(t), A_{n-2}x(t), \cdots, A_1x(t), x(t)) \geq \frac{b}{m_{n-1}}, \quad 0 \leq t \leq 1.$$

So,

$$\begin{aligned}\alpha(Ax) &= \min_{0\leq t\leq 1}(-1)^{n-1}Ax(t) \\ &= \min_{0\leq t\leq 1}\left\{\int_0^1 -g_{n-1}(t,s)(-1)^n f(s, A_{n-1}x(s), A_{n-2}x(s), \cdots, A_1x(s), x(s))ds\right\} \\ &\geq \frac{b}{m_{n-1}}\min_{0\leq t\leq 1}\int_0^1 |g_{n-1}(t,s)|ds \\ &= b.\end{aligned}$$

Therefore, condition (C1) is satisfied.

Finally, we show that condition (C3) is also satisfied. That is, we show that if $x \in P(\alpha, b, c)$ and $\|Ax\| > d = b/\gamma$, then $\alpha(Ax) > b$. This follows since $A : P \to P$, then

$$\alpha(Ax) = \min_{0\leq t\leq 1}(-1)^{n-1}Ax(t) \geq \gamma\|Ax\| > b.$$

Therefore, condition (C3) is also satisfied. So we complete the proof.

4. Example

In this section, we present an example to demonstrate the application of Theorem 3.1. Consider the boundary value problem

$$\begin{cases} x^{(4)}(t) = f(t, x(t), x''(t)), \quad 0 \le t \le 1, \\ x'(0) = \frac{1}{2}x'\left(\frac{1}{2}\right), \quad x(1) = \frac{1}{2}x\left(\frac{1}{2}\right), \\ x^{(3)}(0) = \frac{1}{4}x^{(3)}\left(\frac{1}{2}\right), \quad x''(1) = \frac{3}{4}x''\left(\frac{1}{2}\right). \end{cases} \tag{12}$$

where

$$f(t,x,y) = \begin{cases} \frac{1}{1000}\sin t + 4x + \frac{1}{1000}y^3, \ x \in (-\infty, 1/32], \\ \frac{1}{1000}\sin t - \frac{15584}{25}\left(x - \frac{3}{32}\right)^2 + \frac{64}{25} + \frac{1}{1000}y^3, \ x \in [1/32, 3/32], \\ \frac{1}{1000}\sin t + \frac{32768}{16875}\left(x - \frac{13}{32}\right)^2 + \frac{64}{27} + \frac{1}{1000}y^3, \ x \in [3/32, 13/32], \\ \frac{1}{1000}\sin t + \frac{64}{27} + \frac{1}{1000}y^3, \ x \in [13/32, +\infty). \end{cases}$$

By Lemma 3.3, we have

$$|g_0(t,s)| = \begin{cases} \frac{3}{4} - \frac{1}{2}t, \quad 0 \le t \le 1, \ 0 \le s \le \frac{1}{2}, \ s \le t; \\ \frac{3}{4} - \frac{1}{4}t - \frac{1}{4}s, \quad 0 \le t \le 1, \ 0 \le s \le \frac{1}{2}, \ t \le s; \\ \frac{1}{2} - \frac{1}{4}t - \frac{1}{4}s, \quad 0 \le t \le 1, \ \frac{1}{2} \le s \le 1, \ s \le t; \\ \frac{1}{2} - \frac{1}{2}s, \quad 0 \le t \le 1, \ \frac{1}{2} \le s \le 1, \ t \le s. \end{cases}$$

$$|g_1(t,s)| = \begin{cases} \frac{5}{8} - \frac{1}{4}t, \quad 0 \le t \le 1, \ 0 \le s \le \frac{1}{2}, \ s \le t; \\ \frac{5}{8} - \frac{1}{16}t - \frac{3}{16}s, \quad 0 \le t \le 1, \ 0 \le s \le \frac{1}{2}, \ t \le s; \\ \frac{3}{4} - \frac{3}{16}t - \frac{9}{16}s, \quad 0 \le t \le 1, \ \frac{1}{2} \le s \le 1, \ s \le t; \\ \frac{3}{4} - \frac{3}{4}s, \quad 0 \le t \le 1, \ \frac{1}{2} \le s \le 1, \ t \le s. \end{cases}$$

We first consider the condition $i = 0$.

1) For $0 \le t \le \frac{1}{2}$, we have

$$\begin{aligned} \int_0^1 |g_0(t,s)|ds &= \int_0^t |g_0(t,s)|ds + \int_t^{\frac{1}{2}} |g_0(t,s)|ds + \int_{\frac{1}{2}}^1 |g_0(t,s)|ds \\ &= \int_0^t \left(\frac{3}{4} - \frac{1}{2}t\right) ds + \int_t^{\frac{1}{2}} \left(\frac{3}{4} - \frac{1}{4}t - \frac{1}{4}s\right) ds + \int_{\frac{1}{2}}^1 \left(\frac{1}{2} - \frac{1}{2}s\right) ds \\ &= \frac{13}{32} - \frac{1}{8}t - \frac{1}{8}t^2. \end{aligned}$$

2) For $\frac{1}{2} \le t \le 1$, we have

$$\begin{aligned} \int_0^1 |g_0(t,s)|ds &= \int_0^{\frac{1}{2}} |g_0(t,s)|ds + \int_{\frac{1}{2}}^t |g_0(t,s)|ds + \int_t^1 |g_0(t,s)|ds \\ &= \int_0^{\frac{1}{2}} \left(\frac{3}{4} - \frac{1}{2}t\right) ds + \int_{\frac{1}{2}}^t \left(\frac{1}{2} - \frac{1}{4}t - \frac{1}{4}s\right) ds + \int_t^1 \left(\frac{1}{2} - \frac{1}{2}s\right) ds \\ &= \frac{13}{32} - \frac{1}{8}t - \frac{1}{8}t^2. \end{aligned}$$

So,

$$M_0 = \max_{0\le t\le 1}\int_0^1 |g_0(t,s)|ds = \frac{13}{32}, \qquad m_0 = \min_{0\le t\le 1}\int_0^1 |g_0(t,s)|ds = \frac{5}{32}.$$

Next, we consider the condition $i = 1$.

3) For $0 \le t \le \frac{1}{2}$, we have

$$\begin{aligned}\int_0^1 |g_1(t,s)|ds &= \int_0^t |g_1(t,s)|ds + \int_t^{\frac{1}{2}} |g_1(t,s)|ds + \int_{\frac{1}{2}}^1 |g_1(t,s)|ds \\ &= \int_0^t \left(\frac{5}{8} - \frac{1}{4}t\right) ds + \int_t^{\frac{1}{2}} \left(\frac{5}{8} - \frac{1}{16}t - \frac{3}{16}s\right) ds + \int_{\frac{1}{2}}^1 \left(\frac{3}{4} - \frac{3}{4}s\right) ds \\ &= \frac{49}{128} - \frac{1}{32}t - \frac{3}{32}t^2.\end{aligned}$$

4) For $\frac{1}{2} \le t \le 1$, we have

$$\begin{aligned}\int_0^1 |g_1(t,s)|ds &= \int_0^{\frac{1}{2}} |g_1(t,s)|ds + \int_{\frac{1}{2}}^t |g_1(t,s)|ds + \int_t^1 |g_1(t,s)|ds \\ &= \int_0^{\frac{1}{2}} \left(\frac{5}{8} - \frac{1}{4}t\right) ds + \int_{\frac{1}{2}}^t \left(\frac{3}{4} - \frac{3}{16}t - \frac{9}{16}s\right) ds + \int_t^1 \left(\frac{3}{4} - \frac{3}{4}s\right) ds \\ &= \frac{49}{128} - \frac{1}{32}t - \frac{3}{32}t^2.\end{aligned}$$

So,

$$M_1 = \max_{0\le t\le 1}\int_0^1 |g_1(t,s)|ds = \frac{49}{128}, \qquad m_1 = \min_{0\le t\le 1}\int_0^1 |g_1(t,s)|ds = \frac{33}{128}.$$

As $\gamma = \frac{3}{5}$, $m_1/M_1 = \frac{33}{49}$, so we can let $a = \frac{1}{13}, b = \frac{3}{5}, c = 1$, then

$$\begin{aligned} f(t,x,y) &< a/M_1 = \frac{128}{637} \quad \text{for} \quad (t,|x|,|y|) \in [0,1]\times[0,1/32]\times[0,1/13], \\ f(t,x,y) &< c/M_1 = \frac{128}{49} \quad \text{for} \quad (t,|x|,|y|) \in [0,1]\times[0,13/32]\times[0,1], \\ f(t,x,y) &\ge b/m_1 = \frac{128}{55} \quad \text{for} \quad (t,|x|,|y|) \in [0,1]\times[3/32,13/32]\times[3/5,1]. \end{aligned}$$

By Theorem 3.1, problem (12) has at least three positive solutions.

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