Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **42**, 1-9; **http://www.math.u-szeged.hu/ejqtde/**

Existence of $\Psi-$bounded solutions for nonhomogeneous Lyapunov matrix differential equations on R

Aurel Diamandescu
University of Craiova

Abstract

In this paper, we give a necessary and sufficient condition for the existence of at least one $\Psi-$ bounded solution of a linear nonhomogeneous Lyapunov matrix differential equation on $\mathbb{R}$. In addition, we give a result in connection with the asymptotic behavior of the $\Psi-$ bounded solution of this equation.

Key Words: $\Psi-$ bounded solution, Lyapunov matrix differential equation.
2000 Mathematics Subject Classification: 34D05, 34C11.

1. Introduction.

This paper deals with the linear nonhomogeneous Lyapunov matrix differential equation

$$X' = A(t)X + XB(t) + F(t) \tag{1}$$

where A, B and F are continuous n×n matrix-valued functions on $\mathbb{R}$.

Recently, the existence of at least one $\Psi-$ bounded solution on $\mathbb{R}$ of equation (1) for every Lebesgue $\Psi-$ integrable matrix function F on $\mathbb{R}$ has been studied in [8].

Our aim is to determine necessary and sufficient condition for the existence of at least one $\Psi-$ bounded solution on $\mathbb{R}$ of equation (1), for every continuous and $\Psi-$ bounded matrix function F on $\mathbb{R}$.

Here, Ψ is a matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

In order to be able to solve our problem, we use a bounded input - bounded output approach which has been used in the past few years (see [2], [10], [11] and [12]).

The approach used in our paper is essentially based on a trichotomic type decomposition of the space $\mathbb{R}^n$ at the initial moment (which has been used in the past few years both in the finite-dimensional spaces (see [4], [5] and [8]) and in general case of Banach spaces (see [6], [7] and [13])) and the technique of Kronecker product of matrices (which has been successfully applied in various fields of matrix theory).

Thus, we obtain results which extend the recent results regarding the boundedness of solutions of the equation (1) (according to [4]).

2. Preliminaries.

In this section we present some basic definitions and results which are useful later on.

Let $\mathbb{R}^n$ be the Euclidean n - space. For $x = (x_1, x_2, x_3, ...,x_n)^T \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, ..., |x_n|\}$ be the norm of x (^{T} denotes transpose).

Let $\mathbb{M}_{m\times n}$ be the linear space of all m×n real valued matrices.

For a n×n real matrix $A = (a_{ij})$, we define the norm $|A|$ by $|A| = \sup_{\|x\|\leq 1} \|Ax\|$.

It is well-known that $|A| = \max_{1\leq i\leq n}\{\sum_{j=1}^{n} |a_{ij}|\}$.

Definition 1. ([1]) Let $A = (a_{ij}) \in \mathbb{M}_{m\times n}$ and $B = (b_{ij}) \in \mathbb{M}_{p\times q}$. The Kronecker product of A and B, written A⊗B, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Obviously, $A\otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 1. The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions are defined:

1). $A\otimes(B\otimes C) = (A\otimes B)\otimes C$;
2). $(A\otimes B)^T = A^T\otimes B^T$;
3). $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$;
4). $(A\otimes B)\cdot(C\otimes D) = AC\otimes BD$;
5). $A\otimes(B + C) = A\otimes B + A\otimes C$;
6). $(A + B)\otimes C = A\otimes C + B\otimes C$;
7). $I_p\otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8). $(A(t)\otimes B(t))' = A'(t)\otimes B(t) + A(t)\otimes B'(t)$; (here, $'$ denotes derivative $\frac{d}{dt}$).

Proof. See in [1].

Definition 2. The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by

$$\mathcal{V}ec(A) = (a_{11},a_{21},\cdots,a_{m1},a_{12},a_{22},\cdots,a_{m2},\cdots,a_{1n},a_{2n},\cdots,a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

Lemma 2. The vectorization operator $\mathcal{V}ec : \mathbb{M}_{n\times n} \longrightarrow \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ and $\mathcal{V}ec^{-1}$ are continuous operators.

Proof. See in [3].

Remark. Obviously, if F is a continuous matrix function on $\mathbb{R}$, then $f = \mathcal{V}ec(F)$ is a continuous vector function on $\mathbb{R}$ and vice-versa.

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations (see [9]):

Lemma 3. If A, B, $M \in \mathbb{M}_{n\times n}$, then
1). $\mathcal{V}ec(AMB) = (B^T \otimes A)\cdot\mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T \otimes I_n)\cdot\mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n \otimes A)\cdot\mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T \otimes A)\cdot\mathcal{V}ec(I_n)$.

Proof. It is a simple exercise.

Let $\Psi_i : \mathbb{R} \longrightarrow (0,\infty)$, i = 1,2,...,n, be continuous functions and

$$\Psi = \text{diag}\,[\Psi_1, \Psi_2, \cdots \Psi_n].$$

Definition 3. ([3]). A function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be $\Psi-$ bounded on $\mathbb{R}$ if Ψf is bounded on $\mathbb{R}$ (i.e. $\sup\limits_{t\in\mathbb{R}} \| \Psi(t)f(t)\| < +\infty$).

Definition 4. ([3]). A matrix function $M : \mathbb{R} \longrightarrow \mathbb{M}_{n\times n}$ is said to be $\Psi-$ bounded on $\mathbb{R}$ if the matrix function ΨM is bounded on $\mathbb{R}$ (i.e. $\sup\limits_{t\in\mathbb{R}} | \Psi(t)M(t)| < +\infty$).

We shall assume that A, B and F are continuous n×n - matrices on $\mathbb{R}$.

By a solution of (1), we mean a continuous differentiable n×n - matrix function X satisfying the equation (1) for all $t \in \mathbb{R}$.

The following lemmas play a vital role in the proofs of the main results.

Lemma 4. ([3]). The matrix function X(t) is a solution of (1) on the interval $J \subset \mathbb{R}$ if and only if the vector valued function $x(t) = \mathcal{V}ecX(t)$ is a solution of the differential system

$$x' = (I_n \otimes A(t) + B^T(t) \otimes I_n)x + f(t), \tag{2}$$

where $f(t) = \mathcal{V}ecF(t)$, on the same interval J.

Definition 5. The above system (2) is called 'corresponding Kronecker product system associated with (1)'.

Lemma 5. ([3]). The matrix function M(t) is $\Psi-$ bounded on $\mathbb{R}$ if and only if the vector function $\mathcal{V}ecM(t)$ is $I_n \otimes \Psi$ − bounded on $\mathbb{R}$.

Proof. From the proof of Lemma 2, it results that

$$\frac{1}{n} | A | \ \leq \ \| \mathcal{V}ecA \|_{\mathbb{R}^{n^2}} \ \leq \ | A |,$$

for every $A \in \mathbb{M}_{n\times n}$.

Setting $A = \Psi(t)M(t)$, $t \in \mathbb{R}$ and using Lemma 3, we have the inequality

$$\frac{1}{n} \mid \Psi(t)M(t) \mid \ \leq \ \| \, (I_n \otimes \Psi(t)) \cdot \mathcal{V}ecM(t) \, \|_{\mathbb{R}^{n^2}} \ \leq \ \mid \Psi(t)M(t) \mid , \, t \ \in \mathbb{R} \quad (3)$$

for all matrix function M(t).

Now, the Lemma follows immediately.

Lemma 6. ([3]). Let X(t) and Y(t) be fundamental matrices for the systems

$$x'(t) = A(t)x(t) \quad (4)$$

and

$$y'(t) = y(t)B(t) \quad (5)$$

respectively.

Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the system

$$z'(t) = (I_n \otimes A(t) + B^T(t) \otimes I_n)z(t). \quad (6)$$

If, in addition, $X(0) = I_n$ and $Y(0) = I_n$, then $Z(0) = I_{n^2}$.

Now, let Z(t) be the above fundamental matrix for the system (6) with $Z(0) = I_{n^2}$.

Let the vector space $\mathbb{R}^{n^2}$ represented as a direct sum of three subspaces X_-, X_0 and X_+ defined as follows: a solution z of the sistem (6) is $I_n \otimes \Psi-$ bounded on $\mathbb{R}$ if and only if $z(0) \in X_0$; let $\widetilde{X}$ denote the subspace of $\mathbb{R}^{n^2}$ consisting of all vectors which are values of $I_n \otimes \Psi-$ bounded solutions of (6) on $\mathbb{R}_+$ for $t = 0$; let X_- denote an arbitrary fixed subspace of $\widetilde{X}$ supplementary to $X_0 : \widetilde{X} = X_- \oplus X_0$; finally, the subspace X_+ is an arbitrary fixed subspace of $\mathbb{R}^{n^2}$, supplementary to $X_- \oplus X_0$. Let P_-, P_0 and P_+ denote the corresponding projections of $\mathbb{R}^{n^2}$ onto X_-, X_0 and X_+ respectively.

3. The main results.

The main results of this paper are the following.

Theorem 1. If A and B are continuous n×n real matrices on $\mathbb{R}$ then, the equation (1) has at least one $\Psi-$ bounded solution on $\mathbb{R}$ for every continuous and $\Psi-$ bounded matrix function $F : \mathbb{R} \longrightarrow \mathbb{M}_{n\times n}$ if and only if there exists a positive constant K such that

$$\int_{-\infty}^{t} \mid \left(Y^T(t) \otimes (\Psi(t)X(t))\right)P_- \left((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))\right) \mid ds +$$

$$+ \int_{t}^{0} \mid \left(Y^T(t) \otimes (\Psi(t)X(t))\right) (P_0 + P_+) \left((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))\right) \mid ds +$$

$$+ \int_{0}^{\infty} \mid \left(Y^T(t) \otimes (\Psi(t)X(t))\right)P_+ \left((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))\right) \mid ds \leq K, t < 0;$$

(7)

$$\int_{-\infty}^{0} \mid \left(\mathrm{Y^T(t)} \otimes (\Psi(\mathrm{t})\mathrm{X(t)})\right)\mathrm{P_-}\left((\mathrm{Y^T(s)})^{-1} \otimes (\mathrm{X^{-1}(s)}\Psi^{-1}(\mathrm{s}))\right) \mid ds+$$

$$+\int_{0}^{t} \mid \left(\mathrm{Y^T(t)} \otimes (\Psi(\mathrm{t})\mathrm{X(t)})\right)(\mathrm{P_0}+\mathrm{P_-})\left((\mathrm{Y^T(s)})^{-1} \otimes (\mathrm{X^{-1}(s)}\Psi^{-1}(\mathrm{s}))\right) \mid ds+$$

$$+\int_{t}^{\infty} \mid \left(\mathrm{Y^T(t)} \otimes (\Psi(\mathrm{t})\mathrm{X(t)})\right)\mathrm{P_+}\left((\mathrm{Y^T(s)})^{-1} \otimes (\mathrm{X^{-1}(s)}\Psi^{-1}(\mathrm{s}))\right) \mid ds \leq K, t \geq 0.$$

Proof. First, we prove the 'if' part.

Suppose that (7) holds for some $K > 0$.

Let $\mathrm{F} : \mathbb{R} \longrightarrow \mathbb{M}_{\mathrm{n}\times\mathrm{n}}$ be a continuous and $\Psi-$ bounded matrix function on $\mathbb{R}$. From Lemma 5, it follows that the vector function $\mathrm{f(t)} = \mathcal{V}\mathrm{ecF(t)}$ is $\mathrm{I_n} \otimes \Psi -$ bounded on $\mathbb{R}$. From Theorem 1.1 ([4]), it follows that the differential system (2) has at least one $\mathrm{I_n} \otimes \Psi -$ bounded solution on $\mathbb{R}$ (because condition (7) for the system (1) becomes condition (1.3) in Theorem 1.1 ([4]) for system (6)).

Let z(t) be this solution.

From Lemma 4 and Lemma 5, it follows that the matrix function $\mathrm{Z(t)} = \mathcal{V}\mathrm{ec}^{-1}\mathrm{z(t)}$ is a $\Psi-$ bounded solution of the equation (1) on $\mathbb{R}$ (because $\mathrm{F(t)} = \mathcal{V}\mathrm{ec}^{-1}\mathrm{f(t)}$).

Thus, the linear nonhomogeneous Lyapunov matrix differential equation (1) has at least one $\Psi -$ bounded solution on $\mathbb{R}$ for every continuous and $\Psi-$ bounded matrix function F on $\mathbb{R}$.

Now, we prove the 'only if' part.

Suppose that the equation (1) has at least one $\Psi -$ bounded solution on $\mathbb{R}$ for every continuous and $\Psi-$ bounded matrix function $\mathrm{F} : \mathbb{R} \longrightarrow \mathbb{M}_{\mathrm{n}\times\mathrm{n}}$ on $\mathbb{R}$.

Let $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}^{n^2}$ be a continuous and $\mathrm{I_n} \otimes \Psi-$ bounded function on $\mathbb{R}$. From Lemma 5, it follows that the matrix function $\mathrm{F(t)} = \mathcal{V}\mathrm{ec}^{-1}\mathrm{f(t)}$ is continuous and $\Psi-$ bounded on $\mathbb{R}$. From the hypothesis, the equation

$$\frac{\mathrm{dZ}}{\mathrm{dt}} = \mathrm{A(t)Z} + \mathrm{ZB(t)} + \mathcal{V}\mathrm{ec}^{-1}\mathrm{f(t)}$$

has at least one $\Psi -$ bounded solution Z(t) on $\mathbb{R}$.

From Lemma 4 and Lemma 5, it follows that the vector valued function $\mathrm{z(t)} = \mathcal{V}\mathrm{ecZ(t)}$ is a $\mathrm{I_n} \otimes \Psi-$ bounded solution on $\mathbb{R}$ of the differential system

$$\frac{\mathrm{dz}}{\mathrm{dt}} = (\mathrm{I_n}\otimes\mathrm{A(t)} + \mathrm{B^T(t)}\otimes\mathrm{I_n})\mathrm{z} + \mathrm{f(t)}.$$

Thus, this system has at least one $\mathrm{I_n} \otimes \Psi-$ bounded solution on $\mathbb{R}$ for every continuous and $\mathrm{I_n} \otimes \Psi-$ bounded function $\mathrm{f} : \mathbb{R} \longrightarrow \mathbb{R}^{n^2}$.

Theorem 1.1 ([4]) tell us that there exists a positive constant K such that the fundamental matrix U(t) of the differential system

$$\frac{\mathrm{dz}}{\mathrm{dt}} = (\mathrm{I_n}\otimes\mathrm{A(t)} + \mathrm{B^T(t)}\otimes\mathrm{I_n})\mathrm{z}$$

satisfies the condition (1.3) of Theorem 1.1 ([4]).

Lemma 6 tell us that $U(t) = Y^T(t) \otimes X(t)$. After computation, it follows that (7) holds.

The proof is now complete.

Remark. Theorem 1 generalizes Theorem 1.1 ([4]).

As a particular case, we have the following result:

Corollary 1. If A and B are continuous n×n real matrices on $\mathbb{R}$ and the equation

$$Z' = A(t)Z + ZB(t) \tag{8}$$

has no nontrivial $\Psi-$ bounded solution on $\mathbb{R}$, then, the equation (1) has a unique $\Psi-$ bounded solution on $\mathbb{R}$ for every continuous and $\Psi-$ bounded matrix function $F : \mathbb{R} \longrightarrow \mathbb{M}_{n\times n}$ if and only if there exists a positive constant K such that for $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} \mid \left(Y^T(t) \otimes (\Psi(t)X(t))\right) P_- \left((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))\right) \mid ds + \tag{9}$$

$$+ \int_{t}^{\infty} \mid \left(Y^T(t) \otimes (\Psi(t)X(t))\right) P_+ \left((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))\right) \mid ds \leq K.$$

Proof. Indeed, in this case, $P_0 = O_n$.

The next result shows us that the asymptotic behavior of $\Psi-$ bounded solutions of (1) is determined completely by the asymptotic behavior of F(t) as $t \longrightarrow \pm\infty$.

Theorem 3. Suppose that

(1). The fundamental matrices X and Y for the systems (4) and (5) respectively satisfy:

(a). the condition (7) for some $K > 0$;

(b). the condition $\lim_{t\to\pm\infty} |Y^T(t) \otimes (\Psi(t)X(t))P_0 \mid = 0$;

(2). The continuous and $\Psi-$ bounded matrix function $F : \mathbb{R} \longrightarrow M_{n\times n}$ is such that

$$\lim_{t\to\pm\infty} \Psi(t)F(t) = O_n.$$

Then, every $\Psi-$ bounded solution Z on $\mathbb{R}$ of the equation (1) satisfies the condition

$$\lim_{t\to\pm\infty} \Psi(t)Z(t) = O_n.$$

Proof. Let Z(t) be a $\Psi-$ bounded solution on $\mathbb{R}$ of the equation (1). From Lemma 4 and Lemma 5, it follows that the vector valued function $z(t) = \mathcal{V}ecZ(t)$ is a $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}$ of the differential system

$$\frac{dz}{dt} = (I_n \otimes A(t) + B^T(t) \otimes I_n)z + f(t),$$

where $f(t) = \mathcal{V}ecF(t)$.

Also, from Lemma 5, the function f is continuous and $I_n \otimes \Psi-$ bounded on $\mathbb{R}$.

From Theorem 1.3 ([4]), it follows that

$$\lim_{t \to \pm\infty} \|(I_n \otimes \Psi(t))z(t)\|_{\mathbb{R}^{n^2}} = 0.$$

Now, from the inequality (3), we have

$$|\Psi(t)Z(t)| \leq n\|(I_n \otimes \Psi(t))z(t)\|_{\mathbb{R}^{n^2}},\ t \in \mathbb{R}$$

and then,

$$\lim_{t \to \pm\infty} \Psi(t)Z(t) = O_n.$$

The proof is now complete.

Remark. Theorem generalizes Theorem 1.3 ([4]).

As a particular case, we have

Corollary 2. Suppose that

(1). The homogeneous equation (8) has no nontrivial $\Psi-$ bounded solution on $\mathbb{R}$;

(2). The fundamental matrices X and Y for the systems (4) and (5) respectively satisfy the condition (9) for some $K > 0$;

(3). The continuous and $\Psi-$ bounded matrix function $F : \mathbb{R} \longrightarrow M_{n \times n}$ is such that

$$\lim_{t \to \pm\infty} \Psi(t)F(t) = O_n.$$

Then, the equation (1) has a unique solution Z on $\mathbb{R}$ such that

$$\lim_{t \to \pm\infty} \Psi(t)Z(t) = O_n.$$

Proof. Indeed, in this case, we have $P_0 = O_n$ in Theorem 3.

Remark. If the function F does not fulfill the condition 2 of theorem 3, then, the $\Psi-$ bounded solution Z(t) of equation (1) may be such that $\lim_{t \to \pm\infty} \Psi(t)Z(t) \neq O_n$. This is shown in the next simple example.

Example. Consider the linear equation (1) with

$$A(t) = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix},\ B(t) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

The fundamental matrices for the systems (4) and (5) are

$$\mathrm{X(t)} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{pmatrix}, \mathrm{Y(t)} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

respectively.

Consider

$$\Psi(\mathrm{t}) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

It is easy to see that the conditions of theorem 3 are satisfied with

$$\mathrm{P_0 = O_4},\ \mathrm{P_-} = \text{diag } [1,0,1,0],\ \mathrm{P_+} = \text{diag } [0,1,0,1] \text{ and } \mathrm{K} = \tfrac{5}{12}.$$

In addition, the matrix function F is $\Psi-$ bounded on $\mathbb{R}$.

On the other hand, the solutions of the equation (1) are

$$\mathrm{Z(t)} = \begin{pmatrix} c_1e^{-3t} + \frac{1}{6}e^{3t} & c_2e^{-3t} \\ c_3e^{2t} & c_4e^{2t} - \frac{1}{4}e^{-2t} \end{pmatrix},$$

where c_1, c_2, c_3, $c_4 \in \mathbb{R}$.

There exists a unique $\Psi-$ bounded solution on $\mathbb{R}$, namely

$$\mathrm{Z(t)} = \begin{pmatrix} \frac{1}{6}e^{3t} & 0 \\ 0 & -\frac{1}{4}e^{-2t} \end{pmatrix},$$

but $\lim\limits_{t\to\pm\infty} |\ \Psi(\mathrm{t})\mathrm{Z(t)}| = \frac{1}{4}$.

Note that the asymptotic properties of the components of the solutions are not the same. On the other hand, we see that the asymptotic properties of the components of the solutions are the same, via matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.

This example shows that the hypothesis (2) of theorem 3 is an essential condition for the validity of the theorem.

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Aurel Diamandescu,
University of Craiova, Department of Applied Mathematics,
13, "Al. I. Cuza" st., 200585 Craiova, Romania
E - mail address: adiamandescu@central.ucv.ro

(Received April 2, 2010)