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# An existence result for a quasilinear degenerate problem in $\mathbb{R}^N$

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**Abstract.** In this paper we study an existence result of the quasilinear problem $-\mathrm{div}[\phi'(|\nabla u|^2)\nabla u] + a(x)|u|^{\alpha-2}u = |u|^{\gamma-2}u + |u|^{\beta-2}u$ in $\mathbb{R}^N (N \geq 3)$, where $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p/2}$ for large $t$, $a$ is a positive potential, $1 < p < q < N$, $1 < \alpha \leq p^*q'/p'$ and $\max\{\alpha, q\} < \gamma < \beta < p^* = pN/(N-p)$, with $p'$ and $q'$ the conjugate exponents of $p$, respectively $q$. Our main result is the proof of the existence of a weak solution, based on the mountain pass theorem.

**Keywords:** quasilinear degenerate problem, weak solutions, nonhomogeneous operator, mountain pass theorem.

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## 1 Introduction and preliminary results

In this paper we are interested for a new type of operator, introduced by Azzollini in some recent papers [4, 5] and by N. Chorfi and V. Rădulescu in [8]. Their studies are based on the nonhomogeneous operators of the type

$$\mathrm{div}[\phi'(|\nabla u|^2)\nabla u],$$

where $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ has a different growth near zero and at infinity. Such a type of behavior occurs when $\phi(t) = 2[(1+t)^{1/2} - 1]$, which corresponds to the operator

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

known as the prescribed mean curvature operator or the capillary surface operator.

More precisely, $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p/2}$ for large $t$, where $1 < p < q < N$. Such behavior occurs, for example, when

$$\phi(t) = \frac{2}{p}\left[\left(1 + t^{q/2}\right)^{p/q} - 1\right]$$

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which generates the differential operator

$$\operatorname{div}[(1+|\nabla u|^q)^{(p-q)/q}\,|\nabla u|^{q-2}\,\nabla u].$$

In [8] N. Chorfi and V. Rădulescu approached the quasilinear Schrödinger equation

$$-\operatorname{div}[\phi'(|\nabla u|^2)\nabla u] + a(x)|u|^{\alpha-2}u = f(x,u) \quad \text{in } \mathbb{R}^N \quad (N \geq 3). \tag{1.1}$$

Their interest was studying the problem

$$-\Delta u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $a$ is a positive potential and $f$ has a subcritical growth, problem studied by P. Rabinowitz in [15], in the new abstract setting introduced by Azzollini in [4,5] (see also [17]).

The importance of the Schrödinger type equation is obvious. This equation is fundamental for quantum mechanics, which together with general relativity represents the most useful current theories about the physical universe.

In 1927, elaborating research of many physicists, Erwin Schrödinger wrote a differential equation for any quantum waves function, namely

$$i\hbar\frac{\partial}{\partial u}\Psi = \hat{H}\Psi,$$

where $\hbar$ is the Planck constant divided by $2\Pi$, $\Psi$ is the wave function, $i$ is the square root of minus one and $\hat{H}$ is the Hamiltonian operator.

The classical wave equation defines waves in space and the solution is a numerical function depending on space and time. The same happens with the Schrödinger equation, but in this case the values of the wave function $\Psi$ are also complex, not just real.

The applications of this equations are numerous, varying from Bose–Einstein condensates and nonlinear optics, propagation of the electric field in optical fibers, stability of Stokes waves in water to the behavior of deep water waves and freak waves in the ocean. For more applications to nonlinear equations with variable or constant exponents we refer [1,7,9,16,18–20].

In this paper we are interested to study problem (1.1) in the particular case

$$f(x,u) = |u|^{\gamma-2}u + |u|^{\beta-2}u.$$

More precisely, we consider the quasilinear degenerate problem

$$-\operatorname{div}[\phi'(|\nabla u|^2)\nabla u] + a(x)|u|^{\alpha-2}u = |u|^{\gamma-2}u + |u|^{\beta-2}u \quad \text{in } \mathbb{R}^N \quad (N \geq 3), \tag{1.2}$$

where $a$ is a positive potential, $1 < p < q < N$, $1 < \alpha \leq p^*q'/p'$ and $\max\{\alpha, q\} < \gamma < \beta < p^* = pN/(N-p)$, with $p'$ and $q'$ the conjugate exponents, respectively, of $p$ and $q$.

Our purpose is to prove, by means of the mountain pass theorem (see [12–14]), that problem (1.2) admits at last one weak solution.

Now, we define the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{L^p+L^q} := \inf\left\{\|v\|_p + \|w\|_q\,;\ v \in L^p(\mathbb{R}^N),\ w \in L^q(\mathbb{R}^N),\ u = v + w\right\}. \tag{1.3}$$

We set $\|u\|_{p,q} = \|u\|_{L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)}$.

The property that $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ are Orlicz spaces, as well as others properties of these spaces, has been proved by M. Badiale, L. Pisani and S. Rolando in [6].

In order to use them throughout this paper, we state the following result that is also found in [5].

**Proposition 1.1.** *Let $\Omega \in \mathbb{R}^N$, $u \in L^p(\Omega) + L^q(\Omega)$. We have:*

*(i) if $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^p(\Omega')$;*

*(ii) if $\Omega' \subset \Omega$ is such that $u \in L^\infty(\Omega')$, then $u \in L^q(\Omega')$;*

*(iii) $|[|u(x) > 1|]| < +\infty$;*

*(iv) $u \in L^p([|u(x)| > 1]) \cap L^q([|u(x)| \leq 1])$;*

*(v) the infimum in (1.3) is attained;*

*(vi) $L^p(\Omega) + L^q(\Omega)$ is reflexive and $(L^p(\Omega) + L^q(\Omega))' = L^{p'}(\Omega) \cap L^{q'}(\Omega)$;*

*(vii) $\|u\|_{L^p(\Omega)+L^q(\Omega)} \leq \max\left\{ \|u\|_{L^p([|u(x)|>1])}, \|u\|_{L^q([|u(x)|\leq 1])} \right\}$;*

*(viii) if $B \in \Omega$, then $\|u\|_{L^p(\Omega)+L^q(\Omega)} \leq \|u\|_{L^p(B)+L^q(B)} + \|u\|_{L^p(\Omega\setminus B)+L^q(\Omega\setminus B)}$.*

Finally, we define the function space

$$X := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| = \|\nabla u\|_{p,q} + \left( \int_{\mathbb{R}^N} a(x)|u|^\alpha dx \right)^{1/\alpha}.$$

We remark that $X$ is continuously embedded in $W$ defined by Azzollini in [5], where

$$W := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

$$\|u\| = \|\nabla u\|_{p,q} + \|u\|_\alpha .$$

In the next section we introduce the main hypotheses and we state the basic results of this paper. The proof of the main result are developed in Section 3.

## 2 The main results

We assume that $a$ in problem (1.2) is a singular potential satisfying the following hypotheses:

$(a_1)$ $a \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$;

$(a_2)$ $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$;

$(a_3)$ $\lim_{x\to 0} a(x) = \lim_{|x|\to\infty} a(x) = +\infty$.

For example, $a(x) = \exp(|x|)/|x|$, for $x \in \mathbb{R}^N \setminus \{0\}$ is such a potential.

In the following, we assume that the function $\phi$, which generates the differential operator in problem (1.2), has the next properties:

$(\phi_1)$ $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$;

$(\phi_2)$ $\phi(0) = 0$;

$(\phi_3)$ there exists $c_1 > 0$ such that

$$\begin{cases} c_1 t^{p(x)/2} \leq \phi(t) & \text{if } t \geq 1, \\ c_1 t^{q(x)/2} \leq \phi(t) & \text{if } 0 \leq t \leq 1; \end{cases}$$

$(\phi_4)$ there exists $c_2 > 0$ such that

$$\begin{cases} \phi(t) \leq c_2 t^{p(x)/2} & \text{if } t \geq 1, \\ \phi(t) \leq c_2 t^{q(x)/2} & \text{if } 0 \leq t \leq 1; \end{cases}$$

$(\phi_5)$ there exists $0 < \mu < 1$ such that $2t\phi'(t) \leq \gamma\mu\phi(t)$ for all $t \geq 0$;

$(\phi_6)$ the mapping $t \longmapsto \phi(t^2)$ is strictly convex.

Our first hypothesis which asserts that $\phi'$ approaches 0 ensures us that problem (1.2) is degenerate and no ellipticity condition is assumed.

We also remark that, because of the presence of the general potential $a$, the solutions of problem (1.2) cannot be reduced to radially symmetric solutions, like in [5]. A frequently used property in [5] by Azzollini was the continuously embedding of the space $W$ in $L^{p^*}(\mathbb{R}^N)$, provided that $1 < p < \min\{q, N\}$, $1 < p^* q'/p'$ and $\alpha \in (1, p^* q'/p')$. By interpolation, for every $r \in [\alpha, p^*]$, $W$ is continuously embedded in $L^r(\mathbb{R}^N)$ .

**Definition 2.1.** A *weak solution* of problem (1.2) is a function $u \in X \setminus \{0\}$ such that

$$\int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)(\nabla u \cdot \nabla v) + a(x)|u|^{\alpha-2}uv - |u|^{\gamma-2}uv - |u|^{\beta-2}uv\right] dx = 0,$$

for any $v \in X$.

We define the energy functional $I : X \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx - \frac{1}{\gamma}\int_{\mathbb{R}^N} |u|^\gamma dx - \frac{1}{\beta}\int_{\mathbb{R}^N} |u|^\beta dx.$$

**Proposition 2.2.** *The functional $I$ is well-defined on $X$ and $I \in C^1(X, \mathbb{R})$, with the Gâteaux derivative given by*

$$I'(u)(v) = \int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)(\nabla u \cdot \nabla v) + a(x)|u|^{\alpha-2}uv - |u|^{\gamma-2}uv - |u|^{\beta-2}uv\right] dx,$$

*for all $u, v \in X$.*

This result can be easily ensured by standard arguments and [3, Lemma 2.2].

We notice that our hypotheses imply that

$$\phi(|\nabla u|^2) \approx \begin{cases} |\nabla u|^p, \text{ if } |\nabla u| >> 1; \\ |\nabla u|^q, \text{ if } |\nabla u| << 1. \end{cases}$$

Now, we give a version of the mountain pass lemma of A. Ambrosetti and P. Rabinowitz [2] (see also [8]).

**Lemma 2.3.** *Let $X$ be a Banach space and assume that $I \in C^1(X, \mathbb{R})$ satisfies the following geometric hypotheses:*

*(a)* $I(0) = 0$

*(b) there exist two positive numbers $a$ and $r$ such that $I(u) \geq a$ for any $u \in X$ with $\|u\| = r$;*

*(c) there exists $e \in X$ with $\|e\| > r$ such that $I(e) < 0$.*

*Let*

$$P := \{p \in C([0,1]; X);\ p(0) = 0,\ p(1) = e\}$$

*and*

$$c := \inf_{p \in P} \sup_{t \in [0,1]} I(p(t)).$$

*Then there exists a sequence $(u_n) \subset X$ such that*

$$\lim_{n\to\infty} I(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \left\|I'(u_n)\right\|_{X^*} = 0.$$

*Moreover, if $I$ satisfies the Palais–Smale condition at the level $c$, then $c$ is a critical value of $I$.*

Finally, the main result of this paper is given by the following theorem.

**Theorem 2.4.** *Suppose that $1 < p < q < N$, $1 < \alpha \leq p^* q'/p'$, $\max\{\alpha, q\} < \gamma < \beta < p^*$, $(a_1)$–$(a_3)$ and $(\phi_1)$–$(\phi_6)$ are satisfied. Then problem (1.2) has at last one weak solution.*

## 3 Proof of Theorem 2.4

It is obvious that $I(0) = 0$.

Now we check (b), the first geometrical condition of the mountain pass lemma, more exactly the existence of a "mountain" around the origin. Let be $u \in X$, $r \in (0,1)$ a fixed point and $\|u\| = r$.

Using $(\phi_3)$, $(iv)$ of Proposition 1.1 and the continuously embeddings of the spaces $L^\gamma(\mathbb{R}^N)$ and $L^\beta(\mathbb{R}^N)$ in $L^{p^*}(\mathbb{R}^N)$ we obtain

$$\begin{aligned}
I(u) &\geq \frac{c_1}{2} \int_{[|\nabla u| \leq 1]} |\nabla u|^q dx + \frac{c_2}{2} \int_{[|\nabla u| > 1]} |\nabla u|^p dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} a(x)|u|^\alpha dx \\
&\quad - \frac{1}{\gamma} \int_{\mathbb{R}^N} |u|^\gamma dx - \frac{1}{\beta} \int_{\mathbb{R}^N} |u|^\beta dx \\
&\geq C \max\left\{ \int_{[|\nabla u| \leq 1]} |\nabla u|^q dx, \int_{[|\nabla u| > 1]} |\nabla u|^p dx \right\} + \frac{1}{\alpha} \int_{\mathbb{R}^N} a(x)|u|^\alpha dx \\
&\quad - \frac{1}{\gamma} \int_{\mathbb{R}^N} |u|^\gamma dx - \frac{1}{\beta} \int_{\mathbb{R}^N} |u|^\beta dx \\
&\geq C \|\nabla u\|_{p,q}^q + \frac{1}{\alpha} \int_{\mathbb{R}^N} a(x)|u|^\alpha dx - c_3 \|u\|_{p^*}^\gamma - c_4 \|u\|_{p^*}^\beta \\
&\geq C \|\nabla u\|_{p,q}^q + \frac{1}{\alpha} \int_{\mathbb{R}^N} a(x)|u|^\alpha dx - \widetilde{C} \|u\|_{p^*}^{p^*}
\end{aligned} \tag{3.1}$$

where $c_3$ and $c_4$ are two positive constants.

Since we set $r \in (0,1)$ and we have the hypothesis that $\max\{\alpha, q\} < p^*$, we obtain by relation (3.1) that there exists $a > 0$ such that

$$I(u) \geq a, \quad \text{for every } u \in X \text{ with } \|u\| = r. \tag{3.2}$$

Now, we verify (c). We fix $u \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. Then, by $(\phi_4)$, we have

$$\begin{aligned} I(tu) \leq & \frac{c_1}{2}\int_{[|\nabla(tu)|\leq 1]} |\nabla(tu)|^q dx + \frac{c_2}{2}\int_{[|\nabla(tu)|>1]} |\nabla(tu)|^p dx + \frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx \\ & - \frac{t^\gamma}{\gamma}\int_{\mathbb{R}^N} |u|^\gamma dx - \frac{t^\beta}{\beta}\int_{\mathbb{R}^N} |u|^\beta dx \\ \leq & C\left(t^q \int_{\mathbb{R}^N} |\nabla u|^q dx + t^p \int_{\mathbb{R}^N} |\nabla u|^p dx\right) + \frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx \\ & - \frac{t^\gamma}{\gamma}\int_{\mathbb{R}^N} |u|^\gamma dx - \frac{t^\beta}{\beta}\int_{\mathbb{R}^N} |u|^\beta dx. \end{aligned} \tag{3.3}$$

Taking into account the hypotheses of problem (1.2), relations (3.2) and (3.3), since $u$ is fixed, we obtain that

$$\lim_{t\to\infty} I(tu) = -\infty.$$

So, there exists $t_0 > 0$ such that $I(tu) < 0$. Thus, we have checked the second geometrical hypothesis of the mountain pass lemma, or the existence of a "valley" over the chain of mountains.

Now, we prove that the corresponding setting is non-degenerate, namely, the associated min-max value given by Lemma (2.3) is positive.

Let us define

$$c := \inf_{p\in P} \max_{t\in[0,1]} I(p(t)),$$

where

$$P := \{p \in C([0,1], X);\ p(0) = 0,\ p(1) = t_0 u\}.$$

We notice that

$$c \geq I(p(0)) = I(0) = 0,$$

for all $p \in P$.

We claim that

$$c > 0. \tag{3.4}$$

By contradiction, we suppose that $c = 0$, that is, for all $\epsilon > 0$ there exists $q \in P$ such that

$$0 \leq \max_{t\in[0,1]} I(q(t)) < \epsilon.$$

If we fix $\epsilon < a$, where $a$ is given by (3.2), then $q(0) = 0$ and $q(1) = t_0 u$. Therefore,

$$\|q(0)\| = 0 \quad \text{and} \quad \|q(1)\| > r.$$

Since $q$ is continuous, there exists $t_1 \in (0,1)$ such that $\|q(t_1)\| = r$, so

$$\|I(q(t_1))\| = a > \epsilon.$$

The above inequality is a contradiction, which shows that our claim (3.4) is true.

By Lemma (2.3), we obtain a Palais–Smale sequence $(u_n) \in X$ for the level $c > 0$ such that

$$\lim_{n\to\infty} I(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0. \tag{3.5}$$

Finally, we prove that this sequence $(u_n)$ is bounded in $X$. Using relation (3.5), we obtain that

$$\begin{aligned}
&c + O(1) + o(\|u_n\|) \\
&\quad = I(u_n) - \frac{1}{\gamma} I'(u_n)[u_n] \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u_n|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx - \frac{1}{\gamma}\int_{\mathbb{R}^N} |u_n|^\gamma dx \\
&\qquad - \frac{1}{\beta}\int_{\mathbb{R}^N} |u_n|^\beta dx - \frac{1}{\gamma}\int_{\mathbb{R}^N} \phi'(|\nabla u_n|^2)|\nabla u_n|^2 dx - \frac{1}{\gamma}\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \\
&\qquad + \frac{1}{\gamma}\int_{\mathbb{R}^N} |u_n|^\gamma dx + \frac{1}{\gamma}\int_{\mathbb{R}^N} |u_n|^\beta dx \\
&\quad = \int_{\mathbb{R}^N} \left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{\gamma}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx \\
&\qquad + \left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N} |u_n|^\beta dx.
\end{aligned} \tag{3.6}$$

By relation $(\phi_5)$ and hypothesis $\max\{\alpha, q\} < \gamma < \beta < p^*$ we have

$$\begin{aligned}
c + O(1) + o(\|u_n\|) &= I(u_n) - \frac{1}{\gamma} I'(u_n)[u_n] \\
&\geq \int_{\mathbb{R}^N} \left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{\gamma}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx \\
&\quad + \left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u_n|^2)dx - \frac{\mu}{2}\int_{\mathbb{R}^N} \phi(|\nabla u_n|^2)dx \\
&\quad + \left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \\
&= \frac{1-\mu}{2}\int_{\mathbb{R}^N} \phi(|\nabla u_n|^2)dx + \left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \\
&\geq c_0\left[\min\left\{\|\nabla u_n\|_{p,q}^q, \|\nabla u_n\|_{p,q}^p\right\} + \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx\right],
\end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$, with $c_0 > 0$ arbitrary.

By the above inequality we deduce that $(u_n)$ is bounded in $X$.

We know that $X$ is a closed subset of $W$. Then, by Proposition 2.5 in [5] we deduce that

$$u_n \rightharpoonup u_0 \quad \text{in } X, \tag{3.8}$$

$$u_n \to u_0 \quad \text{in } L^\gamma(\mathbb{R}^N), \tag{3.9}$$

$$u_n \to u_0 \quad \text{in } L^\beta(\mathbb{R}^N). \tag{3.10}$$

Now, we are concerned to prove that $u_0$ is a solution of problem (1.2).

Fix $\varphi \in C_0^\infty(\mathbb{R}^N)$ and set $\Omega := \mathrm{supp}(\varphi)$. We can write

$$I(u) = A(u) - B(u)$$

and for this purpose, we define the energy functionals

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx$$

and

$$B(u) = \frac{1}{\gamma}\int_{\mathbb{R}^N} |u|^\gamma dx + \frac{1}{\beta}\int_{\mathbb{R}^N} |u|^\beta dx.$$

Relation (3.5) yelds to

$$A(u_n) - B(u_n) \to c \tag{3.11}$$

and

$$A'(u_n)(\varphi) - B'(u_n)(\varphi) \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

By (3.9) and (3.10) we obtain

$$B(u_n) \to B(u_0) \quad \text{and} \quad B'(u_n)(\varphi) \to B'(u_0)(\varphi) \quad \text{as } n \to \infty. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$A'(u_n)(\varphi) \to B'(u_0)(\varphi) \quad \text{as } n \to \infty \tag{3.14}$$

Since $A$ is convex (by ($\phi_6$)),

$$A(u_n) \le A(u_0) + A'(u_n)(u_n - u_0) \quad \text{for every } n \in \mathbb{N}. \tag{3.15}$$

This both (3.14) and (3.8) yields to

$$\limsup_{n\to\infty} A(u_n) \le A(u_0). \tag{3.16}$$

The functional $A$ is convex and continuous, thus it is lower semicontinuous, so

$$A(u_0) \le \liminf_{n\to\infty} A(u_n). \tag{3.17}$$

Combining (3.16) and (3.17) we obtain that

$$A(u_n) \to A(u_0) \quad \text{as } n \to \infty.$$

Making use of the same arguments as in [5, p. 210], we deduce that

$$\nabla u_n \to \nabla u_0 \quad \text{as } n \to \infty \quad \text{in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \to \int_{\mathbb{R}^N} a(x)|u_0|^\alpha dx \quad \text{as } n \to \infty.$$

We can conclude now that

$$\int_{\mathbb{R}^N} \phi'(|\nabla u_0|^2)\nabla u_0 \nabla\varphi dx + \int_{\mathbb{R}^N} a(x)|u_0|^{\alpha-2}u_0\varphi dx - \int_{\mathbb{R}^N} |u_0|^{\gamma-2}u_0\varphi dx - \int_{\mathbb{R}^N} |u_0|^{\beta-2}u_0\varphi dx = 0,$$

for all $\varphi \in X$, then $u_0$ is a solution of problem (1.2).

*Proof of Theorem 2.4 completed.* We have previously shown, by means of mountain pass lemma, that problem (1.2) has a weak solution. It remains to argue that the solution $u_0$ found above is nontrivial. So, in order to complete the proof of theorem (2.4), we use some methods developed in [10] and [11].

Since the secquence $(u_n)$ satisfies the Palais–Smale condition, relation (3.5) leads to

$$\begin{aligned}\frac{c}{2} &\leq I(u_n) - \frac{1}{\gamma} I'(u_n)[u_n] \\ &= \int_{\mathbb{R}^N} [\phi(|\nabla u_n|^2) - \phi'(|\nabla u_n|^2)|\nabla u_n|^2] + \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} |u_n|^\gamma dx + \left(\frac{1}{2} - \frac{1}{\beta}\right) \int_{\mathbb{R}^N} |u_n|^\beta dx,\end{aligned} \tag{3.18}$$

for $n$ a positive integer large enough.

By $(\phi_6)$ we deduce that

$$\phi(t^2) - \phi(0) \leq \phi'(t^2)t^2$$

and applying now $(\phi_2)$,

$$\phi(t^2) \leq \phi'(t^2)t^2,$$

which means we can write that

$$\phi(|\nabla u_n|^2) \leq \phi'(|\nabla u_n|^2)|\nabla u_n|^2. \tag{3.19}$$

From now, we split the proof in two cases. First, we suppose that $\alpha \geq 2$. Combining relations (3.18) and (3.19) we obtain

$$\begin{aligned}\frac{c}{2} &\leq \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} |u_n|^\gamma dx + \left(\frac{1}{2} - \frac{1}{\beta}\right) \int_{\mathbb{R}^N} |u_n|^\beta dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^\gamma dx + \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^\beta dx \\ &\leq c_3 \|u_n\|_{p^*}^{p^*} + c_4 \|u_n\|_{p^*}^{p^*} \\ &\leq \bar{c} \|u_n\|_{p^*}^{p^*},\end{aligned} \tag{3.20}$$

where $c_3$ and $c_4$ are positive constants.

Our aim is to show that $u_0 \neq 0$. For this purpose we suppose by contradiction that $u_0 = 0$. This both relation (3.9) implies that

$$u_n \to 0 \quad \text{in } L^\gamma(\mathbb{R}^N), \tag{3.21}$$

hence,

$$u_n \to 0 \quad \text{in } L^{p^*}(\mathbb{R}^N). \tag{3.22}$$

By (3.20) and (3.22) it follows that $c = 0$, which is a contradiction.

It remains to study the case $\alpha \in (0, 2)$. By (3.18) and (3.19) we obtain that for $n$ large enough,

$$\begin{aligned}\frac{c}{2} &\leq I(u_n) - \frac{1}{\gamma} I'(u_n)[u_n] \\ &\leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} |u_n|^\gamma dx + \left(\frac{1}{2} - \frac{1}{\beta}\right) \int_{\mathbb{R}^N} |u_n|^\beta dx \\ &\leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \left(\frac{1}{2} - \frac{1}{\gamma}\right) c_3 \|u_n\|_{p^*}^{p^*} + \left(\frac{1}{2} - \frac{1}{\beta}\right) c_4 \|u_n\|_{p^*}^{p^*}.\end{aligned} \tag{3.23}$$

We argue again by contradiction and assume that $u_0 = 0$. In particular, this implies that

$$u_n \to 0 \quad \text{in } L^\alpha_{\text{loc}}(\mathbb{R}^N). \tag{3.24}$$

Thus, by (3.22), relation (3.23) becomes

$$\frac{c}{2} \leq \left(\frac{1}{\alpha} - \frac{1}{2}\right) \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx. \tag{3.25}$$

We consider $k$ a positive integer and define

$$\omega := \left\{x \in \mathbb{R}^N;\ 1/k < |x| < k\right\}.$$

Using (3.24) and the continuously embedding of the space $L^\alpha(\omega, a)$ in $L^\alpha(\omega)$ we obtain that

$$C_0 \int_\omega a(x)|u_n|^\alpha dx \leq \frac{c}{2},$$

for all $n \geq n_0$ and $k$ large enough. Thus,

$$\begin{aligned} \frac{c}{2} &\leq \int_{\mathbb{R}^N \setminus \omega} a(x)|u_n|^\alpha dx \\ &\leq \frac{C_0}{\inf_{|x|\leq 1/k} a(x)} \int_{|x|\leq 1/k} a(x)|u_n|^\alpha dx + \frac{C_0}{\inf_{|x|\geq k} a(x)} \int_{|x|\geq k} a(x)|u_n|^\alpha dx \\ &\leq C_0 M \left[\frac{1}{\inf_{|x|\leq 1/k} a(x)} + \frac{1}{\inf_{|x|\geq k} a(x)}\right], \end{aligned} \tag{3.26}$$

where $M = \sup_n \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx$.

If we choose $k$ large enough and take into account hypothesis $(a_2)$, we obtain by (3.26) that $c = 0$, a contradiction.

Resuming, we have obtained that $u_0$ is a nontrivial solution of problem (1.2) and this concludes our proof.

□

## References

[1] M. J. Ablowitz, B. Prinari, A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, Cambridge University Press, Cambridge, 2004. MR2040621; url

[2] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14**(1973), 349–381. MR0370183; url

[3] T. D'Aprile, G. Siciliano, Magnetostatic solutions for a semilinear perturbation of the Maxwell equations, *Adv. Differential Equations* **16**(2011), 435–466. MR2816112

[4] A. Azzollini, Minimum action solutions for a quasilinear equations, *J. Lond. Math. Soc. (2)* **92**(2015), 583–595. MR3431651; url

[5] A. Azzollini, P. d'Avenia, A. Pomponio, Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz–Sobolev embeddings, *Calc. Var. Partial Differential Equations* **49**(2014), 197–213. MR3148112; url

[6] M. Badiale, L. Pisani, S. Rolando, Sum of weighted Lebesgue spaces and nonlinear elliptic equations, *NoDEA Nonlinear Differential Equations Appl.* **18**(2011), 369–405. MR2825301; url

[7] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, Vol.10, American Mathematical Society, Providence, RI, 2003. MR2002047; url

[8] N. Chorfi, V. D. Rădulescu, Standing waves solutions of a quasilinear degenerate Schrödinger equation with unbounded potential, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 37, 1–12. MR3513973; url

[9] Y. Fu, Y. Shan, On the removability of isolated singular points for elliptic equations involving variable exponent, *Adv. Nonlinear Anal.* **5**(2016), No. 2, 121–132. MR3510816; url

[10] F. Gazzola, V. D. Rădulescu, A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^N$, *Differential Integral Equations* **13**(2000), 47–60. MR1811948

[11] M. Mihăilescu, V. D. Rădulescu, Ground state solutions of nonlinear singular Schrödinger equations with lack of compactness, *Math. Methods Appl. Sciences* **26**(2003), 897–906. url

[12] P. Pucci, V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, *Boll. Unione Mat. Ital. (3)* **9**(2010), 543–582. MR2742781

[13] P. Pucci, J. Serrin, Extensions of the mountain pass theorem, *J. Funct. Anal.* **59**(1984), 185–210. MR0766489; url

[14] P. Pucci, J. Serrin, A mountain pass theorem, *J. Differential Equations* **60**(1985), 142–149. MR0808262; url

[15] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43**(1992), 270–291. MR1162728; url

[16] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121**(2015), 336–369. MR3348928; url

[17] V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents*, CRC Press, Taylor and Francis Group, Boca Raton FL, 2015. MR3379920; url

[18] D. Repovš, Stationary waves of Schrödinger-type equations with variable exponent, *Anal. Appl. (Singap.)* **13**(2015), 645–661. MR3376930; url

[19] C. Sulem, P. L. Sulem, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences, Vol. 139, Springer-Verlag, New York, 1999. MR1696311; url

[20] Z. Yücedağ, Solutions of nonlinear problems involving $p(x)$-Laplace operator, *Adv. Nonlinear Anal.* **4**(2015), No. 4, 285–293. MR3420320; url