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On the Dirichlet problem for a Duffing type equation

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Abstract

We use direct variational method in order to investigate the dependence on parameter for the solution for a Duffing type equation with Dirichlet boundary value conditions.

1 Introduction

Recently the classical variational problem for a Duffing type equation received again some attention. In [1], [2], [7], some variational approaches were used in order to receive the existence of solutions for both periodic and Dirichlet type boundary value problems. Mainly direct method is applied under various conditions pertaining to at most quadratic growth imposed on the nonlinear term given in [2] and further relaxed in [7]. Dirichlet problems for such equations could also be considered by some other methods, for example min-max theorem due to Manashevich, [8]. In [6] the author gives some historical results concerning the Dirichlet problem for Duffing type equations and discusses the methods which are used in reaching the existence results which are different from the ones which we use and comprise the classical variational approach, the topological method.

In the boundary value problems for differential equations it is also important to know whether the solution, once its existence is proved, depends continuously on a functional parameter. This question has a great impact on future applications of any model since it is desirable to know whether the solution to the small deviation from the model would return, in a continuous way, to the solution of the original model. This is known in differential equation as stability or continuous dependence on parameter, see [4]. We will investigate the dependence on a functional parameter for a Duffing type equations basing on some results developed for different kind of problems in [4]. However we provide some general principle which will allow for investigation of dependence on parameters for other problems also. In [4] and also in other papers by these authors, it is required that each problem should be investigated separately as far as the dependence on parameters is concerned. Here we aim at providing some hint how to obtain a general rule, which will allow to investigate the dependence on parameters for various types of nonlinear problems. We will demonstrate our results on the Duffing type boundary value problem.

To be precise, in this paper we will consider the Dirichlet problem for a forced Duffing type equation with a functional parameter u. We investigate the problem

$$\begin{aligned} &\tfrac{d^2}{dt^2}x\left(t\right)+r\left(t\right)\tfrac{d}{dt}x\left(t\right)+F_x^2\left(t,x\left(t\right)\right)u\left(t\right)-F_x^1\left(t,x\left(t\right)\right)=f\left(t\right), \\ &x\left(0\right)=x\left(1\right)=0 \end{aligned} \tag{1}$$

with $u:[0,1]\to R$ belonging to the set

$$L_M=\{u:[0,1]\to R:\ u \text{ is measurable, } |u\left(t\right)|\leq m \text{ for } a.e.\ t\in[0,1]\}$$

and where $m>0$ is a fixed real number. Here $f\in L^2\left(0,1\right)$ is the forcing term and $r\in C^1\left(0,1\right)$ denotes the friction; $r\left(\tau\right)\geq 0$ for $\tau\in[0,1]$. Here we do not assume anything about the monotonicity of r, but instead we require that

$$\frac{1}{4}r^2\left(t\right)+\frac{1}{2}\frac{d}{dt}r\left(t\right)>0 \tag{2}$$

for all $t\in[0,1]$. We denote $w\left(t\right)=\frac{1}{4}r^2\left(t\right)+\frac{1}{2}\frac{d}{dt}r\left(t\right)$. Of course, when r is nondecreasing we obviously have (2). Following [7] we denote $R\left(t\right)=e^{\int_0^t\frac{1}{2}r(\tau)d\tau}$. Since $r\left(\tau\right)\geq 0$ on $[0,1]$ we see that

$$R_{\max}=e^{\max_{\tau\in[0,1]}r(\tau)}\geq R\left(t\right)\geq R\left(0\right)=1. \tag{3}$$

Upon putting $y=R\left(t\right)x$ boundary problem (1) reads

$$\begin{aligned} &-\tfrac{d^2}{dt^2}y\left(t\right)+w\left(t\right)y\left(t\right)=R\left(t\right)F_x^2\left(t,\tfrac{y(t)}{R(t)}\right)u\left(t\right)-R\left(t\right)F_x^1\left(t,\tfrac{y(t)}{R(t)}\right)-R\left(t\right)f\left(t\right), \\ &y\left(0\right)=y\left(1\right)=0. \end{aligned} \tag{4}$$

Therefore instead of (1) in this paper we will investigate (4). In what follows $(F^1)^*$ denotes the Fenchel-Young transform (see for example [5]) of a function F^1 with respect to the second variable, namely

$$\left(F^1\right)^*(t,v)=\sup_{x\in R}\left\{xv-F^1(t,x)\right\} \text{ for a.e. } t\in[0,1].$$

As an application, we finally consider the existence to some optimal control problem.

2 The assumptions and examples

In order to apply a direct variational method to a Dirichlet problem (4) we will employ the following assumptions besides the assumptions given at the beginning of the paper.

F1 *$F^1, F^1_x, F^2, F^2_x : [0,1]\times R \to R$ are Caratheodory functions; $t \to F^1(t,0)$ is integrable on $[0,1]$; for any $d>0$ there exists a function $f_d \in L^2(0,1)$ (depending on d), $f_d(t)>0$ for a.e. $t\in[0,1]$, such that*

$$\left|F^1_x(t,x)\right| \le f_d(t) \text{ for all } x\in[-d,d], \text{ for a.e. } t\in[0,1]; \tag{5}$$

F2 *either $t\to (F^1)^*(t,0)$ is integrable on $[0,1]$ or else F^1 is convex in x for a.e. $t\in[0,1]$;*

F3 *there exist functions $a,b\in L^2(0,1)$ such that*

$$\left|F^2(t,x)\right|\le a(t),\ \left|F^2_x(t,x)\right|\le b(t) \quad \text{for a.e. } t\in[0,1] \text{ and all } x\in R. \tag{6}$$

With assumptions **F1, F2**, **F3** we get for any fixed $u\in L_M$ the existence of an argument of a minimum for an Euler functional $J_u : H^1_0(0,1)\to R$

$$J_u(x)=\frac{1}{2}\int_0^1\left(\tfrac{d}{dt}x(t)\right)^2dt+\frac{1}{2}\int_0^1 w(t)\,x^2(t)\,dt+$$

$$-\int_0^1 R^2(t)\,F^2\left(t,\tfrac{x(t)}{R(t)}\right)u(t)\,dt+\int_0^1 R^2(t)\,F^1\left(t,\tfrac{x(t)}{R(t)}\right)dt-\textstyle\int_0^1 R(t)\,f(t)\,x(t)\,dt.$$

A weak solution to (4) is understood as such a function $x \in H_0^1(0,1)$ that for all $g \in H_0^1(0,1)$ the following relation holds:

$$\int_0^1 \tfrac{d}{dt}x(t)\tfrac{d}{dt}g(t)\,dt + \int_0^1 w(t)\,x(t)\,g(t)\,dt - \int_0^1 R(t)\,F_x^2\left(t,\tfrac{x(t)}{R(t)}\right)u(t)\,g(t)\,dt$$

$$+\int_0^1 R(t)\,F_x^1\left(t,\tfrac{x(t)}{R(t)}\right)g(t)\,dt - \int_0^1 R(t)\,f(t)\,g(t)\,dt = 0. \tag{7}$$

Lemma 1 *We assume* ***F1, F2, F3.*** *For any fixed $u \in L_M$ functional J_u is well defined and Gâteaux differentiable onto $H_0^1(0,1)$. Moreover, weak solutions to (4) correspond to critical points of J_u.*

Proof. Let us fix any $x \in H_0^1(0,1)$. Since $|u(t)| \le m$ we see that $F_x^2(\cdot, x(\cdot))\,u(\cdot) \in L^2(0,1)$. We further observe by inequality

$$\max_{t\in[0,1]} |x(t)| \le \|\dot{x}\|_{L^2(0,1)}$$

that there exists a number $d_u > 0$ such that $|x(t)| \le d_u$. Hence by the Mean Value Theorem, by integrability of $t \to F(t,0)$ it follows by (5) that

$$\int_0^1 \left|R^2(t)\,F^1\left(t,\frac{x(t)}{R(t)}\right)\right| dt \le \int_0^1 \left|R^2(t)\,F^1(t,0)\right| dt + d_u\int_0^1 \left|R^2(t)\,f_{d_u}(t)\right| dt \tag{8}$$

the integral $\int_0^1 R^2(t)\,F^1\left(t,\tfrac{x(t)}{R(t)}\right)dt$ is finite. By (6) we have also the integral $\int_0^1 R^2(t)\,F^2\left(t,\tfrac{x(t)}{R(t)}\right)u(t)\,dt$ exists. Thus J_u is well defined. The Gâteaux differentiability follows since $F_x^1(\cdot,x(\cdot)) \in L^2(0,1)$ and since $F_x^2(\cdot,x(\cdot)) \in L^2(0,1)$ by (6). A direct calculation shows $\left\langle \frac{d}{dx}J_u(x_u), g\right\rangle = 0$ equals exactly (7). ■

We conclude this section with examples of nonlinearities satisfying our assumptions.

Let $F^2(t,x) = f(t)\,g(x)$, where $g \in C^1(R)$ has a bounded derivative and

$$F^1(t,x) = \frac{1}{2s}g_1(t)\,x^{2s} - \frac{1}{s}g_2(t)\,x^s,$$

where s is an even number, $f \in L^2(0,1)$, $g_1, g_2 \in L^\infty(0,1)$, $g_1(t), g_2(t) > 0$ for a.e. $t \in [0,1]$. Then

$$\left|F_x^2(t,x)\right| = \left|f(t)\frac{d}{dx}g(x)\right| \leq |f(t)| \sup_{x\in R}\left|\frac{d}{dx}g(x)\right| = a(t) \text{ and } a \in L^2(0,1),$$

and

$$F_x^1(t,x) = g_1(t)x^{2s-1} - g_2(t)x^{s-1}.$$

Again for any fixed $d > 0$ function $t \to \max_{x\in[-d,d]}\left(|g_1(t)|x^{2s-1} + |g_2(t)|x^{s-1}\right)$ belongs to $L^2(0,1)$. We remark that F^1 need not be convex on R and that $t \to (F^1)^*(t,0)$ is integrable. Indeed, for a.e. (fixed) $t \in [0,1]$ function $x \to -\frac{1}{2s}g_1(t)x^{2s} + \frac{1}{s}g_2(t)x^s$ has its maximum x_M satisfying $g_1(t)x^{2s-1} - g_2(t)x^{s-1} = 0$ so either

$$x_M = 0 \text{ and } \left(F^1\right)^*(t,0) = \sup_{x\in R}\left\{-\frac{1}{2s}g_1(t)x^{2s} + \frac{1}{s}g_2(t)x^s\right\} = 0$$

or

$$x_M^s = \frac{g_2(t)}{g_1(t)} \text{ and } \left(F^1\right)^*(t,0) = -\frac{1}{2}\frac{(g_2(t))^2}{g_1(t)}.$$

3 Dependence on parameters for action functionals

In order to derive the results concerning the dependence on parameters for problem (4), we employ the following general principle. Let E be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and with the induced norm $\|\cdot\|$. Let C be a Banach space with norm $\|\cdot\|_C$. Let us consider a family of action functionals $x \to J(x,u)$, where $x \in E$ and where $u \in C$ is a parameter.

Theorem 2 *Assume that $E \ni x \to J(x,u)$ satisfies Palais-Smale condition, is weakly lower semicontinuous and bounded from below for any fixed $u \in M$, where $M \subset C$. Then $x \to J(x,u)$ has the argument of a minimum over E. Suppose further that there exists a constant $\alpha > 0$ such that the set $\{(x,u) : J(x,u) \leq \alpha\}$ is bounded in E uniformly in $u \in M$. Let $\{u_n\}_{n=1}^\infty \subset M$ be a weakly convergent sequence of parameters, where a weak limit $\lim_{n\to\infty} u_n = \overline{u} \in M$. Let $\{x_n\}_{n=1}^\infty \subset E$ be the corresponding sequence of the arguments of minimum to $E \ni x \to J(x,u_n)$. Then, there* *is a convergent subsequence* $\{x_{n_i}\}_{i=1}^{\infty} \subset E$ *and an element* $\overline{x} \in E$ *such that* $\lim_{i\to\infty} x_{n_i} = \overline{x}$. *If additionally*

$$J(\overline{x}, u_{n_i}) \to J(\overline{x}, \overline{u}) \text{ and } J(x_{n_i}, u_{n_i}) \to J(\overline{x}, u_{n_i}) \text{ as } i \to \infty, \tag{9}$$

we obtain that $\overline{x}$ *is an argument of a minimum to* $x \to J(x, \overline{u})$.

Proof. Let us fix $u \in M$. Since $x \to J(x,u)$ satisfies Palais-Smale condition, is weakly lower semicontinuous and bounded from below, it follows that $J(\cdot, u)$ has an argument of a minimum.

Let $\{u_n\}_{n=1}^{\infty} \subset M$ be a weakly convergent sequence of parameters with $\lim_{n\to\infty} u_n = \overline{u}$. Now since the set $\{x : J(x,u) \leq \alpha\}$ is bounded it follows that sequence $\{x_n\}_{n=1}^{\infty} \subset \{x : J(x,u) \leq \alpha\}$ of the arguments of a minimum to $x \to J(x, u_n)$ has a weakly convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty} \subset E$. Let us denote $\overline{x} = \lim_{i\to\infty} x_{n_i}$, where $\overline{x}$ denotes the weak limit.

We will prove that $\overline{x}$ is an argument of a minimum to $x \to J(x, \overline{u})$. We see that there exists $x_0 \in E$ such that $J(x_0, \overline{u}) = \inf_{y\in E} J(y, \overline{u})$ and there are two possibilities: either $J(x_0, \overline{u}) < J(\overline{x}, \overline{u})$ or $J(x_0, \overline{u}) = J(\overline{x}, \overline{u})$. If we have $J(x_0, \overline{u}) = J(\overline{x}, \overline{u})$, then we have the assertion. Let us suppose that $J(x_0, \overline{u}) < J(\overline{x}, \overline{u})$, so there exists $\delta > 0$ such that

$$J(\overline{x}, \overline{u}) - J(x_0, \overline{u}) > \delta > 0. \tag{10}$$

We investigate the inequality

$$\begin{aligned} &\delta < (J(x_{n_i}, u_{n_i}) - J(x_0, \overline{u})) - (J(x_{n_i}, u_{n_i}) - J(\overline{x}, u_{n_i})) \\ &- (J(\overline{x}, u_{n_i}) - J(\overline{x}, \overline{u})) \end{aligned} \tag{11}$$

which is equivalent to (10). In view of (9) we see that the second and third term converge to 0. Finally, since x_{n_i} minimizes $x \to J(x, u_{n_i})$ over E we get $J(x_{n_i}, u_{n_i}) \leq J(x_0, u_{n_i})$ and next

$$\lim_{i\to\infty} (J(x_{n_i}, u_{n_i}) - J(x_0, \overline{u})) \leq \lim_{i\to\infty} (J(x_0, u_{n_i}) - J(x_0, \overline{u})) = 0.$$

Summarizing, we see that we have $\delta \leq 0$ in (11), which is a contradiction. ■

4 Existence result

Theorem 3 *Let* $u \in L_M$ *be arbitrarily fixed. Assume* ***F1, F2, F3 .*** *There exists* $x_u \in H_0^1(0,1)$ *such that* $J_u(x_u) = \inf_{x\in H_0^1(0,1)} J_u(x)$ *and*

$$x_u \in V_u = \left\{ x \in H_0^1(0,1) : J_u(x) = \inf_{v\in H_0^1(0,1)} J_u(v) \text{ and } \frac{d}{dx}J_u(x) = 0 \right\}$$

Moreover, x_u *satisfies (4) for a.e.* $t \in [0,1]$.

Proof. First we show that J_u is weakly l.s.c. on $H_0^1(0,1)$. Let us take any sequence $\{x_n\}_{n=1}^{\infty} \subset H_0^1(0,1)$ such that x_n converges weakly in $H_0^1(0,1)$ to x. Then $\{x_n\}_{n=1}^{\infty}$ contains by the Arzela-Ascoli Theorem a subsequence convergent uniformly and which we denote by $\{x_n\}_{n=1}^{\infty}$. Now since $\{x_n\}_{n=1}^{\infty}$ is convergent in $C(0,1)$ it follows that there exist a number d such that $\max_{t\in[0,1]} |x_n(t)| \leq d$ for sufficiently large n. By (6) and by the Lebesgue Dominated Convergence Theorem that

$$\int_0^1 R^2(t)\, F^2\left(t, \frac{x_n(t)}{R(t)}\right) u(t)\, dt \to \int_0^1 R^2(t)\, F^2\left(t, \frac{x(t)}{R(t)}\right) u(t)\, dt \text{ as } n \to \infty.$$

Now by (8) we see that

$$\lim_{n\to\infty} \int_0^1 R^2(t)\, F^1\left(t, \frac{x_n(t)}{R(t)}\right) dt = \int_0^1 R^2(t)\, F^1\left(t, \frac{x(t)}{R(t)}\right) dt \text{ as } n \to \infty.$$

Since the remaining terms of J_u are convex and defined on $H_0^1(0,1)$, these are also weakly l.s.c. on $H_0^1(0,1)$. Thus J_u is weakly l.s.c. on $H_0^1(0,1)$.

We observe that J_u is coercive on $H_0^1(0,1)$ in both cases.

Indeed, in case F^1 is convex for any $v \in R$ we get

$$F^1(t,v) \geq F^1(t,0) + F_x(t,0)\, v \tag{12}$$

and further since $t \to F_x^1(t,0)$ is integrable with square on $[0,1]$, the same follows for $t \to R(t)\, F_x^1(t,0)$. Thus for any $x \in H_0^1(0,1)$

$$\int_0^1 R^2(t)\, F^1\left(t, \tfrac{x(t)}{R(t)}\right) dt \geq \int_0^1 R^2(t)\, F^1(t,0)\, dt + \int_0^1 R(t)\, F_x^1(t,0)\, x(t)\, dt \geq$$

$$\int_0^1 R^2(t)\, F^1(t,0)\, dt - \left\| R(\cdot)\, F_x^1(\cdot,0) \right\|_{L^2(0,1)} \|x\|_{L^2(0,1)}$$

and by (6)

$$-\int_0^1 \left| R^2(t)\, F^2\left(t, \frac{x(t)}{R(t)}\right) u(t) \right| dt \geq -m\,(R_{\max})^2 \int_0^1 |a(t)|\, dt. \qquad (13)$$

In case $t \to (F^1)^*(t,0)$ is integrable we obtain by inequality Fenchel-Young inequality

$$\int_0^1 R^2(t)\, F^1\left(t, \frac{x(t)}{R(t)}\right) dt \geq -\int_0^1 R^2(t)\, \left(F^1\right)^*(t,0)\, dt. \qquad (14)$$

It follows that there exists $x_u \in H_0^1(0,1)$ such that $J_u(x_u) = \inf_{x \in H_0^1(0,1)} J_u(x)$ and obviously x_u is a weak solution to (4). Applying the fundamental lemma of the calculus of variations we obtain that x_u satisfies (4) for a.e. $t \in [0,1]$. ■

5 Dependence on a functional parameter

Theorem 4 *We assume* ***F1, F2, F3****. Let $\{u_k\}_{k=1}^{\infty}$, $u_k \in L_M$, be such a sequence that $\lim_{k\to\infty} u_k = \overline{u}$ weakly in $L^2(0,1)$. For each $k = 1,2,...$ the set V_{u_k} is nonempty and for any sequence $\{x_k\}_{k=1}^{\infty}$ of solutions $x_k \in V_{u_k}$ to the problem (4) corresponding to u_k, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty} \subset H_0^1(0,1)$ and an element $\overline{x} \in H_0^1(0,1)$ such that $\lim_{n\to\infty} x_{k_n} = \overline{x}$ (strongly in $L^2(0,1)$, weakly in $H_0^1(0,1)$, strongly in $C(0,1)$) and $J_{\overline{u}}(\overline{x}) = \inf_{x\in H_0^1(0,1)} J_{\overline{u}}(x)$. Moreover, $\overline{x} \in V_{\overline{u}}$, i.e.*

$$-\tfrac{d^2}{dt^2}\overline{x}(t) + w(t)\,\overline{x}(t) = R(t)\, F_x^2\left(t, \tfrac{\overline{x}(t)}{R(t)}\right) \overline{u}(t) - R(t)\, F_x^1\left(t, \tfrac{\overline{x}(t)}{R(t)}\right) - R(t)\, f(t),$$

$$\overline{x}(0) = \overline{x}(1) = 0. \qquad (15)$$

Proof. We will verify the assumptions of Theorem 2. By Theorem 3 for each $k = 1,2,...$ there exists a solution $x_k \in V_{u_k}$ to (4). We see that $x_k \in V_{u_k} \subset S_k = \{x : J_{u_k}(x) \leq J_{u_k}(0)\}$. We shall show that sequence

$\{x_k\}_{k=1}^{\infty}$ is bounded in $H_0^1(0,1)$. In case F^1 is convex for any $x \in S_k$ we have

$$-\int_0^1 R^2(t)\,F^2(t,0)\,u_k(t)\,dt + \int_0^1 R^2(t)\,F^2(t,x(t))\,u_k(t)\,dt \leq \tag{16}$$
$$2m\left(R_{\max}\right)^2 \int_0^1 |a(t)|\,dt.$$

By (12) and since $F_x^1(\cdot,0) \in L^2(0,1)$ we see by Poincaré inequality

$$\|x\|_{L^2(0,1)} \leq \frac{1}{\pi}\|\dot{x}\|_{L^2(0,1)}$$

that

$$\int_0^1 R^2(t)\,F^1(t,0)\,dt - \int_0^1 R^2(t)\,F^1\left(t,\tfrac{x(t)}{R(t)}\right)dt \leq$$
$$-\int_0^1 R(t)\,F_x^1(t,0)\,x(t)\,dt \leq \tfrac{1}{\pi}\|R(\cdot)\,F_x^1(\cdot,0)\|_{L^2(0,1)}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)}.$$

Therefore writing $0 \leq J_{u_k}(0) - J_{u_k}(x)$ explicitly and using Schwartz and Poincaré inequalities we have

$$\frac{1}{2}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)}^2 - \tfrac{1}{\pi}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)}\|f\|_{L^2(0,1)} \tag{17}$$
$$-\tfrac{1}{\pi}\|R(\cdot)\,F_x^1(\cdot,0)\|_{L^2(0,1)}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)} \leq 2m\left(R_{\max}\right)^2\int_0^1 |a(t)|\,dt.$$

Thus we see that the term $\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)}$ is in fact bounded disregarding of u_k. In case $t \to (F^1)^*(t,0)$ is integrable we also have (16) and by (14) we see that

$$\frac{1}{2}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)}^2 - \tfrac{1}{\pi}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)}\|f\|_{L^2(0,1)}$$
$$-\tfrac{1}{\pi}\|R(\cdot)\,F_x^1(\cdot,0)\|_{L^2(0,1)}\left\|\tfrac{d}{dt}x\right\|_{L^2(0,1)} \leq 2m\left(R_{\max}\right)^2\int_0^1 |a(t)|\,dt + \tag{18}$$
$$\int_0^1 R^2(t)\,F^1(t,0)\,dt + \int_0^1 R^2(t)\,(F^1)^*(t,0)\,dt.$$

Therefore, either by (17) or by (18) there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty} \subset H_0^1(0,1)$ weakly convergent in $H_0^1(0,1)$ to $\overline{x} \in H_0^1(0,1)$, which up to a subsequence may be assumed uniformly convergent and thus strongly convergent to $L^2(0,1)$.

Next, by Lebesgue Dominated Convergence Theorem we see that

$$\lim_{k_n\to\infty} \int_0^1 R^2(t)\, F^1\left(t, \frac{x_{k_n}(t)}{R(t)}\right) dt = \int_0^1 R^2(t)\, F^1\left(t, \frac{\overline{x}(t)}{R(t)}\right) dt, \tag{19}$$

and $\lim\limits_{k_n\to\infty} \int\limits_0^1 R^2(t)\, F^2\left(t, \frac{x_{k_n}(t)}{R(t)}\right) \overline{u}(t)(t)\, dt = \int\limits_0^1 R^2(t)\, F^2\left(t, \frac{\overline{x}(t)}{R(t)}\right) \overline{u}(t)\, dt.$

Thus

$$\lim_{k_n\to\infty} \left(J_{u_{k_n}}(\overline{x}) - J_{\overline{u}}(\overline{x})\right).$$

By the generalized Krasnosel'skij Theorem, see [3], and by (6) we see that

$$\lim_{k_n\to\infty} R^2(\cdot)\, F^2\left(\cdot, \frac{x_{k_n}(\cdot)}{R(\cdot)}\right) = R^2(\cdot)\, F^2\left(\cdot, \frac{\overline{x}(\cdot)}{R(\cdot)}\right)$$

strongly in $L^2(0,1)$. Thus $\lim_{k\to\infty} u_k = \overline{u}$ weakly in $L^2(0,1)$ provides that

$$\lim_{n\to\infty} \int_0^1 R^2(t)\, F^2\left(t, \frac{x_{k_n}(t)}{R(t)}\right) u_{k_n}(t)\, dt = \int_0^1 R^2(t)\, F^2\left(t, \frac{\overline{x}(t)}{R(t)}\right) \overline{u}(t)\, dt.$$

So by (19) we have

$$\lim\nolimits_{k_n\to\infty} \left(J_{u_{k_n}}(x_{k_n}) - J_{u_{k_n}}(\overline{x})\right) = 0.$$

The same arguments lead to conclusion that

$$\lim_{k_n\to\infty} \left(J_{u_{k_n}}(x_0) - J_{\overline{u}}(x_0)\right) = 0.$$

Hence all the assumptions of Theorem 2 are satisfied. Thus $\overline{x} \in V_{\overline{u}}$ and so $\overline{x}$ necessarily satisfies (15). ■

6 Applications to optimal control

We now show the existence of an optimal process for an optimal control problem in which the dynamics is described by the Duffing equation, i.e. we will minimize the following action functional

$$J(x,u) = \int_0^1 f_0(t, x(t), u(t))\, dt \tag{20}$$

subject to (4) and where

f0 *$f_0 : [0,1] \times R \times M \to R$ is measurable with respect to the first variable and continuous with respect to the two last variables and convex in u. Moreover, for any $d > 0$ there exists a function $\psi_d \in L^1(0,1)$ such that $|f_0(t,x,u)| \le \psi(t)$ a.e. on $[0,1]$ for all $x \in [-d,d]$ and for all $u \in M$.*

We define a set A consisting of pairs $(x_u, u) \in V_u \times L_M$ on which we consider the existence of an optimal process to (20)-(4); x_u is a solution to (4) corresponding to u. We mention here that since the functions from L_M are equibounded we get $\lim_{k\to\infty} u_k = \overline{u}$ weakly in $L^2(0,1)$, up to a subsequence, for any sequence $\{u_k\}_{k=1}^{\infty} \subset L_M$. Moreover, any sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \in V_{u_k}$ or $x_k \in X$, of solutions to (4) corresponding to such $\{u_k\}_{k=1}^{\infty}$ is necessarily bounded in $H_0^1(0,1)$ as follows from the proof of Theorem 4. Thus there exists a $d > 0$ such that $x_k(t) \in [-d,d]$ for all $k = 1,2,...$ and for *a.e.* $t \in [0,1]$.

Theorem 5 *We assume **f0, F1, F2, F3.** There exists a pair $(\overline{x}, \overline{u}) \in A$ such that $J(\overline{x}, \overline{u}) = \inf_{(x_u,u)\in A} J(x_u, u)$.*

Proof. Since any bounded sequence in $H_0^1(0,1)$ has a uniformly convergent subsequence and by convexity of f_0 with respect to u we see that J is weakly l.s.c. on $H_0^1(0,1) \times L^2(0,1)$. Assumption **f0** and remarks proceeding the formulation of the theorem provide that the functional J is bounded from below on A. Thus we may choose a minimizing sequence $\left\{x_u^k, u^k\right\}_{k=1}^{\infty}$ for a functional J such that $\left\{u^k\right\}_{k=1}^{\infty}$ is weakly convergent in $L^2(0,1)$ to a certain $\overline{u} \in L_M$. Theorem 4 asserts that $\left\{x_u^k\right\}_{k=1}^{\infty}$ converges, possibly up to a subsequence, strongly in $H_0^1(0,1)$, weakly in $H_0^1(0,1)$, strongly in $C(0,1)$ to a certain $\overline{x}$ solving (4) for $\overline{u}$. Thus

$$J(\overline{x}, \overline{u}) = \lim_{k\to\infty} \inf J\left(x_u^k, u^k\right) \ge J(\overline{x}, \overline{u}) \ge \inf_{(x,u)\in A} J(x,u).$$

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