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A topological classification of plane polynomial systems having a globally attracting singular point

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Abstract. In this paper, plane polynomial systems having a singular point attracting all orbits in positive time are classified up to topological equivalence. This is done by assigning a combinatorial invariant to the system (a so-called "feasible set" consisting of finitely many vectors with components in the set $\{n/3 : n = 0, 1, 2, \ldots\}$), so that two such systems are equivalent if and only if (after appropriately fixing an orientation in $\mathbb{R}^2$ and a heteroclinic separatrix) they have the same feasible set. In fact, this classification is achieved in the more general setting of continuous flows having finitely many separatrices.

Polynomial representatives for each equivalence class are found, although in a non-constructive way. Since, to the best of our knowledge, the literature does not provide any concrete polynomial system having a non-trivial globally attracting singular point, an explicit example is given as well.

Keywords: global unstable attraction, plane polynomial systems, topological equivalence.

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1 Introduction and statements of the main results

Classifying the phase portraits of plane polynomial systems, that is, those of the form

$$\begin{cases} x' = P(x,y), \\ y' = Q(x,y), \end{cases}$$

with $P(x,y)$ and $Q(x,y)$ polynomials in the variables x and y, is a classical problem (many would say the problem *par excellence*) of the qualitative theory of differential equations. As a whole it is a daunting, probably insurmountable, task, which if completed would provide, as a by-product, an answer to the famous (second part of the) Hilbert 16th problem asking for a bound $H(n)$ on the number of limit cycles of the system in terms of the maximum degree n of $P(x,y)$ and $Q(x,y)$. Presently, this bound is unknown even in the quadratic case $n = 2$; in

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fact, although there are strong reasons to conjecture $H(2) = 4$, not even the finiteness of $H(2)$ has been established.

Understandably, researchers in this area have added dynamical and/or analytic restrictions to the problem, as in [21], where it is shown that any C^1-structurally stable system with finitely many singular points and limit cycles is topologically equivalent to a polynomial system, or as in [4], where complex polynomial systems (that is, polynomial systems such that $P(x, y)$ and $Q(x, y)$ satisfy the Cauchy–Riemann conditions) are fully described in terms of appropriate combinatorial and analytic data.

No doubt fuelled by the search of a proof for the elusive equality $H(2) = 4$, quadratic systems have got the lion's share of this work. Here, among many others, cordal [10], Lotka–Volterra [20], those having a center [22], homogeneous [6], Hamiltonian [3], and bounded systems [7] have been classified up to topological equivalence. The monograph [18] is specifically devoted to this subject; interestingly, in p. 303 there, the number of possible portrait phases for quadratic systems (under the hypothesis $H(2) = 4$) is estimated to be around 2000.

Somewhat surprisingly, the most natural problem of classifying polynomial systems with a globally attracting singular point, that is, those whose orbits tend in positive time to the same singular point (which we can assume, without loss of generality, to be the origin $\mathbf{0}$), has not been studied yet. A possible explanation for this is that such a classification is pretty trivial in the quadratic realm: these systems are equivalent to the linear attracting node $x' = -x, y' = -y$. The reason is the following. As we will see below (Remark 4.2), in order to avoid the above trivial case, the finite sectorial decomposition at $\mathbf{0}$ must include both an elliptic and a hyperbolic sector. Such a local behaviour is certainly possible for quadratic systems: an explicit example with an *elliptic saddle* (that is, a decomposition consisting exactly of one elliptic sector and one hyperbolic sector) can be found in [2, p. 368]. Nevertheless, global attraction implies that the system is bounded (that is, it has bounded positive semiorbits), and for these systems the existence of elliptic sectors at singular points is excluded by [7]. Incidentally, if a C^1-system is locally holomorphic at $\mathbf{0}$, that is, the Cauchy–Riemann conditions hold near $\mathbf{0}$, then either $\mathbf{0}$ is a topological node or the sectorial decomposition consists of exactly evenly many elliptic sectors [5]. Therefore, non-trivial global attraction is also impossible in this case.

In the present paper we fulfil this gap by classifying polynomial global attraction up to topological equivalence. Indeed we work in the much more general setting of (continuous) flows with finitely many separatrices (or equivalently, see Remark 5.1, those having the finite sectorial decomposition property at $\mathbf{0}$, or those having finitely many unstable orbits), when their separatrix skeletons (the union of all separatrices and exactly one orbit from each region in the complementary set) are also finite. To begin with, there is a dichotomy: global attraction is trivial if and only if $\mathbf{0}$ is positively stable, that is, there are no regular homoclinic orbits (Proposition 3.9 (i)). Hence we concentrate in what follows in the "non-positively stable" case, when at least (as implied by Proposition 3.9 (ii)) one heteroclinic separatrix must exist. We rely on a well-known result by Markus [15], later extended by Neumann [16] (see also [9]), stating that two flows are equivalent if and only if there is a plane homeomorphism preserving the orbits and time directions of their separatrix skeletons (Theorem 2.7). As it turns out, a weaker so-called *compatibility* condition (just assuming preservation of orbits, see Subsection 2.2) suffices, provided that at least one heteroclinic separatrix is preserved as well. Moreover, after fixing an orientation in $\mathbb{R}^2$ (counterclockwise or clockwise) and a heteroclinic separatrix, and using the skeleton combinatorial structure, there is a canonical way to associate a so-called feasible set (a finite vectorial set as described in Definition 4.4) to the flow, and this

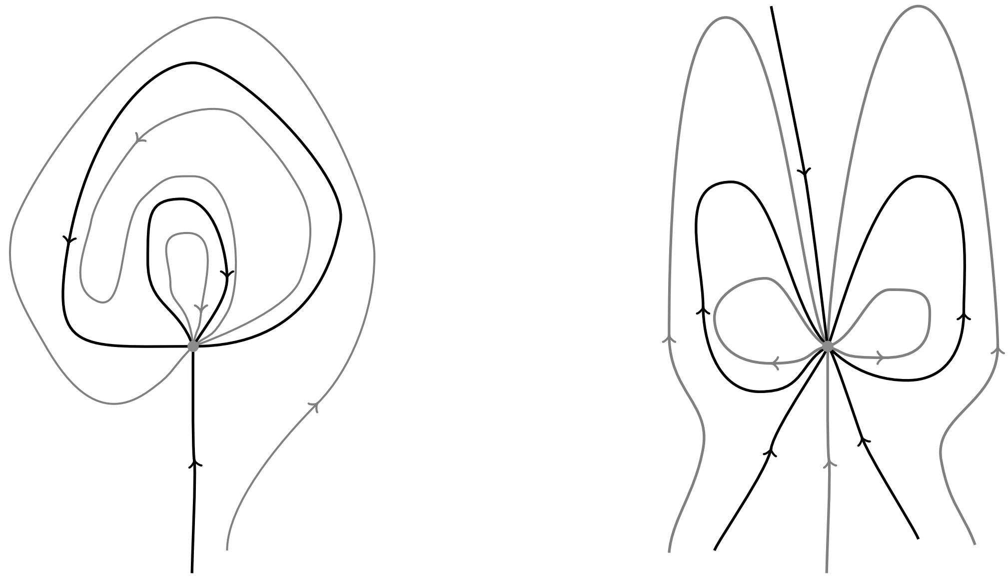

Figure 1.1: Two non-equivalent phase portraits with the same sectorial decomposition (elliptic-elliptic-hyperbolic-attracting-hyperbolic in counterclockwise sense) at the origin.

labelling characterizes equivalence: topologically equivalent flows have the same canonical feasible set. We emphasize that although the separatrix skeleton is not uniquely defined, no ambiguity arises because the corresponding canonical feasible sets are the same (this follows from Lemma 3.8).

Our first theorem summarizes these results.

Theorem A. Assume that $\mathbf{0}$ is a global attractor, non-positively stable, for two flows Φ and Φ', both having finitely many separatrices, and let $\mathcal{X}$ and $\mathcal{X}'$ denote their separatrix skeletons. Then the following statements are equivalent.

(i) Φ and Φ' are topologically equivalent.

(ii) $\mathcal{X}$ and $\mathcal{X}'$ are compatible and the compatibility bijection $\xi : \mathcal{X} \to \mathcal{X}'$ maps some heteroclinic separatrix of Φ to a heteroclinic separatrix of Φ'.

(iii) There are respective orientations Θ, Θ' in $\mathbb{R}^2$ and heteroclinic separatrices Σ, Σ' such that the associated canonical feasible sets are the same.

Contrary to what one might initially expect, the index of the global attractor plays no role in this topological classification. In fact, after extending the flow to the Riemann sphere, we get that ∞ is a repelling (topological) node (Remark 2.3). Hence, its index is 1 and, by the Poincaré–Hopf theorem [8, p. 179], the index of the attractor is 1 as well. Moreover, sharing (up to homeomorphisms) the same finite sectorial decomposition is a necessary but not sufficient condition for two such flows being topologically equivalent, see Figure 1.1. Likewise, compatibility alone is not enough to guarantee topological equivalence, see Figure 1.2.

Although the lemmas in Section 3 do not require finiteness of separatrices, no attempt has been done to find a more general version of Theorem A disposing of this restriction. Anyway, we are mainly interested in polynomial (local) flows, that is, those associated to polynomial vector fields, hence finiteness of separatrices is guaranteed (Remarks 2.1 and 2.4). Our next

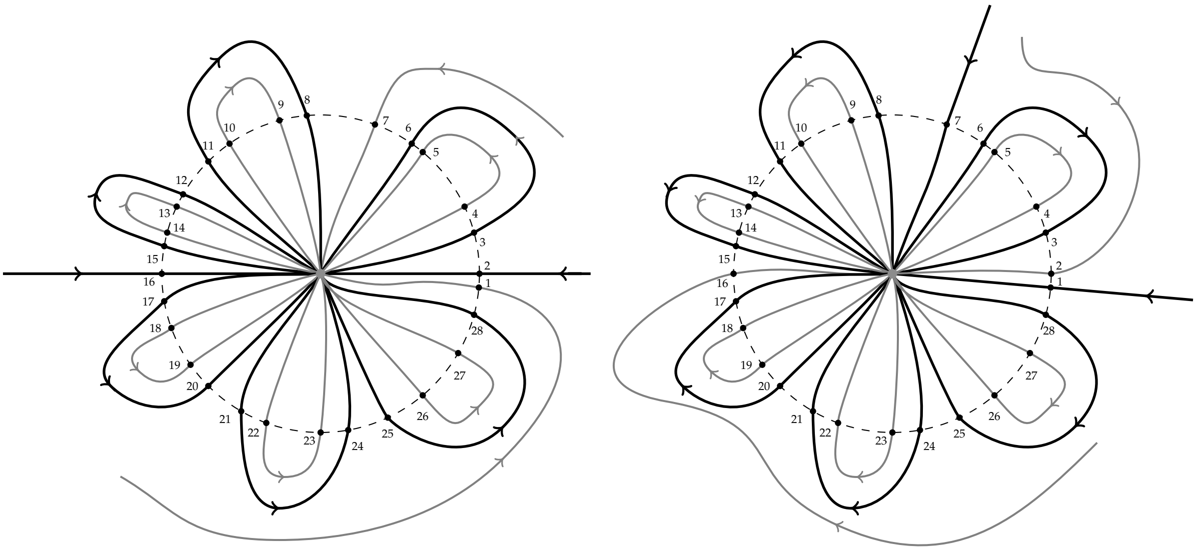

Figure 1.2: Two non-equivalent phase portraits with compatible separatrix skeleton (numbering indicating the compatibility bijection).

result, together with Theorem A, implies that if a flow has a globally attracting singular point, then it is equivalent to a polynomial flow.

Theorem B. Let L be a feasible set. Then there are a polynomial flow Φ (having $\mathbf{0}$ as a non-positively stable global attractor) and a heteroclinic separatrix Σ of Φ such that L is the canonical feasible set associated to Φ, the counterclockwise orientation in $\mathbb{R}^2$ and Σ.

Our proof of Theorem B depends heavily on the paper [19], where sufficient conditions are given allowing the associated flow to a C^1-vector field to be equivalent to a polynomial flow. It is worth emphasizing that these conditions, as explained in that paper, are not necessary: fortunately, the partial result in [19] turns out to be enough for our purposes. Still, this is not fully satisfying, because the arguments in [19] are essentially non-constructive. In fact, to the best of our knowledge, the literature provides no explicit examples of polynomial flows having a non-trivial globally attracting singular point. For this reason we finally prove:

Theorem C. The origin is both a global attractor and an elliptic saddle for the system

$$\begin{cases} x' = -((1+x^2)y + x^3)^5, \\ y' = y^2(y^2+x^3). \end{cases} \tag{1.1}$$

2 Preliminary notions

A number of standard topological notions will be of repeated use in this paper. We say that a topological space is an *arc* (respectively, *open arc*, *circle*, *disk*) if it is homeomorphic to $[0,1]$ (respectively, $\mathbb{R}$, the unit circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$, the unit disk $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$). If T is arc, and $h : [0,1] \to T$ is a homeomorphism, then $h(0)$ and $h(1)$ are called the *endpoints* of T. A *region* of a topological space X is an open, connected subset of X.

A *local flow* on a metric space (X,d) is a continuous map $\Phi : \Lambda \subset \mathbb{R} \times X \to X$ satisfying:

- Λ is open in $\mathbb{R} \times X$; moreover, for any $z \in X$ the set of numbers t for which $\Phi(t,z)$ is defined is an open interval $I_z = (a_z, b_z)$, with $-\infty \leq a_z < 0 < b_z \leq \infty$;
- $\Phi(0,z) = z$ for any $z \in X$;
- if $\Phi(t,z) = u$, then $I_u = \{s - t : s \in I_z\}$; moreover, $\Phi(r,u) = \Phi(r, \Phi(t,z)) = \Phi(r+t, z)$ for every $r \in I_u$.

In the particular case $\Lambda = \mathbb{R} \times X$, we call Φ a *flow* on X. Observe that if X is compact, then $I_z = \mathbb{R}$ for any $z \in X$, that is, any local flow on X is a flow. We write $\Phi_z(t) = \Phi_t(z) = \Phi(t,z)$ whenever it makes sense, when observe that if Φ is a flow, then the map $\Phi_t : X \to X$ is a homeomorphism for every t. We call $\varphi_\Phi(z) := \Phi_z(I_z)$. Here (as for the subsequent notions) we typically omit Φ in the subindex and write $\varphi(z)$ instead. If $\varphi(z) = \{z\}$ (when $I_z = \mathbb{R}$), then we call z a *singular point* of Φ; otherwise the orbit, and its points, are called *regular*. Since orbits foliate the space, that is, distinct orbits are disjoint, no point can be regular and singular at the same time. When the orbit $\varphi(z)$ is a circle (equivalently, the map $\Phi_z(t)$ is periodic), it is called *periodic*. If $I \subset I_z$ is an interval, then we call $\Phi_z(I)$ a *semi-orbit* of $\varphi(z)$. In the particular cases $I = [a,b]$ (with $\Phi_z(a) = p$, $\Phi_z(b) = q$) $I = [0, b_z)$ or $I = (a_z, 0]$, we rewrite $\Phi_z(I)$ as $\varphi(p,q)$, $\varphi(z,+)$ or $\varphi(-,z)$, respectively. We define the *ω-limit set* of the orbit $\varphi(z)$ (or the point z) as the set

$$\omega(z) = \{u \in X : \exists t_n \to b_z;\ \Phi_z(t_n) \to u\}.$$

The *α-limit set* $\alpha(z)$ is analogously defined (now $t_n \to a_z$).

We say that an orbit Γ is *positively* (respectively, *negatively*) *stable* if for any $p \in \Gamma$ and any $\epsilon > 0$ there is a number $\delta > 0$ (depending of p and ϵ) such that if $d(p,q) < \delta$, then all points from $\varphi(q,+)$ (respectively, $\varphi(-,q)$) stay at a distance less than ϵ from $\varphi(p,+)$ (respectively, $\varphi(-,p)$). We say that Γ is *stable* if it is both positively and negatively stable, and we say that it is *unstable* if it is not stable. It is worth emphasizing that these notions are not purely topological: they depend on the metric d.

A set $\Omega \subset X$ is *invariant* for Φ if it is the union of some orbits of Φ. If the restriction of Φ to $\Lambda \cap (\mathbb{R} \times \Omega)$ is a local flow on Ω (for instance, if Ω is invariant), then we call it, more simply if somewhat incorrectly, the *restriction of* Φ *to* Ω.

Let Φ and Ψ be respective local flows on the spaces X and Y. We say that Φ and Ψ are *topologically equivalent* if there is a homeomorphism $h : X \to Y$ such that $h(\varphi_\Phi(z)) = \varphi_\Psi(h(z))$ for every $z \in X$ which preserves the respective (time) directions of Φ and Ψ.

Local flows are associated, in a natural way, to (autonomous) systems of differential equations defined on smooth manifolds M (which will be seen here as embedded in $\mathbb{R}^m$ for some natural number m). Namely, if $\Phi : \Lambda \subset \mathbb{R} \times M \to M$ is a smooth local flow, then the vector field $f : M \to \mathbb{R}^m$ given by $f(z) = \frac{\partial \Phi}{\partial t}(0,z)$ (the *associated vector field* to Φ) is tangent to M and satisfies $\frac{\partial \Phi}{\partial t}(t,z) = f(\Phi(t,z))$, that is, the solution of the system $u' = f(u)$ with initial condition $u(0) = z$ is the map $\Phi_z(t) := \Phi(t,z)$, $t \in I_z$. Conversely, if a vector field $f : M \to \mathbb{R}^m$ is tangent to M and sufficiently smooth (locally Lipschitz is enough), and $\Phi_z(t)$ denotes the solution of $u' = f(u)$ satisfying $u(0) = z$, then $\Phi(t,z) := \Phi_z(t)$ is a local flow on M. While polynomial vector fields are the primary interest of this paper, and their associated flows are usually just local, there is a way to get rid of this restriction. In fact, if X is locally compact, $O \subset X$ is open, and Φ is a local flow on O, then there is a flow Ψ on X whose restriction to O has the same orbits and directions as those of Φ, and having singular points outside O [13, Lemma 2.3]. To simplify the notation we will call Φ, rather than Ψ, this extended flow, hoping that this will not lead to confusion. If Φ is associated to a polynomial vector field, then we also call it (and its extension) *polynomial*, although of course this map is not "polynomial" in the usual sense.

In concrete, we are interested in the case $O = \mathbb{R}^2$ and $X = \mathbb{R}^2_\infty = \mathbb{R}^2 \cup \{\infty\}$ (the one-point compactification of $\mathbb{R}^2$), when after identifying $\mathbb{R}^2_\infty$ with the Euclidean unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ via the stereographic projection $(u,v,w) \mapsto (x,y)$ given by $x = u/(1-w)$, $y = v/(1-w)$, we use in $\mathbb{R}^2_\infty$ (and then in $\mathbb{R}^2$) the distance $d(\cdot,\cdot)$ inherited from the Euclidean distance in $\mathbb{S}^2$. Hence, the topologies on $\mathbb{R}^2$ and $\mathbb{R}^2_\infty$ are the usual ones but $d(z,z') \leq 2$ for any $z,z' \in \mathbb{R}^2_\infty$. Unless otherwise stated, topological properties of subsets of $\mathbb{R}^2$ refer to the topology in $\mathbb{R}^2$. In particular, we mean $A \subset \mathbb{R}^2$ to be *bounded* in the conventional sense, that is, when it is contained in an Euclidean plane ball (while, of course, all sets in $\mathbb{R}^2$ are "bounded" regarding the distance $d(\cdot,\cdot)$).

Sphere and plane local flows have, as it is well known, some particularly good dynamical properties. The reader is assumed to be familiar with the basic facts of the Poincaré-Bendixson theory; for instance, recall that if z is regular, then there is a transversal to z for this flow. (If a local flow Φ can be restricted to a neighbourhood A of z so that it is topologically equivalent to that induced by the constant vector field $f_0 = (1,0)$ on the square $S = (-1,1) \times [-1,1]$, and the arc $T \subset A$ is the image of the vertical arc $\{0\} \times [-1,1]$ by the corresponding homeomorphism $h : S \to A$, with $h(\mathbf{0}) = h(0,0) = z$, then T is called a *transversal* to z for Φ, or just a transversal to Φ —or simply a transversal— when no emphasis on z is required. If all subarcs of an open arc or a circle Q are transversal to Φ, we similarly say that Q is *transversal* to Φ.)

There is a natural way to transport polynomial vector fields from $\mathbb{S}^2$ to $\mathbb{R}^2$. Namely, if $f : \mathbb{S}^2 \to \mathbb{R}^3$ is a polynomial vector field, tangent to $\mathbb{S}^2$ and vanishing at the north pole $(0,0,1)$ of $\mathbb{S}^2$, say $f(u,v,w) = (P(u,v,w), Q(u,v,w), R(u,v,w))$, then we can carry it, via the stereographic projection, to the plane vector field

$$g(x,y) = (1-w)^{-1}(P(u,v,w) + R(u,v,w)x, Q(u,v,w) + R(u,v,w)y)$$

with $u = 2x/(1+x^2+y^2)$, $v = 2y/(1+x^2+y^2)$, $w = (x^2+y^2-1)/(1+x^2+y^2)$, and after multiplying g by a appropriate power of $1+x^2+y^2$ we obtain a polynomial vector field whose associated (polynomial) flow is topologically equivalent to the flow induced by f on $\mathbb{S}^2 \setminus \{(0,0,1)\}$.

2.1 On special flows and regions

The standing assumption in this paper is that $\mathbf{0}$ is a globally attracting singular point for the flows Φ on $\mathbb{R}^2$ we deal with, that is, $\omega(z) = \{\mathbf{0}\}$ for any $z \in \mathbb{R}^2$. This is closely related to the notions of heteroclinicity and homoclinicity. We say that an orbit $\varphi(z)$ of Φ is *homoclinic* (respectively, *heteroclinic*) if (besides $\omega(z) = \{\mathbf{0}\}$) we have $\alpha(z) = \{\mathbf{0}\}$ (respectively, $\alpha(z) = \emptyset$ —that is, $\alpha(z) = \{\infty\}$ when using the extended flow to $\mathbb{R}^2_\infty$). Of course, the singular point $\mathbf{0}$ is trivially homoclinic. If Γ is homoclinic, then we denote by $E(\Gamma)$ the disk enclosed by the circle $\Gamma \cup \{\mathbf{0}\}$ (or just the singleton $\{\mathbf{0}\}$ in the case $\Gamma = \{\mathbf{0}\}$). Since $\mathbf{0}$ as a global attractor, any orbit of Φ is either heteroclinic or homoclinic (Lemma 3.1).

Let $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $1 \leq i \leq 4$, be the vector fields $f_1(x,y) = (x,-y)$, $f_2(x,y) = (-x,-y)$, $f_3(x,y) = (x,y)$, $f_4(x,y) = (x^2 - 2xy, xy - y^2)$ respectively. Also, let

$$A_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x,y < 1, xy < 1/2\},$$

$$A_2 = A_3 = A_4 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x,y < 1, x^2+y^2 < 1\}.$$

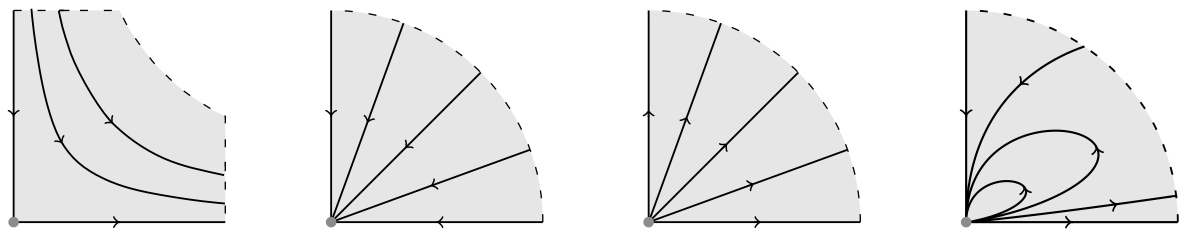

Figure 2.1: From left to right: a hyperbolic, an attracting, a repelling and an elliptic sector.

We remark that although the sets A_i are not open, f_i still induces a local flow Φ_i on A_i, $1 \le i \le 4$. See Figure 2.1. Assume now that B is a set containing $\mathbf{0}$ and Φ induces a local flow on B which is topologically equivalent to Φ_i. Then we say that B is a *hyperbolic*, *attracting*, *repelling* or *elliptic sector* of Φ (at $\mathbf{0}$) when, respectively, $i = 1, 2, 3, 4$. The flow Φ is said to have the *finite sectorial decomposition property* (at $\mathbf{0}$) if either $\mathbf{0}$ is positively stable or has a neighbourhood which is the (minimal) union of at least two, but finitely many, hyperbolic, attracting, repelling and elliptic sectors (since Φ admits no periodic orbits, see also Proposition 3.9, this amounts to the standard definition to be found, for instance, in [8, p. 18]).

Remark 2.1. The typical case for this to happen is that Φ is associated to a vector field (real) analytic at $\mathbf{0}$, see for instance [8, Chapter 3].

We call a region $\Omega \subset \mathbb{R}^2$ *radial* (respectively, a *strip*) if it is invariant for Φ, and, when restricted to Ω, Φ is topologically equivalent to the flow induced by f_2 on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ (respectively, in the upper half-plane $\mathbb{H} = \mathbb{R} \times (0, \infty)$). Needless to say, to define strips, one can equivalently use (as it is usually done) the associated flow to the constant vector field f_0 on $\mathbb{R}^2$. If all orbits of a strip Ω are heteroclinic (respectively, homoclinic), then we call Ω *heteroclinic* (respectively, *homoclinic*) as well. Observe that, in general, the interior of a hyperbolic, attracting or repelling sector is not a strip because it is not invariant (it does not consist of full orbits of Φ). We say that the strip Ω is *strong* if there are orbits Γ_1, Γ_2 in $\operatorname{Bd} \Omega$ such that the restriction of Φ to $\Omega \cup \Gamma_1 \cup \Gamma_2$ is equivalent to that of the flow induced by f_2 on $\operatorname{Cl} \mathbb{H} \setminus \{\mathbf{0}\}$. If, moreover, $\operatorname{Cl} \Omega = \Omega \cup \Gamma_1 \cup \Gamma_2 \cup \{\mathbf{0}\}$, then we say that Ω is *solid*.

Remark 2.2. If Ω is a solid strip, then either all Ω, Γ_1 and Γ_2 are heteroclinic, or all of them are homoclinic. Otherwise, as it is easy to check, either (a) one of orbits, say Γ_1, is heteroclinic, Γ_2 is homoclinic and $\Omega = \mathbb{R}^2 \setminus (\Gamma_1 \cup E(\Gamma_2))$, or (b) both Γ_1 and Γ_2 are homoclinic, with $E(\Gamma_1) \cap E(\Gamma_2) = \{\mathbf{0}\}$, and $\Omega = \mathbb{R}^2 \setminus (E(\Gamma_1) \cup E(\Gamma_2))$. Use Lemma 3.2 to find a heteroclinic orbit $\Gamma \subset \Omega$. Clearly, Γ cannot disconnect $\Omega \cup \Gamma_1 \cup \Gamma_2$, which contradicts that Ω is strong.

If Q is a transversal circle (respectively, open arc) with the property that, for every $z \in Q$, $\varphi(z)$ intersects Q exactly at z, then $\Omega = \bigcup_{z \in Q} \varphi(z)$ is radial (respectively, a strip). To construct the corresponding homeomorphism $h : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \Omega$ (respectively, $h : \mathbb{H} \to \Omega$) just fix a homeomorphism $f : \mathbb{S}^1 \to Q$ (respectively, $f : \mathbb{S}^1 \cap \mathbb{H} \to Q$) and write $h(e^{-t + \mathrm{i}\theta}) = \Phi(t, f(e^{\mathrm{i}\theta}))$. Conversely, if $\Omega \subset \mathbb{R}^2$ is radial (respectively, a strip) then there is a circle (respectively, an open arc) $Q \subset \Omega$, transversal to Φ, having exactly one common point with every orbit in Ω. We call any such set Q a *complete transversal* to Ω. If Ω is a strong strip, then more is true: there is a transversal arc T having exactly one common point with every orbit in Ω and every orbit Γ_1, Γ_2. We call T a *strong transversal* to Ω.

Remark 2.3. If Ω is radial, and the circle C is a complete transversal to Ω, then it must enclose $\mathbf{0}$. Hence all heteroclinic orbits intersect C, that is, Ω is the union set of all heteroclinic orbits of Φ; in other words, Φ admits one radial region at most (later we will see, Proposition 3.9, that such a region does exist). Moreover, the circles $\Phi_t(C)$ tend uniformly to ∞ as $t \to -\infty$. In fact, if $D_t \subset \mathbb{R}^2_\infty$ is the disk containing ∞ and having $\Phi_t(C)$ as its boundary, then $D_t = \{\Phi_s(u) : u \in C, s \leq t\} \cup \{\infty\}$. Since these disks intersect exactly at ∞, we get $\operatorname{diam}(D_t) \to 0$ as $t \to -\infty$, and the uniform convergence to ∞ follows. As a corollary, all heteroclinic orbits are negatively stable.

Similarly, if Ω is a solid strip and T is a strong transversal to Ω, then $\Phi_t(T)$ tends uniformly to $\mathbf{0}$ as $t \to \infty$, and tend uniformly to $\mathbf{0}$ as $t \to -\infty$ in the homoclinic case, and to ∞ in the heteroclinic case. In particular, all orbits of a solid strip are stable, and if it is heteroclinic (respectively, homoclinic), then the flow induced by f_2 on $\mathrm{Cl}\,\mathbb{H}$ (respectively, by f_4 on the union set of $\mathbf{0}$ and all orbits intersecting the diagonal arc $\{(x,x) : 1/2 \leq x \leq 1\}$) is topologically equivalent to the restriction of Φ to $\mathrm{Cl}\,\Omega$.

If an orbit is not contained in any solid strip, then it is called a *separatrix* of Φ. Note that the union set X of all separatrices of Φ is closed. The components of $\mathbb{R}^2 \setminus X$ are called the *canonical regions* of Φ. A family of orbits of Φ consisting of all its separatrices and exactly one orbit from every canonical region is called a *separatrix skeleton* of Φ. Observe that any regular separatrix can belong to the boundary of, at most, two different canonical regions. Therefore, if the number of separatrices is finite, so it the number of canonical regions.

Remark 2.4. As indicated in Remark 2.3, any unstable orbit must be a separatrix. If Φ has the finite sectorial decomposition property, then Γ is a separatrix if and only if it is either the singular point, or includes a semi-orbit limiting a hyperbolic sector. In particular, Φ has finitely many separatrices and Γ is a separatrix if and only if it is unstable.

The next result is a particular case of [15, Theorems 5.2 and 7.1], see also [16] and [9].

Proposition 2.5. *Any canonical region of Φ is either radial or a strip.*

Remark 2.6. A strip (even a strong strip) needs not be either heteroclinic or homoclinic. Nevertheless, if a canonical region is a strip, then it must be either heteroclinic or homoclinic (because, in this case, the set of its heteroclinic orbits and the set of its homoclinic orbits are both open; hence, by connectedness, one of them must be empty).

Theorem 2.7. *Assume that $\mathbf{0}$ is a global attractor for two flows Φ and Φ' and let $\mathcal{X}$ and $\mathcal{X}'$ denote some separatrix skeletons for Φ and Φ'. Then Φ and Φ' are topologically equivalent if and only if there is a homeomorphism from the plane onto itself mapping the orbits of $\mathcal{X}$ onto the orbits of $\mathcal{X}'$ and preserving the flows directions.*

Remark 2.8. Our definition of separatrix is not the standard one (compare to [15], [16], [17, p. 294] or [8, p. 34]), even when we restrict ourselves, as it is the case here (Lemma 3.1), to flows having only heteroclinic or homoclinic orbits. More precisely, our "separatrices" are what we called "separators" in [9] (and our "canonical regions" what we called "standard regions" there). If the boundary of a heteroclinic strip consists of the singular point and two heteroclinic orbits, then it is solid (the corresponding strong transversal can be found with the help of Lemma 3.7). If we replace "heteroclinic" by "homoclinic", this needs not happen unless we additionally assume that the ordering "$\prec$" we introduce below Lemma 3.2 totally orders the orbits of the closure of the strip. This point is missed in the above-mentioned references and, as a consequence, Theorem 2.7, as stated there, does not work, see [9] for the details. Surprisingly, it seems that this fact has passed unnoticed until now.

2.2 On orientations and the extension of homeomorphisms

Let C be a circle around $\mathbf{0}$. If Γ is heteroclinic, we call the last point of Γ in C (that is, the point $q \in \Gamma \cap C$ such that $\Phi_q(t) \notin C$ for any $t > 0$) the *ω-point* of Γ in C. Likewise, if Γ is regular and homoclinic and C is small enough so that there are points of Γ not enclosed by C, then we call the first and last points of Γ in C (that is, the points $p, q \in \Gamma \cap C$ such that $\Phi_p(t) \notin C$ for any $t < 0$ and $\Phi_q(t) \notin C$ for any $t > 0$) the *α-point* and the *ω-point* of Γ in C, respectively.

If $\mathcal{P}$ is a finite family of orbits of Φ, and C is a circle around $\mathbf{0}$ small enough, then we denote by $\Delta_\Phi(\mathcal{P}, C)$ the set of all α- and ω-points in C from the orbits in $\mathcal{P}$ and call it the *configuration* of $\mathcal{P}$ in C. Note that the possibility that the singular point belongs to $\mathcal{P}$ is not excluded, when of course it adds no points to $\Delta_\Phi(\mathcal{P}, C)$. Also, observe that all configurations of $\mathcal{P}$ are essentially the same, that is, if C and C' are small circles around $\mathbf{0}$, then there is an orientation preserving homeomorphism $h : C \to C'$ mapping the α- and ω-points in C of every orbit $\Gamma \in \mathcal{P}$ to the α- and ω-points in C' of that same orbit Γ.

We call a triplet (A, B, C) of arcs in $\mathbb{R}^2_\infty$ sharing a common endpoint p (and no other point) a *triod*. The point p is called the *vertex* of the triod, the other endpoints of the arcs A, B, C being called its *endpoints*. We say that the triod (A, B, C) is *positive*, when, after taking an open euclidean ball U of center p and radius $\epsilon > 0$ small enough, there is $\theta_0 \in \mathbb{R}$ such that the first intersection points of these arcs with $\operatorname{Bd} U$ can be written as $p + \epsilon e^{\mathbf{i}\theta_A}, p + \epsilon e^{\mathbf{i}\theta_B}, p + \epsilon e^{\mathbf{i}\theta_C}$, with $\theta_0 = \theta_A < \theta_B < \theta_C < \theta_0 + 2\pi$. We say that the triod is *negative* when it is not positive. Observe that the definition above excludes the case when the common endpoint p is ∞. We then say that (A, B, C) is positive when $(G(A), G(B), G(C))$ is negative, $G : \mathbb{R}^2_\infty \to \mathbb{R}^2_\infty$ being defined by $G(z) = 1/\bar{z}$ (here we identify $\mathbb{R}^2$ with $\mathbb{C}$ and mean $G(\infty) = \mathbf{0}$, $G(\mathbf{0}) = \infty$). If C is a circle around $\mathbf{0}$ and (q, q', q'') is a triplet of distinct points in C, then we call it *positive* or *negative* according to whether it is counterclockwise or clockwise oriented in C, that is, there is a positive (negative) triod (A, A', A'') in the disk enclosed by C with vertex $\mathbf{0}$ and endpoints q, q', q''. If Γ is homoclinic, then we say that it is *positive* (respectively, *negative*) when, after taking $\Gamma' \subset \operatorname{Int} E(\Gamma)$ and a small circle C around $\mathbf{0}$, the α- and ω-points p, q of Γ in C, and the ω-point q' of Γ' in C, we get that (p, q', q) is positive (respectively, negative). In simpler words, Γ is positive (negative) when the flow induces the counterclockwise (clockwise) orientation on $\Gamma \cup \{\mathbf{0}\}$.

Let $P, P' \subset \mathbb{R}^2$ (respectively, $P, P' \subset \mathbb{R}^2_\infty$). We say that P and P' are $\mathbb{R}^2$-*compatible* (respectively, $\mathbb{R}^2_\infty$-*compatible*) if there is a homeomorphism H from $\mathbb{R}^2$ (respectively, $\mathbb{R}^2_\infty$) onto itself mapping P onto P'. Clearly, $\mathbb{R}^2$-homeomorphisms amount to $\mathbb{R}^2_\infty$-homeomorphisms mapping ∞ to itself. If $H : \mathbb{R}^2_\infty \to \mathbb{R}^2_\infty$ is a homeomorphism, then, as it is well known, either it preserves the orientation, that is, all pairs of triods (A, B, C) and $(H(A), H(B), H(C))$ have the same sign, or it reverses the orientation, that is, all pairs of triods (A, B, C) and $(H(A), H(B), H(C))$ have opposite sign. As it turns out, see [1], this is the key property to identify compatibility: two Peano sets P and P' (by a *Peano space* we mean a compact, connected, locally connected set) in $\mathbb{R}^2_\infty$ are $\mathbb{R}^2_\infty$-compatible if and only if there is a homeomorphism $h : P \to P'$ either preserving or reversing the orientation, in the former sense, for all pair of triods (A, B, C) and $(h(A), h(B), h(C))$ in P and P' (when h can indeed be homeomorphically extended to the whole $\mathbb{R}^2_\infty$).

The former result can be adapted to the $\mathbb{R}^2$-setting as follows. We say that $P \subset \mathbb{R}^2$ is *nice* if it is unbounded, $P_\infty = P \cup \{\infty\}$ is a Peano subset of $\mathbb{R}^2_\infty$, and for any triod (A, B, C) in P_∞ with vertex ∞ there is a θ-curve in P_∞ including A, B and C (by a *θ-curve* we mean a union of three arcs intersecting exactly at their endpoints). Then we get: two nice sets P, P' are $\mathbb{R}^2$-

compatible if and only if there is a homeomorphism $h : P \to P'$ either preserving or reversing the orientation for all pair of triods (A, B, C) and $(h(A), h(B), h(C))$ in P and P' (when, again, h can indeed be homeomorphically extended to the whole $\mathbb{R}^2$).

Assume that $\mathcal{P}$ and $\mathcal{P}'$ are finite families of orbits of, respectively, Φ and Φ' (we also assume that both of them contain the globally attracting singular point $\mathbf{0}$ and at least one heteroclinic and one homoclinic orbit). Let P and P' be the union sets of these orbits and note that these sets are nice. Then, as it is simple to check, a condition characterizing the $\mathbb{R}^2$-compatibility of P and P' (when we accordingly say that $\mathcal{P}$ and $\mathcal{P}'$ are *compatible*) is the existence of a *compatibility bijection*. By this we mean a bijection $\xi : \mathcal{P} \to \mathcal{P}'$ for which there is a homeomorphism $\mu : C \to C'$, with C and C' small circles around $\mathbf{0}$, mapping $\Delta_\Phi(\mathcal{P}, C)$ onto $\Delta_{\Phi'}(\mathcal{P}', C')$, so that $\mu(C \cap \Gamma) = C' \cap \xi(\Gamma)$ for any $\Gamma \in \mathcal{P}$. In this case we say that μ *preserves orbits for* ξ.

If, additionally, μ maps ω-points onto ω-points (when we say that μ *preserves directions for* ξ), then the corresponding plane homeomorphism preserves the flows directions on $\mathcal{P}$ and $\mathcal{P}'$. If, moreover, these families are the separatrix skeletons of Φ and Φ', Theorem 2.7 implies that the flows are equivalent.

2.3 A lemma on Janiszewski spaces

A compact connected Hausdorff space is called a *continuum*. We say that a topological space X is a *Janiszewski space* it is a locally connected continuum and, moreover, for any subcontinua $C_1, C_2 \subset X$ with the property that $C_1 \cap C_2$ is not connected, there are points $x, y \in X \setminus (C_1 \cup C_2)$ which are simultaneously contained in no subcontinuum in $X \setminus (C_1 \cup C_2)$. By [14, Fundamental Theorem 6, p. 531], a topological space X is homeomorphic to $\mathbb{R}^2_\infty$ if and only if it is a Janiszewski space, contains more than one point, and, for any $x \in X$, the set $X \setminus \{x\}$ is connected. If X is a Janiszewski space, Y is Hausdorff and there is a continuous monotone map mapping X onto Y, then Y is Janiszewski as well (we say that $f : X \to Y$ is *monotone* if $f^{-1}(A)$ is connected whenever $A \subset Y$ is connected). In fact, this is proved in [14, Theorem 9, p. 507] additionally assuming that Y is a locally connected continuum; but if X is a locally connected continuum, Y is Hausdorff, and X can be continuously mapped onto Y, then Y is indeed a locally connected continuum, as seen in [14, Theorem 9, p. 259].

Let $K \subset \mathbb{R}^2_\infty$ be a continuum such that $\mathbb{R}^2_\infty \setminus K$ is connected. We define the equivalence relation "$\sim_K$" in $\mathbb{R}^2_\infty$ by $x \sim_K y$ if either $x = y$ or both x and y belong to K. Then we have:

Lemma 2.9. *The quotient space $\mathcal{Q} = \mathbb{R}^2_\infty / \sim_K$ is homeomorphic to $\mathbb{R}^2_\infty$.*

Proof. According to the previous discussion, if $\Pi : \mathbb{R}^2_\infty \to \mathcal{Q}$ is the projection map (when recall that $\mathcal{U}$ is open in $\mathcal{Q}$ if and only if $\Pi^{-1}(\mathcal{U})$ is open in $\mathbb{R}^2_\infty$), then, in order to prove that $\mathcal{Q}$ is homeomorphic to $\mathbb{R}^2_\infty$, we just need to show:

(i) $\mathcal{Q}$ is Hausdorff;

(ii) $\mathcal{Q} \setminus \{X\}$ is connected for any $X \in \mathcal{Q}$;

(iii) $\Pi^{-1}(\mathcal{C})$ is connected for any connected set $\mathcal{C} \subset \mathcal{Q}$.

Statements (i) and (ii) are immediate because of the assumptions on K. To prove (iii) we use that Π is a closed map by (i) and then apply [14, Theorem 9, p. 131] and the fact that any $X \in \mathcal{Q}$ is a connected subset of $\mathbb{R}^2_\infty$. □

3 General results on global attraction

Recall that we assume that $\mathbf{0}$ is a global attractor for Φ.

Lemma 3.1. *All orbits of Φ are either homoclinic or heteroclinic.*

Proof. If the statement of the lemma is not true, then there is some point $z \in \mathbb{R}^2$ such that $\alpha(z)$ contains a regular point u. Let T be a transversal to u. According to some well-known Poincaré–Bendixson theory, we can find $p, q \in \varphi(z) \cap T$ so that $\varphi(p,q) \cup S$ (where S is the arc in T whose endpoints are p and q) is a circle enclosing a disk D in $\mathbb{R}^2_\infty$ which contains $\varphi(-,p)$, and hence $\alpha(z)$, and intersects $\varphi(q,+)$ just at q. This is impossible: on the one hand, $\mathbf{0}$ cannot belong to D, because it is the ω-limit set of $\varphi(q)$; on the other hand, $u \in \alpha(z)$ implies $\omega(u) \subset \alpha(z)$, so $\mathbf{0}$ does belong to D. $\square$

Lemma 3.2. *The union set of all homoclinic orbits of Φ is bounded.*

Proof. Assume the opposite to find a family of homoclinic orbits $\{\varphi(z_n)\}_{n=1}^\infty$ with $z_n \to \infty$ as $n \to \infty$ and fix a circle C around $\mathbf{0}$. Using the continuity of the (extended) flow Φ at ∞, there is no loss of generality in assuming that the semi-orbits $\Phi_{z_n}([-n,0])$ do not intersect the region O encircled by C. Next, find the numbers $a_n \leq -n$, closest to $-n$, such that the points $\Phi_{z_n}(a_n)$ belong to C (using that the orbits $\varphi(z_n)$ are homoclinic) and assume, again without loss of generality, that the points $u_n = \Phi_{z_n}(a_n)$ converge to u. Since $\Phi_{u_n}(t) \in \mathbb{R}^2 \setminus O$ for any $t \in [0,n]$, the continuity of the flow implies that $\varphi(u,+)$ does not intersect O, contradicting that $\mathbf{0}$ is a global attractor. $\square$

Let $\mathcal{H}$ denote the family of homoclinic orbits of Φ. We introduce a partial order in $\mathcal{H}$ by writing $\Gamma \preceq \Sigma$ if $\Gamma \subset E(\Sigma)$, when $\Gamma \prec \Sigma$ means of course $\Gamma \preceq \Sigma$ with $\Gamma \neq \Sigma$. We say that $\Gamma \in \mathcal{H}$ is *maximal* if there is no $\Sigma \in \mathcal{H}$ such that $\Gamma \prec \Sigma$. If $\Gamma, \Sigma \in \mathcal{H}$ and neither $\Gamma \preceq \Sigma$ nor $\Sigma \preceq \Gamma$ is true, then we say that Γ and Σ are *incomparable*. Realize that a family of pairwise incomparable orbits must be countable. Moreover, we have the following lemma.

Lemma 3.3. *If the orbits $\{\Gamma_n\}_{n=1}^\infty$ are pairwise incomparable, then* $\operatorname{diam}(\Gamma_n) \to 0$ *as $n \to \infty$.*

Proof. Suppose the contrary to get a point $u \neq \mathbf{0}$ at which these orbits accumulate. Let T be a transversal to u and find points $u_{n_k} \in \Gamma_{n_k} \cap T$, $k = 1,2,3$, with, say, u_{n_2} lying between u_{n_1} and u_{n_3} in T. Then u_{n_1} and u_{n_3} belong to different regions in $\mathbb{R}^2 \setminus (\Gamma_{n_2} \cup \{\mathbf{0}\})$: we are using here that any homoclinic orbit can intersect a transversal at one point at most. Thus, either $\Gamma_{n_1} \prec \Gamma_{n_2}$ or $\Gamma_{n_3} \prec \Gamma_{n_2}$, contradicting the hypothesis. $\square$

Lemma 3.4. *Let $\Omega \subsetneq \mathbb{R}^2$ be a region invariant for Φ.*

(i) If Ω is bounded, then $\operatorname{Bd}\Omega$ *is the union set of a homoclinic orbit Σ, a (possibly empty) family $\mathcal{G}$ of pairwise incomparable homoclinic orbits satisfying $\Gamma \prec \Sigma$ for every $\Gamma \in \mathcal{G}$, and the singular point.*

(ii) If Ω is unbounded, then its boundary is the union set of at most two heteroclinic orbits, a (possibly empty) family of pairwise incomparable homoclinic orbits, and the singular point.

Proof. Since Ω in invariant, $\operatorname{Bd}\Omega$ is invariant as well, and the statement (ii) follows easily from the connectedness of Ω. To prove (i), assume that the boundary of the bounded region Ω is not as described and realize that then we must have $\operatorname{Bd}\Omega = \{\mathbf{0}\} \cup \bigcup_n \Gamma_n$ for a family $\{\Gamma_n\}_n$ (having at least two elements) of pairwise incomparable homoclinic orbits. Lemma 3.3 implies that $O = \mathbb{R}^2 \setminus \bigcup_n E(\Gamma_n)$ is a region including Ω with the same boundary as Ω. Hence $\Omega = O$, contradicting that Ω is bounded. $\square$

Lemma 3.5. *Let $\Gamma \in \mathcal{H}$. Then there is $\Sigma \in \mathcal{H}$, maximal for "$\prec$", such that $\Gamma \preceq \Sigma$.*

Proof. If Γ is not maximal itself, then the Jordan curve theorem implies that the non-empty family $\mathcal{F} = \{\Gamma' \in \mathcal{H} : \Gamma \preceq \Gamma'\}$ is a totally ordered subset of $\mathcal{H}$; accordingly, it is enough to show that $\mathcal{F}$ has a maximal element for $\preceq$.

Say $\mathcal{F} = \{\Gamma_i\}_i$. Then, because of the total ordering, $\Omega = \bigcup_i \operatorname{Int} E(\Gamma_i)$ is a region invariant for Φ, and because of Lemma 3.2, Ω is bounded. As a result, we can apply Lemma 3.4(i) to obtain the corresponding homoclinic boundary orbit Σ. Then, clearly, Σ is the maximal element of $\mathcal{F}$. $\square$

Remark 3.6. Note that all maximal homoclinic orbits of Φ are separatrices.

Lemma 3.7. *Let z be a regular point. Then there is a transversal T to z such that, for every $u \in T$, $\varphi(u)$ intersects T exactly at u.*

Proof. Fix an arc Q transversal to z. Note that no orbit can intersect Q infinitely many times. Also, if some orbit intersects Q at consecutive times $t < s$ and corresponding points u and v, then no orbit can intersect the open arc in Q with endpoints u and v more than once. Using these two facts it is easy to construct a transversal $T \subset Q$ to z with endpoints p and q such that the orbits $\varphi(p)$ and $\varphi(q)$ intersect T at exactly p and q. This is the transversal we are looking for, because if an orbit Γ consecutively intersects T at points u and v, and D is the disk in $\mathbb{R}^2_\infty$ enclosed by $\varphi(u, v)$ and the arc in T with endpoints u and v such that $\mathbf{0} \in D$, then either $\varphi(p)$ or $\varphi(q)$ does not intersect D, a contradiction. $\square$

Lemma 3.8. *If Ω is a canonical region and Γ, Γ' are distinct orbits in Ω, then there is a solid strip $S \subset \Omega$ such that $\operatorname{Bd} S = \Gamma \cup \Gamma' \cup \{\mathbf{0}\}$.*

Proof. Let Q be a complete transversal to Ω and let $A \subset Q$ be an arc with endpoints belonging to Γ and Γ'. Since Ω includes no separatrices, for any point $z \in Q$ there is a solid strip in Ω, containing z, whose closure intersects Q at a small arc in Q (this small arc thus being a strong transversal to the strip). Taking this into account, and applying a simple compactness argument to A, the lemma follows. $\square$

Recall that Φ admits one radial region at most, that consisting of all heteroclinic orbits of Φ (Remark 2.3). Indeed, such is the case:

Proposition 3.9. *Let R be the union set of all heteroclinic orbits of Φ. Then it is radial. Moreover:*

(i) If $R = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, that is, all regular orbits of Φ are heteroclinic, then Φ is topologically equivalent to the associated flow to $f_2(x, y) = (-x, -y)$ in $\mathbb{R}^2$ (hence $\mathbf{0}$ is positively stable and it is the only separatrix of Φ).

(ii) If $R \neq \mathbb{R}^2 \setminus \{\mathbf{0}\}$, then R includes a separatrix of Φ.

Proof. First we assume $R = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. To prove that R is radial and (i) holds, it suffices to show that $\mathbf{0}$ is the only separatrix of Φ (Proposition 2.5 and Theorem 2.7). Take $z \in R$ and let $T \subset R$ be an arc transversal to z with the property that the orbits of all its points intersect T exactly once (Lemma 3.7). Let p and q be the endpoints of T and let D be the disk in $\mathbb{R}^2_\infty$ enclosed by $\varphi(p)$, $\varphi(q)$, $\mathbf{0}$ and ∞ and including T. If $u \in \operatorname{Int} D$, then $\varphi(u)$ intersect T (because it is heteroclinic). Therefore, $\operatorname{Int} D$ is a heteroclinic solid strip; in particular, $\varphi(z)$ is not a separatrix.

Assume now $R \neq \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Applying Lemma 2.9 to the union set $K = \mathbb{R}^2 \setminus R$ of all sets $E(\Gamma)$ with Γ maximal for "$\prec$" (recall also Lemmas 3.3 and 3.5), and using (i), we can construct a topological equivalence between the restriction of Φ to R and the restriction (to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$) of the associated flow to f_2. In particular, R is radial.

To prove the last statement of the proposition, assume that R includes no separatrices (hence it is a canonical region by Proposition 2.5), fix a complete transversal circle C to R and apply Lemma 3.8 (recall also Remark 2.3) to conclude the uniform convergence of $\Phi_t(C)$ to $\mathbf{0}$ and ∞ as $t \to \pm\infty$. Then $R = \bigcup_{t \in \mathbb{R}} \Phi_t(C) = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, a contradiction. □

4 Proof of Theorem A

In this section we assume, besides global attraction, that $\mathbf{0}$ is not positively stable and Φ has finitely many separatrices.

Let $\mathcal{X}$ be a separatrix skeleton for Φ, fix a small circle C around $\mathbf{0}$ and let $X = \Delta_\Phi(\mathcal{X}, C)$ be the configuration of Φ in C. Also, fix an orientation Θ (counterclockwise or clockwise) in $\mathbb{R}^2$ and a heteroclinic separatrix Σ in $\mathcal{X}$ (such an orbit exists because of Proposition 3.9 (ii)). Let q be the ω-point of Σ in C. Find disjoint open arcs $J, J' \subset C$ whose closures have q as their common endpoint (small enough so that they do not contain any points from X), take points $p \in J$, $p' \in J'$, and assume that they are labelled so that the orientation of (p, q, p') in C is that given by Θ (that is, (p, q, p') is positive if and only if Θ is the counterclockwise orientation). Finally, after removing J' from C, we get an arc A with endpoints a (the other endpoint of the closure of J') and q, and order the points from A in the natural way so that $a < q$.

We call positive (negative) homoclinic orbits *even* when Θ is the counterclockwise (clockwise) orientation, and *odd* when Θ is the clockwise (counterclockwise) orientation. Thus, a homoclinic orbit from $\mathcal{X}$ is even if and only if its α-point v and its ω-point w satisfy $v < w$. By convention, all heteroclinic orbits are even. We say that two orbits have the *same parity* when both are even or both are odd.

According to Proposition 2.5 and, again, Proposition 3.9 (ii), all canonical regions are indeed strips, so we will call them *canonical strips*. Recall (Remark 2.6) that any canonical strip must be either heteroclinic or homoclinic. By Lemma 3.4, the boundary of any heteroclinic canonical strip Ω consists of (apart from $\mathbf{0}$) two heteroclinic separatrices (or just Σ, when $\Omega = R \setminus \Sigma$ is the union set of all heteroclinic orbits except Σ) and several (possibly zero) maximal homoclinic separatrices (Remark 3.6), when Ω is called *elementary* if and only if this last set is empty. Likewise, the boundary of a homoclinic canonical strip Ω consists of, apart from $\mathbf{0}$, a homoclinic separatrix Γ enclosing it and possibly some others, all of them less than Γ in the $\prec$-ordering, when we again call Ω *elementary* if this last family is empty. Note that is quite possible for a canonical strip to be elementary, but at least one heteroclinic canonical strip cannot be elementary (otherwise Φ would have no homoclinic separatrices, and consequently all its regular orbits would be heteroclinic, contradicting Proposition 3.9(i)).

Remark 4.1. The following statements are easy to prove:

- a heteroclinic canonical strip is elementary if and only if it is solid;
- a homoclininic canonical strip is elementary if and only if the restriction of Φ to its closure is topologically equivalent to the flow induced by the "elliptic vector field"

$f_4(x,y) = (x^2 - 2xy, xy - y^2)$ on the union set A'_4 of all orbits intersecting the diagonal arc $\{(x,x), 0 \leq x \leq 1\}$.

Remark 4.2. If a regular homoclinic separatrix Γ is minimal, that is, $E(\Gamma)$ is an elementary homoclinic canonical strip, then there is an elliptic sector intersecting $E(\Gamma)$ (Remark 4.1). Thus, due to Remark 2.4, if $\mathbf{0}$ is not positively stable, and the finite sectorial decomposition property holds, then the decomposition must include both an elliptic and a hyperbolic sector.

There are two natural ways to associate an orbit from $\mathcal{X}$ to each canonical strip Ω of Φ. Firstly, $\gamma'(\Omega)$ will denote the orbit from $\mathcal{X}$ included in Ω. Next, $\gamma(\Omega)$ will denote (when Ω is homoclinic) the separatrix $\Gamma \subset \operatorname{Bd}\Omega$ enclosing Ω, and (when Ω is heteroclinic) the heteroclinic separatrix $\Gamma \subset \operatorname{Bd}\Omega$ whose ω-point w (in C, and then in A) satisfies $v < w$, v being the ω-point of $\gamma'(\Omega)$. Note that $\mathcal{X}$ consists of all orbits $\gamma(\Omega), \gamma'(\Omega)$ together with $\mathbf{0}$. Also, observe that $\gamma'(\Omega)$ decomposes Ω into two components Ω_l and Ω_u, Ω_u being the component of $\Omega \setminus \gamma'(\Omega)$ including $\gamma(\Omega)$ in its boundary (an ambiguity arises in the case $\Omega = R \setminus \Sigma$, where Ω_u consists of the orbits whose ω-points are greater than the ω-point of $\gamma'(\Omega)$).

Lemma 4.3. *Let Ω be a canonical strip and let Γ be a regular orbit in* $\operatorname{Bd}\Omega$*. Then Γ has the same parity as $\gamma'(\Omega)$ if and only if either $\Gamma = \gamma(\Omega)$ or $\Gamma \in \operatorname{Bd}\Omega_l$.*

Proof. We present the proof under the hypothesis that Ω is a heteroclinic strip whose boundary includes two heteroclinic orbits, $\gamma(\Omega)$ and $\gamma''(\Omega)$. The case when Ω is heteroclinic but Σ is the only heteroclinic separatrix of Φ, and the homoclinic case, can be dealt with in analogous fashion. We will also assume that the fixed orientation Θ is counterclockwise so the even (respectively odd) homoclinic orbits coincide with the positive (respectively negative) ones.

If Ω is elementary, then there is nothing to prove: both Γ and $\gamma'(\Omega)$ are heteroclinic and consequently even. Otherwise, let $\Gamma_1, \ldots, \Gamma_j$ be the maximal homoclinic orbits in $\operatorname{Bd}\Omega$, where these orbits are labelled in such a way that if $q_1, \ldots, q_j$ are the corresponding ω-points, then $q_1 < \cdots < q_j$ (in A). The corresponding α-points will be denoted by p_k, $1 \leq k \leq j$. Finally, let u, v and w be the ω-points of $\gamma''(\Omega)$, $\gamma'(\Omega)$ and $\gamma(\Omega)$, respectively (so $u < v < w$). We can assume without loss of generality that there are small subarcs of C, neighbouring all these points, which are transversal to the flow.

Let $1 \leq k \leq j-1$. We claim that it is not possible that Γ_k is negative and Γ_{k+1} is positive. Assume by contradiction $q_k < p_k < p_{k+1} < q_{k+1}$. Find points $p_k < b < b' < p_{k+1}$ in C, very close to p_k and p_{k+1}, respectively, so that $T = \{t \in C : p_k \leq t \leq b\}$, $T' = \{t \in C : b' \leq t \leq p_{k+1}\}$ are transversal to the flow. Also, let $Q = \{t \in C : b \leq t \leq b'\}$. Since Γ_k is negative, backward semi-orbits starting from points from $T \setminus \{p_k\}$ enter the disk D enclosed by C and, since Γ_{k+1} is positive, then they escape from the disk through Q. Accordingly, take a a decreasing sequence $(b_n)_{n=1}^{\infty}$ in $T \cap \Omega$ tending to p_k and find maximal semi-orbits $\varphi(a_n, b_n)$ fully included in D, when observe that the sequence $(a_n)_n$, besides lying in Q, is increasing. Call a^* its limit. Clearly, $a^* \in \operatorname{Cl}\Omega$. Since the full forward orbit $\varphi(a^*, +)$ lies in D, and Γ_k and Γ_{k+1} are consecutive, we easily get that, in fact, $a^* \in \Omega$ and there is a solid heteroclinic strip S neighbouring a^*. This is impossible because points b_n belong to S if n is large enough, hence $\Gamma_k \subset \operatorname{Bd} S$.

Further, if Γ_k and Γ_{k+1} have the same sign, then $\gamma'(\Omega)$ cannot lie between them. In fact, assume, say, $q_k < p_k < v < q_{k+1} < p_{k+1}$, take b, T and $(b_n)_n$ as before but consider now $Q = \{t \in C : b \leq t \leq v\}$. Find similarly the points a_n and a^* in Q to obtain the analogous contradiction. We prove that if Γ_1 is positive, then $\gamma'(\Omega)$ cannot lie between $\gamma''(\Omega)$ and Γ_1, and if Γ_j is negative, then $\gamma'(\Omega)$ cannot lie between Γ_j and $\gamma(\Omega)$, in the same way.

As a conclusion, we get that either (a) all orbits Γ_k are positive and $\gamma'(\Omega)$ lies between Γ_j and $\gamma(\Omega)$, or (b) all orbits Γ_k are negative and $\gamma'(\Omega)$ lies between $\gamma''(\Omega)$ and Γ_1, or (c) there is $1 \leq m \leq j-1$ such that all orbits Γ_k with $k \leq m$ are positive, all orbits with $k > m$ are negative, and $\gamma'(\Omega)$ lies between Γ_m and Γ_{m+1}. This implies the lemma. $\square$

We say that a finite, non-empty set V of vectors of positive integers is *complete* when, for any $(i_1, \ldots, i_l) \in V$, we have $(i_1, \ldots, i_m) \in V$ for every $1 \leq m \leq l$, and $(i_1, \ldots, i_{l-1}, i) \in V$ for every $1 \leq i \leq i_l$. If $v \in V$, then we denote by $\lambda(v)$ the largest number j such that $(v, j) \in V$, $\lambda(v) = 0$ meaning that there is no j such that $(v, j) \in V$. Likewise, $\lambda(\emptyset)$ stands for the largest number t such that $(t) \in V$. Of course we should write λ_V instead of λ (and similarly ρ_L, σ_L instead of ρ, σ below) to emphasize that this map depends on V, but hopefully this will not lead to confusion.

Let $\mathbb{M} = \{n/3 : n = 0, 1, 2, \ldots\}$.

Definition 4.4. We say that a set L of vectors of numbers from $\mathbb{M}$ is *feasible* with *base* a complete set V if its elements have the structure (v, k), with $v \in V$ and $k \in \mathbb{M}$, and the following conditions hold:

(i) for each $(i) \in V$ of length 1 there are exactly two elements in L: $(i, \lambda(i)+1)$ and $(i, s+2/3)$ for some integer $s = \sigma(i)$, $0 \leq s \leq \lambda(i)$;

(ii) for each $v \in V$ of length at least 2 there are exactly four elements in L: $(v, 0)$, $(v, \lambda(v)+1)$, and $(v, r+1/3)$, $(v, s+2/3)$ for some integers $r = \rho(v)$, $s = \sigma(v)$, $0 \leq r \leq s \leq \lambda(v)$;

(iii) $(i, \lambda(i)+2/3)$ and $(i+1, 2/3)$ cannot simultaneously belong to L (where we mean $i+1 = 1$ when $i = \lambda(\emptyset)$);

(iv) if $\lambda(v) = 1$, then $(v, 1/3)$, $(v, 5/3)$, $(v, 1, 1/3)$ and $(v, 1, \lambda(v, 1)+2/3)$ cannot simultaneously belong to L.

Note that property (iii) above implies that $\lambda(i) \geq 1$ for some i, hence V contains at least one sequence of length 2. If V is the base of a feasible set L, then we assign a parity (even or odd) to each $v \in V$ as follows. All vectors of length 1 in V have parity even. If $(i) \in V$, then we assign even or odd parity to (i, j) depending on whether $j \leq \sigma(i)$ or not. Inductively, once the parity of $v \in V$ is established, we assign to (v, j) the same parity as v, or the other one, depending on whether $\rho(v) < j \leq \sigma(v)$ or not. Finally, if $w = (v, h) \in L$, then we say that w is an *α-vector* if either v is even and $h = 0$ or $h = \rho(v) + 1/3$, or v is odd and $h = \lambda(v)+1$ or $h = \sigma(v) + 2/3$. Otherwise, we say that w is a *ω-vector*.

We next explain how to associate canonically a feasible set L to Φ. To construct the base V we proceed inductively, biunivocally associating to each canonical strip Ω (and the ω-point of $\gamma(\Omega)$) a vector from V. First of all, order the heteroclinic canonical strips of Φ as $\Omega_1, \ldots, \Omega_t$, this meaning that the corresponding ω-points q_i of the orbits $\gamma(\Omega_i)$, $1 \leq i \leq t$, satisfy $q_1 < \ldots < q_t$. Then the 1-length vectors from V will be those of the type (i), $1 \leq i \leq t$. If, additionally, the strip Ω_i is not elementary, and $\Omega_{i,1}, \ldots, \Omega_{i,j}$ are the homoclinic canonical strips Ω such that $\gamma(\Omega) \subset \operatorname{Bd} \Omega_i$ (again assuming $q_{i,1} < \ldots < q_{i,j}$ for their corresponding ω-points), then we add the 2-vectors (i, k) to V, $1 \leq k \leq j$. In general, if a vector v has been added to V, with corresponding canonical strip Ω_v, and Ω_v is not elementary, then we consider as before the homoclinic canonical strips Ω such that $\gamma(\Omega) \subset \operatorname{Bd} \Omega_v$, call them $\Omega_{v,1}, \ldots, \Omega_{v,j'}$ (so that $q_{v,1} < \ldots < q_{v,j'}$ for the corresponding ω-points), and add the vectors (v, k), $1 \leq k \leq j'$, to V. Clearly, the set V so defined is complete.

V	L
(1)	$(1,2)$, $(1,\frac{5}{3})$
$(1,1)$	$(1,1,0)$, $(1,1,2)$, $(1,1,\frac{1}{3})$, $(1,1,\frac{2}{3})$
$(1,1,1)$	$(1,1,1,0)$, $(1,1,1,1)$, $(1,1,1,\frac{1}{3})$, $(1,1,1,\frac{2}{3})$

Table 4.1: The elements of the feasible set L and its base V from the left flow of Figure 1.1.

V	L
(1)	$(1,2)$, $(1,\frac{2}{3})$
$(1,1)$	$(1,1,0)$, $(1,1,1)$, $(1,1,\frac{1}{3})$, $(1,1,\frac{2}{3})$
(2)	$(2,1)$, $(2,\frac{2}{3})$
(3)	$(3,2)$, $(3,\frac{5}{3})$
$(3,1)$	$(3,1,0)$, $(3,1,1)$, $(3,1,\frac{1}{3})$, $(3,1,\frac{2}{3})$

Table 4.2: The elements of the feasible set L and its base V from the right flow of Figure 1.1 (Σ is the "upper" heteroclinic separatrix).

Now we define L (and biunivocally associate to its vectors all points from X). We just must explain how to choose the numbers $\sigma(i)$ and the pairs $\rho(v), \sigma(v)$ in Definition 4.4 (i) and (ii), and then check that (iii) and (iv) hold. As for the first numbers, let (with the notation of the previous paragraph) $1 \leq i \leq t$. Then $s = \sigma(i)$ is the largest number such that $q_{i,s} < y_i$, y_i being the ω-point of $\gamma'(\Omega_i)$ (or $s = 0$ if Ω_i is elementary or no such number exists, that is, $y_i < q_{i,j}$ for all j). Also, we redefine the points y_i and q_i as $c_{i,\sigma(i)+2/3}$ and $c_{i,\lambda(i)+1}$, respectively. In the general case we denote by x_v and y_v the α- and ω-points of $\gamma'(\Omega_v)$ when this orbit is even, reversing the notation when $\gamma'(\Omega_v)$ is odd, and take $r = \rho(v)$ and $s = \sigma(v)$ as the largest numbers satisfying $q_{v,r} < x_v$ and $q_{v,s} < y_v$, respectively (or $r = s = 0$ when Ω_v is elementary, and $r = 0$ or $s = 0$ when the corresponding number does not exist). Finally, we redenote x_v and y_v as $c_{v,\rho(v)+1/3}$ and $c_{v,\sigma(v)+2/3}$, while $c_{v,0}$ and $c_{v,\lambda(v)+1}$ stand for the α- and ω-points (or conversely in the odd case) of $\gamma(\Omega_v)$.

We claim that (iii) in Definition 4.4 holds. Indeed if, say, both $(i, \lambda(i) + 2/3)$ and $(i+1, 2/3)$ belong to L for some i, the orbits $\gamma'(\Omega_i)$ and $\gamma'(\Omega_{i+1})$ would bound, together with $\mathbf{0}$, a solid strip (Remark 2.8). Since this strip includes the separatrix $\gamma(\Omega_i)$, we get a contradiction.

Assume now that Definition 4.4 (iv) does not hold, that is, there is $v \in V$ with $\lambda(v) = 1$ such that all vectors $(v, 1/3)$, $(v, 5/3)$, $(v, 1, 1/3)$ and $(v, 1, \lambda(v, 1) + 2/3)$ belong to L. Then, again by Remark 2.8, the orbits $\gamma'(\Omega_v)$, $\gamma'(\Omega_{v,1})$ bound, together with $\mathbf{0}$, a solid strip including $\gamma(\Omega_{v,1})$, which is impossible.

Thus we have shown that L is feasible. Although L has been constructed with the help of the circle C, it depends only on Θ and Σ. We call it the *canonical feasible set* associated to Φ, the orientation Θ and the separatrix Σ.

As some examples, we present in Tables 4.1 and 4.2 the feasible sets associated to the flows on Figure 1.1 under the counterclockwise orientation.

Remark 4.5. The simplest feasible set

$$L = \{(1,5/3),(1,2),(1,1,0),(1,1,1/3),(1,1,2/3),(1,1,1)\}$$

(equivalent to

$$L = \{(1,2/3),(1,2),(1,1,0),(1,1,1/3),(1,1,2/3),(1,1,1)\}$$

after reversing the orientation) correspond to the case when there are exactly three separatrices (one heteroclinic, another one regular homoclinic, and the singular point), which occurs when "$\prec$" is a total ordering in $\mathcal{H}$ ($\mathbf{0}$ becoming an elliptic saddle for the flow).

Observe that the bijection from L to X given by $w \mapsto c_w$ preserves *orders* (when the lexicographical order is used in L), *orbits* (that is, two points c_w and $c_{w'}$ belongs to the same orbit if and only if $w = (v,h)$ and $w' = (v,h')$ for some $v \in V$ and $h + h'$ is an integer) and *directions* (that is, w is a ω-vector if and only if c_w is a ω-point; this follows from Lemma 4.3, which implies that the parity of $v \in V$ is the same as that of $\gamma(\Omega_v)$ and $\gamma'(\Omega_v)$). There are many feasible sets L' which can be bijectively mapped onto X so that ordering is preserving: since both orderings are total, one just needs that both cardinalities of L and L' are the same. As it turns out, if orbits are preserved, then directions are preserved as well:

Lemma 4.6. *If L' is feasible, and there is a bijection $\psi : L' \to X$ preserving orders and orbits, then $L' = L$.*

Proof. Let V' the base of L' and redenote $\lambda_{V'} = \lambda'$, $\rho_{L'} = \rho'$, $\sigma_{L'} = \sigma'$. Since ψ preserves orbits, it maps vectors $(i', \lambda'(i') + 1)$ and $(i', \sigma'(i') + 2/3)$ to ω-points of heteroclinic orbits, and pairs $(v', 0)$ and $(v', \lambda'(v') + 1)$, as well as pairs $(v', \rho'(v') + 1/3)$ and $(v', \sigma'(v') + 2/3)$, to pairs of points of homoclinic orbits. Since orders are preserved as well, we get that vectors $(i', \lambda'(i') + 1)$ are precisely those mapped to heteroclinic separatrices, and deduce that vectors of lengths 1 and 2 of V and V', as well as vectors of length 2 of L and L', are the same. Now, as the reader will easily convince himself, to prove the lemma we just have to show this: pairs $(v', 0)$ and $(v', \lambda'(v') + 1)$ are exactly those mapped to homoclinic separatrices.

Assume, to arrive at a contradiction, that $(v', 0)$ and $(v', \lambda'(i') + 1)$ are mapped to one of the orbits $\gamma'(\Omega_v)$ of $\mathcal{X}$. Since $\mathcal{X}$ has no orbits between $\gamma'(\Omega_v)$ and the orbits $\gamma(\Omega_{v,k})$ (regarding the order "$\prec$"), it is clear that $(v', \rho'(v') + 1/3)$ and $(v', \sigma'(v') + 2/3)$ must be mapped to one of the orbits $\gamma(\Omega_{v,k})$ (in particular, v cannot have maximal length in V). Similarly, if $(v', \rho'(v') + 1/3)$ and $(v', \sigma'(v') + 2/3)$ are mapped to an orbit $\gamma(\Omega_w)$, the pair which is mapped to $\gamma'(\Omega_w)$ must be of the type $(w', 0)$ and $(w', \lambda'(w') + 1)$, because the orbit corresponding to $(w', \rho'(w') + 1/3)$ and $(w', \sigma'(w') + 2/3)$ is $\prec$-less than that corresponding to $(w', 0)$ and $(w', \lambda'(w') + 1)$, and there are no orbits of $\mathcal{X}$ between $\gamma(\Omega_w)$ and $\gamma'(\Omega_w)$. We could thus proceed indefinitely, contradicting the finiteness of $\mathcal{X}$. □

Proof of Theorem A. The statement (i)⇒(ii) is obvious (recall Proposition 3.9).

Let us show (ii)⇒(iii). Fix small circles C, C' around $\mathbf{0}$ and let $\mu : C \to C'$ be a homeomorphism preserving orbits for ξ. Use the hypothesis to find heteroclinic separatrices Σ and Σ' such that $\xi(\Sigma) = \Sigma'$, fix Θ as the counterclockwise orientation, and take Θ' as the counterclockwise or the clockwise orientation depending on whether μ preserves or reverses the orientation. Construct the canonical feasible sets L and L' associated to them, and the corresponding bijections $\psi : L \to X$, $\psi' : L' \to X'$ to the configurations of $\mathcal{X}$ and $\mathcal{X}'$ preserving

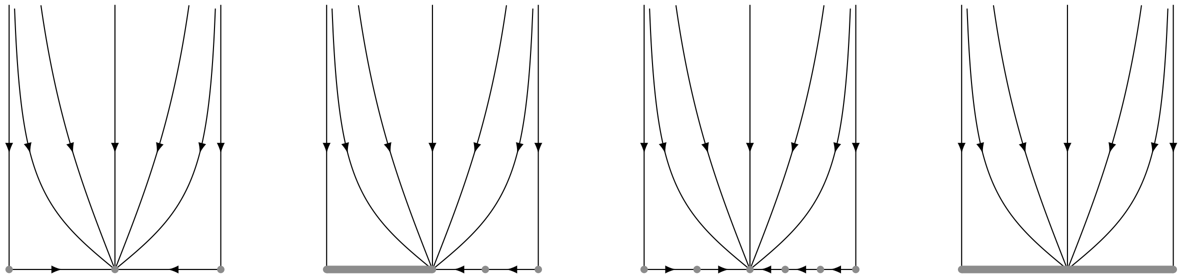

Figure 5.1: From left to right: phase portraits of f (and $f_{1,2}$), $f_{0,2}$, $f_{2,5}$ and $f_{0,0}$.

orders, orbits and directions. Although the hypothesis does not state that μ preserves directions for ξ, we get that $\mu^{-1} \circ \psi' : L' \to X$ preserves orders and orbits anyway. Now Lemma 4.6 applies and (iii) follows.

Finally, to prove (iii)⇒(i), let again C, C' be small circles around $\mathbf{0}$, denote the configurations of $\mathcal{X}$ and $\mathcal{X}'$ in these circles by X and X', and find arcs $A \subset C$ and $A' \subset C'$ containing all points of X and X' and having q and q', the ω-points of Σ and Σ', as their upper endpoints (after using the respective orientations Θ and Θ'). According to the hypothesis, there are a feasible set L and bijections $\psi : L \to X$, $\psi' : L \to X'$ preserving orders, orbits and directions, and hence a bijection $\xi : \mathcal{X} \to \mathcal{X}'$ and a homeomorphism $\mu : C \to C'$ preserving orbits and directions for ξ. Then, as explained in Subsection 2.2, there is a plane homeomorphism preserving the skeletons orbits, which turns out to preserve the flows directions as well. Hence Φ and Φ' are topologically equivalent by Theorem 2.7. □

5 Proof of Theorem B

Let $0 \leq s \leq j$ be non-negative integers. We define a C^1-vector field $f_{s,j}$ as follows. We start from $f(x, y) = (x(x^2 - 1), -y)$. As easily checked, the phase portrait of (the associated local flow to) f in the semi-band $[-1, 1] \times [0, \infty)$ (the only sector we are interested in) consists of three singular points, the attracting node $\mathbf{0}$ and the saddle points $(-1, 0)$ and $(1, 0)$, two horizontal orbits in the x-axis going to $\mathbf{0}$ as time goes to ∞, and three vertical orbits on the semi-lines $x = -1, 0, -1$, each converging in positive time to the corresponding singular point. All other orbits go to $\mathbf{0}$ as $t \to \infty$. Next, let $\kappa(x)$ be a non-negative C^1-function vanishing at points $x = -i/s$, $0 \leq i \leq s$ (or at the whole interval $[-1, 0]$ if $s = 0$), at points $x = i/(j - s)$, $0 \leq i \leq j - s$ (or at the whole interval $[0, 1]$ if $s = j$), and at no other points. Then we define $f_{s,j}(x, y) = (\kappa(x) + y^2) f(x, y)$, thus adding new singular points in the x-axis and leaving unchanged the upper orbits. Figure 5.1 exhibits the phase portrait of $f_{s,j}$ for different values of s and j.

Now, let $0 \leq r \leq s \leq j$ be non-negative integers and define C^1-vector fields $g^+_{r,s,j}, g^-_{r,s,j}$ as follows. This time our starting point is

$$g(x, y) = \left((x^2 - 1) \left(x^2 - \left(1 - \frac{(1 - y)^2}{2} \right)^2 \right), y(y - 1)x \right)$$

and we are interested in its phase portrait in the rectangle $[-1, 1] \times [0, 1]$. We have six singular points: the saddles $(-1, 0)$ and $(1, 0)$, the repelling node $(-1/2, 0)$, the attracting node $(1/2, 0)$ and the semi-hyperbolic singularities $(-1, 1)$ and $(1, 1)$. The boundary of the rectangle is invariant for the flow, hence consisting of the singular points and six regular orbits, all clockwise

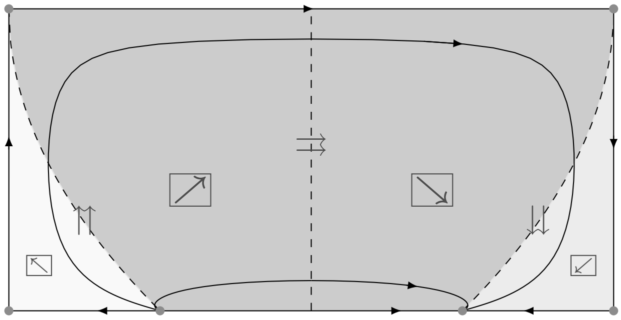

Figure 5.2: Phase portrait of g.

oriented by the flow except that connecting $(-1/2,0)$ and $(1/2,0)$. Additional isoclines exist at the y-axis (for the horizontal direction of the flow) and the parabolas $x = \pm(1-(1-y)^2/2)$ (for the vertical direction of the flow), which ensures that all orbits in the rectangle interior crossing the y-axis go to $(-1/2,0)$ (respectively, $(1/2,0)$) as time goes to $-\infty$ (respectively, ∞). See Figure 5.2.

As it happens, this completes the phase portrait because in fact all interior orbits cross the y-axis. To prove this we must discard the existence of full orbits in the region to the right of the isocline $x = 1-(1-y)^2/2$ or, equivalently (because of the symmetry properties of the vector field) in the region to the left of the isocline $x = -1+(1-y)^2/2$. This follows from the fact that the flow, near $(1,1)$, is equivalent to that associated to $x' = x^2$, $y' = y$ near $\mathbf{0}$ (this is a consequence of [8, Theorem 2.19, pp. 74–75]). Alternatively, one can prove there are no full orbits to the right of $x = 1-(1-y)^2/2$ in a direct way as follows. It clearly suffices to show that the vector field crosses from left to right all lines $y = 1 + a(x-1)$, $a > 0$, in the square $(1/2,1)\times(1/2,1)$, that is, $ag_1(1-t,1-at) - g_2(1-t,1-at) > 0$ whenever $0 < t < 1/2$ and $0 < at < 1/2$, when we mean $g = (g_1, g_2)$. Since

$$
\begin{aligned}
\frac{ag_1(1-t,1-at) - g_2(1-t,1-at)}{at} &= 1 + 3t - at - 4t^2 + at^2 - 2a^2t^2 \\
&\quad + (1+a^2)t^3 + \frac{a^4t^4}{2} - \frac{a^4t^5}{4} \\
&> 1 + 3t - \frac{1}{2} - 2t + at^2 - \frac{1}{2} \\
&\quad + (1+a^2)t^3 + \frac{a^4t^4}{2} - \frac{t}{64} \\
&= \frac{63t}{64} + at^2 + (1+a^2)t^3 + \frac{a^4t^4}{2} \ > \ 0,
\end{aligned}
$$

we are done.

Let $\kappa(x)$ be a non-negative C^1-function vanishing at points $x = -1 + i/(2r)$, $0 \le i \le r$ (or at the whole interval $[-1,-1/2]$ if $r = 0$), at points $x = -1/2 + i/(s-r)$, $0 \le i \le s-r$ (or at the whole interval $[-1/2,1/2]$ if $r = s$), at points $x = 1/2 + i/(2j-2s)$, $0 \le i \le j-s$ (or at the whole interval $[1/2,1]$ if $s = j$), and at no other points. Then we define $g^+_{r,s,j}(x,y) = (\kappa(x)+y^2)(1-x^2)g(x,y)$. In this way, we add some new singular points at the x-axis, and all points from both vertical borders of the rectangle become singular as well, yet the inner orbits remain the same. Finally we put $g^-_{r,s,j}(x,y) = -g_{r,s,j}(x,y)$, getting the same phase portrait with reversed time directions. Some examples of the phase portraits of these vector fields are shown in Figure 5.3.

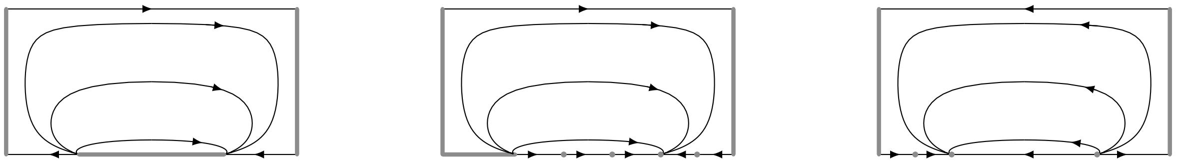

Figure 5.3: Phase portraits of $g^+_{1,1,2}$ (left), $g^+_{0,3,5}$ (center) and $g^-_{2,3,4}$ (right).

Let L be a feasible set with base V. We are ready to explain how to construct a polynomial flow Φ whose associated feasible set, after fixing the counterclockwise orientation and choosing an appropriate heteroclinic separatrix of Φ, is exactly L.

Let n be the length of the largest sequence in V and recall that $n \geq 2$. Also, let $t = \lambda(\emptyset) \geq 1$. Firstly, we define a vector field F on $\mathbb{R}^2$ by gluing (after appropriate translations and dilatations) some vectors fields $f_{r,j}, g^+_{r,s,j}, g^-_{r,s,j}$ (and the null vector field) as prescribed by L.

To begin with, if $(i) \in V$, then we glue at the semi-band $[i-1,i] \times [0,\infty)$ the vector field $f_{\sigma(i),\lambda(i)}$ (better to say, $f_{\sigma(i),\lambda(i)}(2x-2i+1,y)$). Note that the way we defined the maps $f_{s,j}$ ensures that adjacent pieces glue well at the orbits $Y_i := \{i\} \times [0,\infty)$.

Now, the maximal compact intervals I in $I_i := [i-i,i]$ such that $\operatorname{Int} I \times \{0\}$ contains no singular points will be denoted, from left to right, by $I_{i,1},\ldots,I_{i,\lambda(i)}$, the flow travelling to the right on $Y_{i,k} := I_{i,k} \times \{0\}$ if and only if $k \leq \sigma(i)$. Certainly, maximal compact intervals N with $N \times \{0\}$ just consisting of singular points may exist; we call each of them a 0-level null interval.

After F has been defined on $[0,t] \times [0,\infty)$, we define it in $[0,t] \times [-1,0)$. In the rectangles $N \times [-1,0)$, where N is a 0-level null interval, we just define F as zero; and at the rectangles $I_{i,k} \times [-1,0)$ we glue either the vector field $g^+_{\rho(i,k),\sigma(i,k),\lambda(i,k)}$ (more properly,

$$g^+_{\rho(i,k),\sigma(i,k),\lambda(i,k)}((2x-a-b)/(b-a),y+1)$$

with $I_{i,k} = [a,b]$) or the vector field $g^-_{\rho(i,k),\sigma(i,k),\lambda(i,k)}$ according to whether the flow in $Y_{i,k}$ goes to the right or to the left. Similarly as before, the maximal compact intervals I in $I_{i,k}$ such that $\operatorname{Int} I \times \{-1\}$ contains no singular points will be denoted, ordered from left to right, $I_{i,k,1},\ldots,I_{i,k,\lambda(i,k)}$ (write also $Y_{i,k,k'} = I_{i,k,k'} \times \{-1\}$), and the flows travels on $Y_{i,k,k'}$ in the same direction as in $Y_{i,k}$ if and only if $\rho(i,k) < k' \leq \sigma(i,k)$. Any maximal compact interval N such that $N \times \{-1\}$ consists of singular points will be called a 1-level null interval.

Proceeding in this way, we associate inductively to each vector $v \in V$ of length $m \geq 2$ an interval $I_v \subset [0,t]$ (and the corresponding orbit $Y_v = I_v \times \{-m+2\}$), and define the m-level null intervals. Then we define F as zero in $N \times [-m+1,-m+2)$ if N is m-level null, or as $g^+_{\rho(v),\sigma(v),\lambda(v)}$ or $g^-_{\rho(v),\sigma(v),\lambda(v)}$ in $I_v \times [-m+1,-m+2)$ according to the direction of the flow on Y_v. Note that the full lowest segment $[0,t] \times \{-n+1\}$ is null, that is, all its points are singular.

Thus we have completed the definition of F on $[0,t] \times [-n+1,\infty)$. Note that the map so defined is not locally Lipschitz (or even continuous) at the orbits Y_v; this can be easily arranged by multiplying F by appropriate positive C^1-functions $\tau_v(x)$ in the corresponding semi-open rectangles $\operatorname{Int} I_v \times [-m+1,-m+2)$. We keep calling F this modified map; note that, even so, it needs not be continuous at the singular points. To conclude the definition of F, we extend it periodically to the whole semi-plane $\mathbb{R} \times [-n+1,\infty)$ (that is $F(x,y) = F(x+kt,y)$ for any integer k) and define it as zero otherwise.

Before proceeding further, some additional notation must be given. First, let $Y'_i = \{i-1/2\} \times [0,\infty)$, $i = 1,\ldots,t$. Also, for any $v \in V$ with length $m \geq 2$, let Y'_v be the orbit in $I_v \times (-m+1,-m+2)$ corresponding, after translation and dilatation, to the orbit of the

V	L
(1)	$(1,\frac{5}{3})$, $(1,4)$
$(1,1)$	$(1,1,0)$, $(1,1,\frac{1}{3})$, $(1,1,\frac{2}{3})$, $(1,1,1)$
$(1,2)$	$(1,2,0)$, $(1,2,\frac{1}{3})$, $(1,2,\frac{2}{3})$, $(1,2,1)$
$(1,3)$	$(1,3,0)$, $(1,3,\frac{1}{3})$, $(1,3,\frac{2}{3})$, $(1,3,1)$
(2)	$(2,\frac{2}{3})$, $(2,2)$
$(2,1)$	$(2,1,0)$, $(2,1,\frac{1}{3})$, $(2,1,\frac{8}{3})$, $(2,1,3)$
$(2,1,1)$	$(2,1,1,0)$, $(2,1,1,\frac{1}{3})$, $(2,1,1,\frac{2}{3})$, $(2,1,1,1)$
$(2,1,2)$	$(2,1,2,0)$, $(2,1,2,\frac{1}{3})$, $(2,1,2,\frac{2}{3})$, $(2,1,2,1)$
(3)	$(3,\frac{2}{3})$, $(3,1)$
(4)	$(4,\frac{5}{3})$, $(4,2)$
$(4,1)$	$(4,1,0)$, $(4,1,\frac{4}{3})$, $(4,1,\frac{5}{3})$, $(4,1,3)$
$(4,1,1)$	$(4,1,1,0)$, $(4,1,1,\frac{1}{3})$, $(4,1,1,\frac{2}{3})$, $(4,1,1,1)$
$(4,1,2)$	$(4,1,2,0)$, $(4,1,2,\frac{1}{3})$, $(4,1,2,\frac{2}{3})$, $(4,1,2,1)$

Table 5.1: The elements of the feasible set L and its base V from Figure 5.4.

vector field $g(x,y)$ passing through the point $(0,1/2)$. Now it is easy to construct a polygonal arc A with endpoints $(0,1/2)$ and $(t,1/2)$, consisting of alternate horizontal and vertical segments, so that:

- horizontal segments are of type $J \times \{-m+\epsilon_J\}$ for some compact interval J, some $0 < \epsilon_J < 1$ and $0 \leq m < n$;
- any two such intervals J, J' have at most one common point, and the union of all intervals J is $[0,t]$;
- A intersects each orbit Υ_i, Υ'_i at exactly one point, and all other orbits Υ_v, Υ'_v at exactly two points.

Observe that the bijection mapping L to the set of these intersection points that preserves orders (hence mapping $(t,\lambda(t)+1)$ to $(t,1/2)$), also preserves orbits as previously meant, that is, every vector (i,h) is mapped either to $A \cap \Upsilon_i$ or to $A \cap \Upsilon'_i$ and every pair of vectors $(v,h),(v,h')$ with $h+h'$ an integer is mapped either to $A \cup \Upsilon_v$ or to $A \cup \Upsilon'_v$.

Figure 5.4 illustrates the former construction starting from the feasible set L described in Table 5.1. The dotted line indicates the arc A.

Let $\Xi : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be given by $\Xi(r,\theta) = e^{r+\mathrm{i}2\pi\theta/t}$. Although F may not be continuous, the set T of singular points of F is closed and F is locally Lipschitz in the region $O = \mathbb{R}^2 \setminus T$; hence, when restricted to O, it has an associated local flow which can be naturally carried to the region $U = \Xi(O)$ via Ξ: call Ψ' this projected local flow on U. Let Ψ be a flow on $\mathbb{R}^2_\infty$ with the same orbits and time orientations as Ψ', and having singular points outside U,

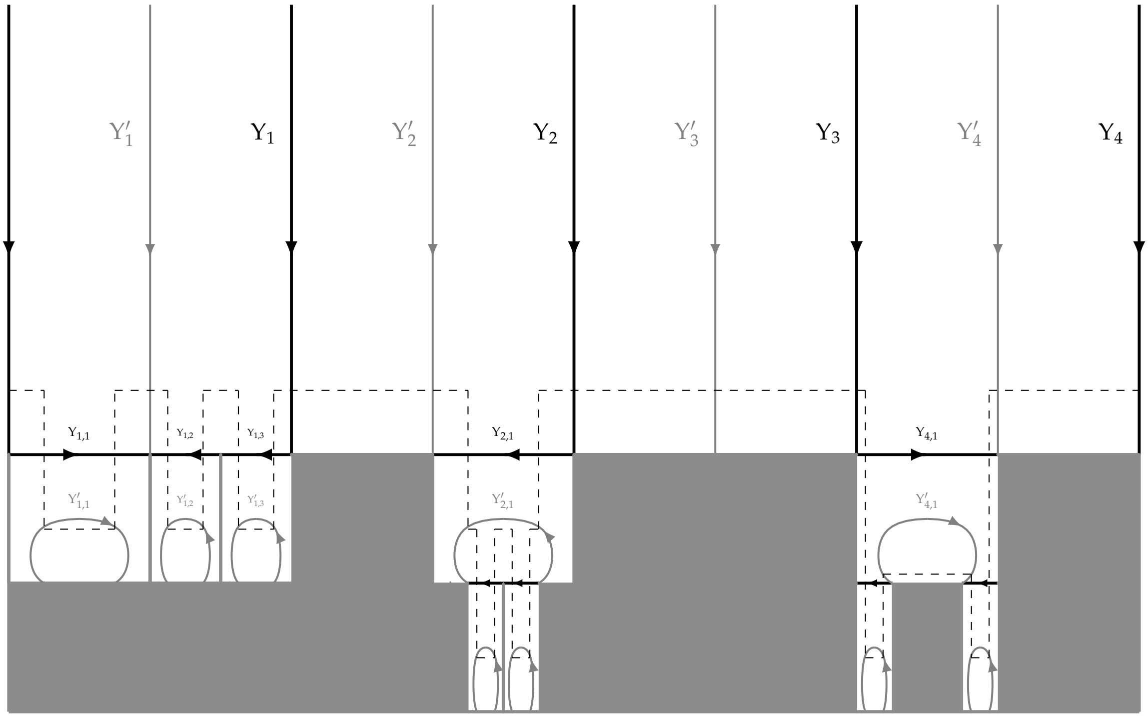

Figure 5.4: Constructing a polynomial flow from a feasible set.

that is, at $K = \Xi(T) \cup \{\mathbf{0}\}$ and ∞. This flow induces in $\mathcal{Q} = \mathbb{R}^2_\infty / \sim_K$, in the natural way, a flow $\Psi_{\sim_K}$ with two singular points, K (now an element of $\mathcal{Q}$) and ∞. Moreover, since $\mathbb{R}^2_\infty \setminus K$ is connected, there is a homeomorphism $H : \mathcal{Q} \to \mathbb{R}^2_\infty$ (Lemma 2.9), when we can assume $H(K) = \mathbf{0}$, $H(\infty) = \infty$. After carrying $\Psi_{\sim_K}$ to $\mathbb{R}^2_\infty$ via H, we get a flow Φ' on $\mathbb{R}^2_\infty$ having (when restricted to $\mathbb{R}^2$) $\mathbf{0}$ as its global attractor, its separatrix skeleton consisting of $\mathbf{0}$ and the curves $(H \circ \Xi)(Y_v), (H \circ \Xi)(Y'_v)$, $v \in V$. Using $C = (H \circ \Xi)(A)$, now a circle around $\mathbf{0}$, choosing an appropriate orientation Θ in $\mathbb{R}^2$, and taking $\Sigma = (H \circ \Xi)(Y_t)$ (recall also Lemma 4.6), we get that L is the canonical feasible set associated to Φ', Θ and Σ. Composing H if necessary with a reversing order homeomorphism, we can in fact get Θ to be the counterclockwise orientation.

We are almost done. Indeed, since Φ' has finitely many unstable orbits, two singular points (the only possible α-limit and ω-limit sets of the flow) and no periodic orbits, [12, Lemma 4.1] (essentially, a corollary of the main results in [11] and [19]) implies that it is topologically equivalent to the associated flow to a polynomial vector field in $\mathbb{S}^2$ and then, as explained in Section 2, to a polynomial flow in $\mathbb{R}^2$. Figure 5.5 shows the resultant flow after collapsing the flow from Figure 5.4.

Remark 5.1. Since any flow having $\mathbf{0}$ as a global attractor and finitely many separatrices is topologically equivalent to a polynomial flow, and polynomial flows have the finite sectorial decomposition property, we get that finiteness of separatrices and sectors are, in fact, equivalent properties in this setting (compare to Remark 2.4).

6 Proof of Theorem C

To study the nature of the phase portrait of (1.1) near $\mathbf{0}$ and at the infinity one could use, in principle, desingularization [8, Chapter 3] and the Poincaré compactification [8, Chapter 5].

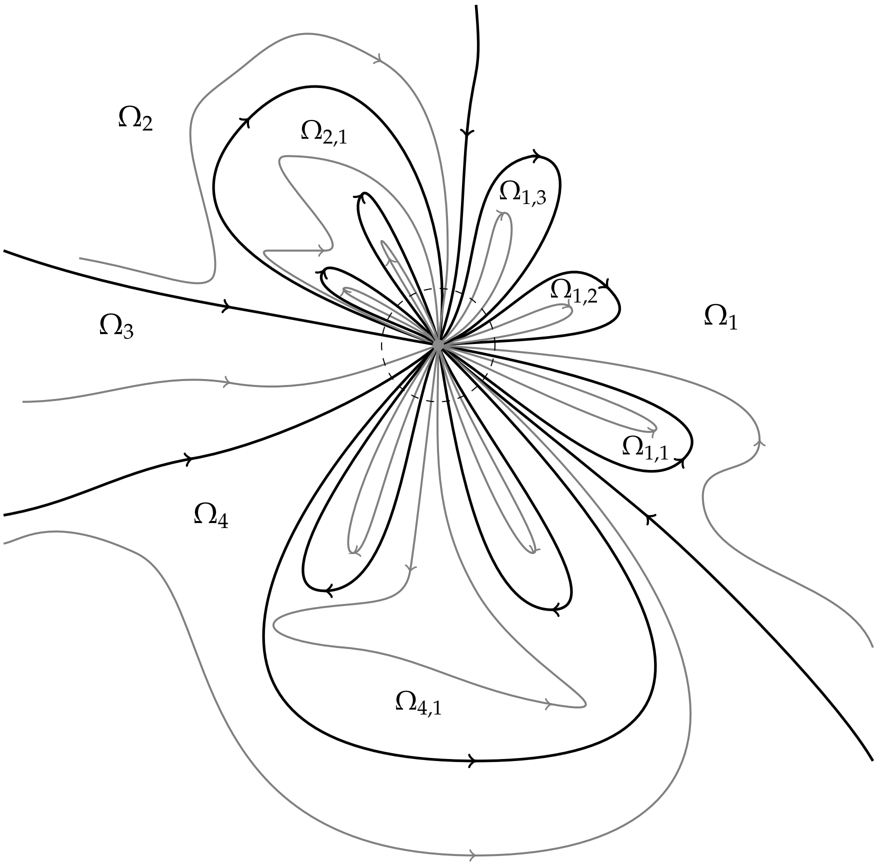

Figure 5.5: The phase portrait of the flow labelled by the feasible set from Table 5.1.

Directions	Regions
$x' < 0, y' > 0$	$U_1 = \{(x,y) : y > 0, y^2 + x^3 > 0\}$
$x' < 0, y' < 0$	$U_2 = \{(x,y) : y^2 + x^3 < 0, (1+x^2)y + x^3 > 0\}$
$x' > 0, y' < 0$	$U_3 = \{(x,y) : (1+x^2)y + x^3 < 0, y > 0\}$
$x' > 0, y' < 0$	$U_4 = \{(x,y) : y < 0, y^2 + x^3 < 0\}$
$x' > 0, y' > 0$	$U_5 = \{(x,y) : y^2 + x^3 > 0, (1+x^2)y + x^3 < 0\}$
$x' < 0, y' > 0$	$U_6 = \{(x,y) : (1+x^2)y + x^3 > 0, y < 0\}$

Table 6.1: Directions of the vector field for the system (1.1).

In the present case this leads, however, to very heavy calculations; thus the need to rely on specific (yet elementary) arguments, as those given below.

Since the polynomial $(1+x^2)^2 + x^3$ has no real zeros, $\mathbf{0}$ is the only singular point of the associated local flow to (1.1). The isocline corresponding to the horizontal direction of the vector field is the union of the curves $y = 0$ and $y^2 + x^3 = 0$. Thus, the x-axis consists of $\mathbf{0}$ and two regular orbits (both going to $\mathbf{0}$ in positive time) and there are no periodic orbits, as they should enclose the singular point. The isocline corresponding to the vertical direction of the vector field is the curve $(1+x^2)y + x^3 = 0$. Finally, the isoclines divide the plane in six regions U_i, $1 \leq i \leq 6$, where the flow has a well-defined direction: see Table 6.1 and Figure 6.1.

Claim 1: The origin is a global attractor of (1.1).

First of all, observe that orbits starting in U_2 go to U_3, and orbits starting in U_3 go to $\mathbf{0}$. Similarly, orbits starting in U_4 go to U_5, orbits starting in U_5 either go to $\mathbf{0}$ or to U_6, and orbits starting in U_6 go to $\mathbf{0}$. As a consequence, in order to prove the claim, it is enough to show that any orbit starting in U_1 meets the curve $y^2 + x^3 = 0$.

Let $P(x,y) = -((1+x^2)y + x^3)^5$ and $Q(x,y) = y^2(y^2 + x^3)$ be the components of the vector field and put $U_1' = U_1 \cap \{(x,y) : y \geq 1\}$. Then we have

$$-1 \leq \frac{Q(x,y)}{P(x,y)} \leq 0 \qquad \text{for any } (x,y) \in U_1' \tag{6.1}$$

because if $x \geq 0$, then

$$Q(x,y) = y^4 + y^2x^3 \leq (1+x^2)^5y^5 + 5(1+x^2)^4y^4x^3 \leq |P(x,y)|,$$

while if $x \leq 0$, we use that $y \geq -x$ holds in U_1' to get

$$Q(x,y) \leq y^4 \leq y^5 \leq (y + yx^2 + x^3)^5 = |P(x,y)|.$$

Now, realize that if an orbit starts in U_1, then either it crosses $y^2 + x^3 = 0$, or goes to U_1'. Therefore, to prove the claim, it suffices to show that if $(x_0, y_0) \in U_1'$, then the orbit (corresponding to the solution) $(x(t), y(t))$ of (1.1) starting at $x(0) = x_0$ and $y(0) = y_0$ meets $y^2 + x^3 = 0$. But, due to (6.1), we have $y'(t) \leq -x'(t)$ and then $y(t) \leq x_0 + y_0 - x(t)$ whenever the orbit stay in U_1'. In other words, the orbit lies below the line $y = x_0 + y_0 - x$ while staying in U_1'. Since this line intersects $y^2 + x^3 = 0$, Claim 1 follows.

Claim 2: The origin is not positively stable for (1.1).

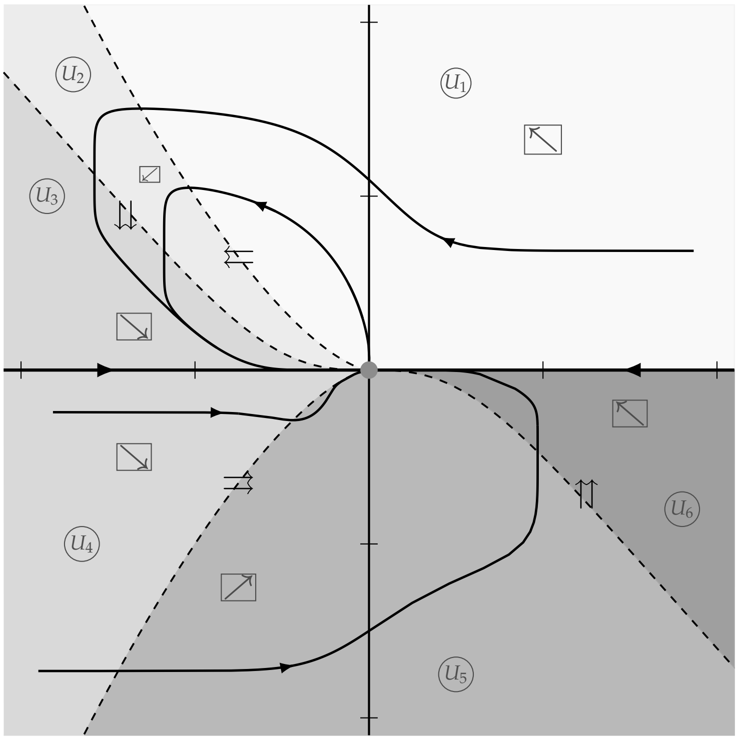

Figure 6.1: Phase portrait of $x' = -((1+x^2)y + x^3)^5$, $y' = y^2(y^2 + x^3)$.

Given any $y_0 > 0$, let $(x(t), y(t))$ be the orbit of (1.1) starting at $x(0) = 0$ and $y(0) = y_0$. According to Claim 1, this orbit must travel to U_2, then to U_3, and finally converge to $\mathbf{0}$. In particular, it meets the line $y = -2x$. Let t_* be the (smallest) positive time for which $y(t_*) = -2x(t_*)$ and define $Y(y_0) = y(t_*)$.

To prove the claim, it suffices to show that $Y(y_0) > 1/4$ (this bound is very conservative; numerical estimations suggest that the optimal bound is approximately 0.831). We proceed by contradiction assuming $Y(y_0) \leq 1/4$. Then $-1/8 \leq x(t) \leq 0$ for any $0 \leq t \leq t_*$.

For the sake of clarity, in this paragraph we assume $0 \leq t \leq t_*$ and shorten $x(t)$ as x and $y(t)$ as y. Since $x \leq 0$, we trivially have

$$y + \frac{x^3}{1+x^2} \leq y + (-x)^{3/2}. \tag{6.2}$$

We assert that

$$y + \frac{x^3}{1+x^2} \leq 2\left(y - (-x)^{3/2}\right) \tag{6.3}$$

is true as well. Observe that (6.3) is equivalent to

$$2(1+x^2)(-x)^{3/2} + x^3 \leq (1+x^2)y$$

and, taking into account that $y \geq -2x$, a sufficient condition for this to happen is

$$(-2x(1+x^2) - x^3)^2 - (2(1+x^2)(-x)^{3/2})^2 \geq 0,$$

which is true indeed:

$$\begin{aligned}(-2x(1+x^2)-x^3)^2-(2(1+x^2)(-x)^{3/2})^2 &= x^2(4+4x+12x^2+8x^3+9x^4+4x^5)\\ &\geq 4x^2(1+x+2x^3+x^5)\\ &\geq 4x^2\left(1-\frac{1}{8}-\frac{1}{256}-\frac{1}{32768}\right)\geq 0.\end{aligned}$$

Finally, we have

$$\frac{1}{(1+x^2)^5}\geq\frac{1}{(1+1/64)^5}>\frac{1}{2}. \tag{6.4}$$

Putting together (6.2), (6.3) and (6.4), we get

$$\frac{Q(x,y)}{P(x,y)}=-\frac{y^2(y+(-x)^{3/2})(y-(-x)^{3/2})}{(1+x^2)^5(y+x^3/(1+x^2))^5}\leq-\frac{1}{4y}.$$

As a consequence, for every $0\leq t\leq t_*$, we have $2y'(t)y(t)\geq -x'(t)/2$ and therefore

$$y(t)^2\geq y_0^2-x(t)/2>-x(t)/2,$$

that is, the orbit lies over the parabola $y^2=-x/2$. Since this parabola intersects $y=-2x$ at the point $(-1/8,1/4)$, we obtain the desired contradiction $Y(t_0)>1/4$, and Claim 2 follows.

Claim 3: The origin is an elliptic saddle for (1.1).

Let R be the union set of all heteroclinic orbits of (1.1), that is, the closed lower half-plane (except $\mathbf{0}$) and all orbits intersecting the positive semi-y-axis. By Claims 1 and 2, R is a radial region strictly included in $\mathbb{R}^2\setminus\{\mathbf{0}\}$ (Proposition 3.9). Moreover, it is clear that this flow does not allow a pair of incomparable homoclinic orbits. Then $\operatorname{Bd}R=\Gamma\cup\{0\}$, Γ being the only regular homoclinic separatrix of the flow (the other separatrices are the positive semi-x-axis and $\mathbf{0}$), and $\mathbf{0}$ is an elliptic saddle (Remark 4.5).

Claims 1, 2 and 3 complete the proof of Theorem C. □

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