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Long-time behaviour of solutions of delayed-type linear differential equations

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. This paper investigates the asymptotic behaviour of the solutions of the retarded-type linear differential functional equations with bounded delays

$$\dot{x}(t) = -L(t, x_t)$$

when $t \to \infty$. The main results concern the existence of two significant positive and asymptotically different solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$ such that $\lim_{t\to\infty} \varphi^{**}(t)/\varphi^*(t) = 0$. These solutions make it possible to describe the family of all solutions by means of an asymptotic formula. The investigation basis is formed by an auxiliary linear differential functional equation of retarded type $\dot{y}(t) = L^*(t, y_t)$ such that $L^*(t, y_t) \equiv 0$ for an arbitrary constant initial function y_t. A commented survey of the previous results is given with illustrative examples.

Keywords: linear differential functional equation of retarded-type, asymptotic behaviour of solutions, asymptotic convergence, asymptotic divergence.

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1 Introduction

1.1 Auxiliary notions

Let $C([a, b], \mathbb{R})$, where $a, b \in \mathbb{R}$, $a < b$, $\mathbb{R} = (-\infty, +\infty)$, be the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}$ with the topology of uniform convergence. In the case of $a = -r < 0$, $b = 0$, we will denote this space by $\mathcal{C}$, that is, $\mathcal{C} = C([-r, 0], \mathbb{R})$ and designate the norm of an element $\varphi \in \mathcal{C}$ by

$$\|\varphi\| = \sup_{-r\le\theta\le0} |\varphi(\theta)|.$$

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Consider a retarded nonlinear functional differential equation

$$\dot{x} = f(t, x_t) \tag{1.1}$$

where $f\colon \Omega \mapsto \mathbb{R}$ is a continuous map that is quasibounded and satisfies a local Lipschitz condition with respect to the second argument in each compact set in $\Omega \subset \mathbb{R} \times \mathcal{C}$ where Ω is specified below. The symbol " $\cdot$ " represents the right-hand derivative.

If $\sigma \in \mathbb{R}, A \geq 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R})$, then, for each $t \in [\sigma, \sigma + A]$, we define $x_t \in \mathcal{C}$ by means of the equation $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, which takes into account the history of the considered process on the past interval of a constant length r, that is, the delays considered in (1.1) are bounded.

In accordance with [29], a function x is said to be a solution of equation (1.1) on $[\sigma - r, \sigma + A)$ with $\sigma \in \mathbb{R}$ and $A > 0$ if $x \in C([\sigma - r, \sigma + A), \mathbb{R})$, $(t, x_t) \in \Omega$ and x satisfies the equation (1.1) for $t \in [\sigma, \sigma + A)$.

For given $\sigma \in \mathbb{R}$, $\varphi \in \mathcal{C}$ and $(\sigma, \varphi) \in \Omega$, we say that $x(\sigma, \varphi)$ is a solution of equation (1.1) through (σ, φ) (or that $x(\sigma, \varphi)$ is defined by the initial point σ and initial function $\varphi \in \mathcal{C}$) if there exists an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of equation (1.1) on $[\sigma - r, \sigma + A)$ and $x_\sigma(\sigma, \varphi) = \varphi$.

In view of the above conditions, each element $(\sigma, \varphi) \in \Omega$ determines a unique solution $x(\sigma, \varphi)$ of equation (1.1) through $(\sigma, \varphi) \in \Omega$ on its maximal interval of existence $I_{\sigma,\varphi} = [\sigma, A)$, $\sigma < A \leq \infty$. If the functional f is linear with respect to the second argument, then $A = \infty$. This solution depends continuously on the initial data [29].

In the sequel, we will use the notation $\mathbb{R}^+ = (0, \infty)$, $I = [t_0 - r, \infty)$ and $I_0 = [t_0, \infty)$ where $t_0 \in \mathbb{R}$. Moreover, we specify $\Omega := \{(t, \varphi) \in I_0 \times \mathcal{C}\}$.

1.2 Outline of investigation

The present paper is devoted to the asymptotic behaviour for $t \to \infty$ of the solutions of the retarded-type linear differential functional equation

$$\dot{x}(t) = -L(t, x_t) \tag{1.2}$$

where $L(t, \cdot)$ is a linear functional with respect to the second argument. As the functional $L(t, \cdot)$ is a particular case of functional $f(t, \cdot)$ on the right-hand side of (1.1), we assume that the properties of $f(t, \cdot)$ formulated above are valid for $L(t, \cdot)$ as well. Moreover, we will assume that $L(t, \cdot)$ is strongly increasing within the meaning of the following definition.

Definition 1.1. We say that a linear functional $L(t, \cdot)$ is strongly increasing if, for arbitrary $\psi_1, \psi_2 \in \mathcal{C}$ satisfying

$$\psi_1(\theta) < \psi_2(\theta), \quad \theta \in [-r, 0), \tag{1.3}$$

we have

$$L(t, \psi_1) < L(t, \psi_2)$$

for every $t \in I_0$.

From this definition, it follows that $L(t, x_t) > 0$ for every $(t, x_t) \in I_0 \times \mathcal{C}$ if $x(t + \theta) > 0$, $\theta \in [-r, 0)$.

The paper investigates the existence of two asymptotically different positive solutions to equation (1.2) and a structure formula describing the asymptotic behaviour of all solutions is derived. As a prototype of this equation, we can take the equation

$$\dot{x}(t) = -\frac{1}{r\mathrm{e}}x(t-r) \tag{1.4}$$

with its two positive solutions

$$x = \varphi^*(t) := t\mathrm{e}^{-t/r}, \qquad x = \varphi^{**}(t) := \mathrm{e}^{-t/r} \tag{1.5}$$

each having a different asymptotic behaviour for $t \to \infty$ and

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = 0. \tag{1.6}$$

We will show that the limit property (1.6) for a pair of positive solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$ is not a specific property of given equation (1.4) only, but is characteristic of retarded-type linear differential functional equations (1.2) in general. More precisely, characteristic of the considered classes of equations is that, if an eventually positive solution $x = \varphi_1(t)$ to (1.2) exists, then there always exists a second, eventually positive, solution $x = \varphi_2(t)$ such that either

$$\lim_{t\to\infty} \frac{\varphi_2(t)}{\varphi_1(t)} = 0$$

or

$$\lim_{t\to\infty} \frac{\varphi_2(t)}{\varphi_1(t)} = \infty.$$

To obtain the desired results, it is necessary to analyze the properties of the solutions of an auxiliary retarded-type linear differential functional equation

$$\dot{y}(t) = L^*(t, y_t), \tag{1.7}$$

connected by a simple transformation with equation (1.2), where $L^*(t,\cdot)$ is a linear functional with respect to the second argument such that $L^*(t, y_t) \equiv 0$ for an arbitrary constant initial function y_t.

The rest of the paper is organized as follows. Initially, in part 2, we will study the structure of solutions of the equation (1.7). An auxiliary lemma on the existence of solutions, staying on their maximal interval of existence in a previously defined domain, is formulated in part 2.1. Then, in parts 2.2, 2.3, the long-time behaviour is described of solutions of equation (1.7) in the so-called divergent and convergent cases. In part 3, the results derived are used for the investigation of the structure of solutions of equation (1.2) and the main results of the paper (Theorems 3.1–3.4) are proved. Two types of applications are demonstrated in part 4, the first one considers an integro-differential equation in part 4.1, and the second one deals with an equation with multiple delays in part 4.2. The paper is finished by part 5 with some concluding remarks (parts 5.1, 5.2) and some open problems formulated in part 5.3.

2 Structure of solutions of auxiliary equation (1.7)

In this section, we consider linear equation (1.7), that is,

$$\dot{y}(t) = L^*(t, y_t)$$

where the functional $L^*(t, \cdot)$ is defined on $I_0 \times \mathcal{C}$ and is linear with respect to the second argument. Since the functional $L^*(t, \cdot)$ is a particular case of functional $f(t, \cdot)$ on the right-hand side of (1.1), we assume that the above-formulated properties of $f(t, \cdot)$ are reduced to $L^*(t, \cdot)$ as well. Below, we require the following additional properties of $L^*(t, \cdot)$:

$$L^*(t, \psi) > 0 \text{ for every } (t, \psi) \in I_0 \times \mathcal{C} \text{ such that } \psi(0) > \psi(\theta), \quad \theta \in [-r, 0), \tag{2.1}$$

$$L^*(t, \psi) < 0 \text{ for every } (t, \psi) \in I_0 \times \mathcal{C} \text{ such that } \psi(0) < \psi(\theta), \quad \theta \in [-r, 0). \tag{2.2}$$

Note that properties (2.1), (2.2) imply the following:

$$L^*(t, M) = 0 \text{ for every } (t, M) \in I_0 \times \mathcal{C} \text{ where } M \text{ is an arbitrary constant function.} \tag{2.3}$$

We will study two cases of asymptotic behaviour of solutions if $t \to \infty$. The first case is referred to as a divergence one. In this case, we prove, provided that there exists a solution $y = Y(t)$, $t \in I$ of (1.7) with the property $\lim_{t\to\infty} Y(t) = \infty$, that, for every solution $y = y^*(t)$, $t \in I$ of (1.7), either there exists a constant $K^* \neq 0$ such that $y^*(t) \sim K^* Y(t)$ for $t \to \infty$ or this solution is bounded. The second case is referred to as a convergence case. In this case, every solution of (1.7) has a finite limit as $t \to \infty$.

2.1 Auxiliary lemma

Here we formulate an auxiliary lemma necessary for the proof of Theorem 2.3 below. As noted above, solutions of the equation (1.1) are defined by the initial points and functions. Below we need to work with systems of initial functions having some common properties. Their descriptions are given below.

If a set $\omega \subset \mathbb{R} \times \mathbb{R}$, then, by $\overline{\omega}$ and $\partial\omega$, we denote, as is customary, the closure and the boundary of ω, respectively.

Let C_1, C_2 be positive constants and $C_1 < C_2$. Define the sets ω and A as follows:

$$\omega := \{(t, y) \in I \times \mathbb{R}, C_1 < y < C_2\} \tag{2.4}$$

and

$$A(t_0) := \{(t_0, y) \in \overline{\omega}\}. \tag{2.5}$$

Definition 2.1. A system of initial functions $p_{A(t_0),\omega}$ with respect to the sets $A(t_0)$ and ω is defined as a continuous mapping $p\colon A(t_0) \to \mathcal{C}$ with the properties:

(α) if $z = (t_0, y) \in A(t_0) \cap \omega$, then $(t_0 + \theta, p(z)(\theta)) \in \omega$, $\theta \in [-\tau, 0]$;

(β) if $z = (t_0, y) \in A(t_0) \cap \partial\omega$, then $(t_0 + \theta, p(z)(\theta)) \in \omega$, $\theta \in [-\tau, 0)$ and $(t_0, p(z)(0)) = z$.

Lemma 2.2. *Let, for all points $(t, y^*) \in \partial\omega$ and for all functions $\pi \in \mathcal{C}$ such that $\pi(0) = y^*$ and $(t + \theta, \pi(\theta)) \in \omega$, $\theta \in [-r, 0)$, it follows that*

$$(2y^* - C_1 - C_2) f(t, \pi) > 0. \tag{2.6}$$

Then, for each given system of initial functions $p_{A(t_0),\omega}$*, there exists a point* $z^{**} = (t_0, y^{**}) \in A(t_0) \cap \omega$ *(depending on the choice of the system of initial functions) such that, for the corresponding solution* $y(t_0, p(z^{**}))$ *of* (1.1)*, we have*

$$(t, y(t_0, p(z^{**}))(t)) \in \omega, \qquad t \in I. \tag{2.7}$$

We omit the proof since it is a simple consequence of Lemma 1 in [8], which describes the main results – Theorem 3.1, Theorem 2.1 and Corollary 3.1 – of the paper [37], proved by an original adaptation of the well-known retract principle for retarded ordinary differential equations (note that its founder T. Ważewski created in [41] this principle originally for ordinary differential equations). To apply Lemma 1 in [8], it is sufficient to set $n = 1$, $p = 1$, $q = 0$, $l_p = l_1 := (y - C_1)(y - C_2)$, $A := A(t_0)$ and to define the set ω as described above.

Definition 2.1 of a system of initial functions, too, is a specification of Definition 2 in [8], which in turn was motivated by Definition 2.2 in [37].

2.2 Long-time behaviour of solutions in the divergent case

Below is the main result related to the long-time behaviour of solutions in the divergent case.

Theorem 2.3. *Let the functional* L^**, defined on* $I_0 \times \mathcal{C}$*, be linear with respect to the second argument and satisfy* (2.1)*,* (2.2)*. Let there exist a positive solution* $y = Y(t)$*,* $t \in I$ *of* (1.7) *such that*

$$\lim_{t\to\infty} Y(t) = \infty. \tag{2.8}$$

Then, for every fixed solution $y = y^*(t)$*,* $t \in I$ *of* (1.7)*, there exists a unique constant* K^* *and a unique bounded solution* $y = \delta(t)$*,* $t \in I$ *of* (1.7) *such that*

$$y^*(t) = K^* Y(t) + \delta(t). \tag{2.9}$$

Proof. i) The case of the solution $y = y^*(t)$ *being eventually strictly monotone.*
Let us discuss the behaviour of the solution $y = y^*(t)$ on I. If there exists an interval $[t^* - r, t^*]$ with $t^* \geq t_0$ such that

$$y^*(t^*) > y^*(t), \qquad t \in [t^* - r, t^*), \tag{2.10}$$

then, by (2.1), this solution is strictly increasing on the interval $[t^*, \infty)$. Similarly, if there exists an interval $[t^* - r, t^*]$ (without loss of generality and to avoid unnecessary extra definitions of auxiliary points, we use here and below the same interval) such that

$$y^*(t^*) < y^*(t), \qquad t \in [t^* - r, t^*),$$

then, by (2.2), this solution is strictly decreasing on the interval $[t^*, \infty)$. Due to property (2.8), the solution $Y(t)$ is either strictly increasing or strictly decreasing on $[t^*, \infty)$.

Without loss of generality again, we can assume (due to the linearity of the functional L^*) that $Y(t)$ is strictly increasing on $[t^*, \infty)$ and $y = y^*(t)$ is strictly decreasing on $[t^*, \infty)$ and that (by property (2.3))

$$Y(t) > y^*(t) > 0, \qquad t \in [t^* - r, t^*]$$

since, if necessary, we can use a solution $y^*(t) + M_{y^*}$ (M_{y^*} is a suitable constant) instead of $y^*(t)$ and $Y(t) + M_Y$ (M_Y is a suitable constant) instead of $Y(t)$ because

$$L^*(t, (y^* + M_{y^*})_t) = L^*(t, y^*_t) + L^*(t, (M_{y^*})_t) = L^*(t, y^*_t)$$

and

$$L^*(t,(Y+M_Y)_t) = L^*(t,Y_t) + L^*(t,(M_Y)_t) = L^*(t,Y_t)$$

where $(M_{y^*})_t$ and $(M_Y)_t$ are constant functions (generated by constants M_{y^*}, M_Y) from $\mathcal{C}$. Set in Lemma 2.2

$$t_0 := t^*, \qquad C_1 := y^*(t^*), \qquad C_2 := Y(t^*)$$

and use C_1, C_2 in the definition of the set ω by (2.4).

Now, define a system of initial functions $p_{A(t^*),\omega}$ by the formula

$$p(z)(\theta) = p(t^*,y)(\theta) = \frac{y-C_1}{C_2-C_1}Y(t^*+\theta) + \frac{C_2-y}{C_2-C_1}y^*(t^*+\theta), \qquad \theta \in [-r,0]. \tag{2.11}$$

Since, obviously,

$$C_1 < p(t^*,y)(\theta) < C_2, \qquad \theta \in [-r,0]$$

for every $y \in (C_1,C_2)$,

$$p(t^*,C_1)(0) = C_1, \qquad p(t^*,C_2)(0) = C_2$$

and

$$C_1 < p(t^*,C_1)(\theta) < C_2, \qquad C_1 < p(t^*,C_2)(\theta) < C_2, \qquad \theta \in [-r,0),$$

assumptions (α), (β) of Definition 2.1 hold.

Let us apply Lemma 2.2 where, as announced above, $t_0 := t^*$. We need to verify that, for all points $(t,y^*) \in \partial\omega$ and for all functions $\pi \in \mathcal{C}$ such that $\pi(0) = y^*$ and $(t+\theta,\pi(\theta)) \in \omega$, $\theta \in [-r,0)$, inequality (2.6) with $f(t,\pi) := L^*(t,\pi)$ holds, that is,

$$(2y^* - C_1 - C_2)L^*(t,\pi) > 0. \tag{2.12}$$

In our case, there are only two possibilities, either $y^* = C_1$ or $y^* = C_2$. If the fist equality is true, the left-hand side of (2.12) equals

$$(2y^* - C_1 - C_2)L^*(t,\pi) = (C_1 - C_2)L^*(t,\pi). \tag{2.13}$$

Since, in the considered case for functions $\pi \in \mathcal{C}$, we have

$$\pi(\theta) > C_1 = \pi(0), \qquad \theta \in [-r,0), \tag{2.14}$$

property (2.2) implies $L^*(t,\pi) < 0$, and from (2.13), we conclude that (2.12) holds. If the second equality is true, we get

$$(2y^* - C_1 - C_2)L^*(t,\pi) = (C_2 - C_1)L^*(t,\pi) \tag{2.15}$$

and, in the considered case for functions $\pi \in \mathcal{C}$, we have

$$\pi(\theta) < C_2 = \pi(0), \qquad \theta \in [-r,0).$$

From (2.1), we have $L^*(t,\pi) > 0$ and (2.15) implies that (2.12) holds again.

All assumptions of Lemma 2.2 hold. Then, by its statement, for our system of initial functions $p_{A(t^*),\omega}$ defined by (2.11), there exists a point

$$z^{**} = (t^*,y^{**}) = (t^*,y^{**}) \in A(t^*) \cap \omega,$$

where $C_1 < y^{**} < C_2$, such that, for the corresponding solution $y(t^*, p(z^{**}))$ of (1.7), property (2.7) holds. In our case, this property says that

$$C_1 < y(t^*, p(z^{**}))(t) < C_2, \qquad t \in [t^* - r, \infty),$$

that is, the solution $y(t^*, p(z^{**}))(t)$ is bounded. In the following, we set

$$\delta^*(t) := y(t^*, p(z^{**}))(t).$$

By (2.11), the initial function that defines this solution is

$$p(z^{**})(\theta) = \frac{y^{**} - C_1}{C_2 - C_1} Y(t^* + \theta) + \frac{C_2 - y^{**}}{C_2 - C_1} y^*(t^* + \theta), \qquad \theta \in [-r, 0] \tag{2.16}$$

and, therefore,

$$\delta^*(t) = \frac{y^{**} - C_1}{C_2 - C_1} Y(t) + \frac{C_2 - y^{**}}{C_2 - C_1} y^*(t), \qquad t \in [t^* - r, \infty). \tag{2.17}$$

Solving (2.17) with respect to $y^*(t)$, we derive

$$y^*(t) = \frac{C_1 - y^{**}}{C_2 - y^{**}} Y(t) + \frac{C_2 - C_1}{C_2 - y^{**}} \delta^*(t), \qquad t \in [t^* - r, \infty). \tag{2.18}$$

Finally, set

$$K^* := \frac{C_1 - y^{**}}{C_2 - y^{**}}$$

and

$$\delta(t) := \frac{C_2 - C_1}{C_2 - y^{**}} \delta^*(t).$$

Because of linearity of (1.7), the function $\delta(t)$ is a bounded solution again. Then, (2.18) can be rewritten as

$$y^*(t) = K^* Y(t) + \delta(t), \qquad t \in [t^* - r, \infty). \tag{2.19}$$

This representation on interval $[t^* - r, \infty)$ coincides with (2.9). It remains to be explained why representation (2.19) holds on I. This follows from (2.17) because $Y(t)$ and $y^*(t)$ are also defined on $[t_0 - t, t^* - r)$, so $\delta^*(t)$ can be defined on this interval by the same formula (2.17). Then, (2.17) as well as (2.19) hold on I. Taking into account that $K^* \neq 0$, we conclude that the statement of Theorem 2.3 given by formula (2.9) holds in the considered case, that is, if (2.10) holds.

ii) The case of the solution $y = y^*(t)$ being eventually not strictly monotone.
We will prove that formula (2.9) holds even if (2.10) does not hold. It means that the solution $y^*(t)$ is not eventually either strictly increasing or strictly decreasing. Moreover, we can easily deduce that $y^*(t)$ is bounded on I and

$$\min_{s \in [t^0 - r, t^0]} y^*(s) \le y^*(t) \le \max_{s \in [t^0 - r, t^0]} y^*(s), \qquad t \in I.$$

In this case, formula (2.9) holds as well since we can put $K^* = 0$ and $\delta(t) \equiv y^*(t)$.

Although the uniqueness follows, in both cases i), and ii), from the method of proof, we remark that, if $y^*(t)$ admits on I two representations of the type (2.9)

$$y^*(t) = K_1^* Y(t) + \delta_1(t), \qquad t \in I$$

and

$$y^*(t) = K_2^* Y(t) + \delta_2(t), \qquad t \in I$$

with $K_1^* \neq K_2^*$ and $\delta_1(t) \not\equiv \delta_2(t)$ on I, then the difference between both expressions yields

$$0 = (K_1^* - K_2^*)Y(t) + (\delta_1(t) - \delta_2(t)), \qquad t \in I.$$

By (2.8), this is only possible if $K_1^* = K_2^*$. But, in this case,

$$\delta_1(t) = \delta_2(t), \qquad t \in I.$$

□

Corollary 2.4. *If all assumptions of Theorem 2.3 are valid, then, from its proof, the following statements also hold.*

1) *If $y^*(t)$ is eventually either strictly increasing or strictly decreasing, then formula (2.9) where $K^* \neq 0$ holds. In other words, every eventually either strictly increasing or strictly decreasing solution $y^*(t)$ is asymptotically equivalent with a multiple $1/K^*$ of $Y(t)$ and*

$$y^*(t) \sim \frac{1}{K^*} Y(t), \qquad t \to \infty.$$

This simultaneously says that there exists no eventually either strictly increasing or strictly decreasing solution $y^*(t)$ with a finite limit $\lim_{t\to\infty} y^*(t)$.

2) *Let two constants C_1, C_2, $C_1 < C_2$ be given. Then, there exists a nonconstant solution $y = y(t)$, $t \in I$ of (1.7), neither strictly increasing nor strictly decreasing such that*

$$C_1 < y(t) < C_2, \qquad t \in I.$$

To prove this statement, it is sufficient to consider a system of initial initial functions $p_{A(t_0),\omega}$ where $p(z^{**})$ is similar to (2.16):

$$p(z^{**})(\theta) = \frac{y^{**} - C_1}{C_2 - C_1} \tilde{Y}(t_0 + \theta) + \frac{C_2 - y^{**}}{C_2 - C_1} \tilde{y}^*(t_0 + \theta), \qquad \theta \in [-r, 0]$$

with the difference that $\tilde{Y} \in \mathcal{C}$ is a strictly increasing function, $\tilde{y} \in \mathcal{C}$ is a strictly decreasing function and $\tilde{Y}$, $\tilde{y}$ are linearly independent.

2.3 Long-time behaviour of solutions in the convergent case

In this part, we will formulate a complement to Theorem 2.3 about the long-time behaviour of solutions in the case of a strictly increasing initial function not defining a solution $Y(t)$ with property (2.8). A simple analysis shows that, in such a case, each solution of (1.7) converges to a finite limit.

Theorem 2.5. *Let the functional L^*, defined on $I_0 \times \mathcal{C}$, be linear with respect to the second argument and satisfy (2.1), (2.2). Let no solution $y = Y(t)$, $t \in I$ exist of (1.7) satisfying (2.8). Then, every fixed solution $y = y^*(t)$, $t \in I$ of (1.7) is convergent, that is, there exists a finite limit*

$$\lim_{t\to\infty} y^*(t) = M^* \in \mathbb{R}.$$

In other words, for every solution, there exists a unique constant M^* and a unique solution $y = \varepsilon(t)$, $t \in I$ of (1.7) such that

$$y^*(t) = M^* + \varepsilon(t) \tag{2.20}$$

and

$$\lim_{t\to\infty} \varepsilon(t) = 0. \tag{2.21}$$

Proof. Every initial problem
$$y^*_{t_0} = \varphi \in \mathcal{C}$$
for (1.7) defines a unique solution $y^*(t_0, \varphi)$ on I. Moreover, $y^*(t_0, \varphi)$, being a continuously differentiable solution of (1.7) on I_0, is absolutely continuous on $[t_0, t_0 + r]$. It is well-known that every absolutely continuous function can be written as the difference of two strictly increasing absolutely continuous functions (cite, for example, [40, p. 315]). Therefore, on $[t_0, t_0 + r]$, we can write
$$y^*(t_0, \varphi)(t) = y^{*1}(t_0, \varphi)(t) - y^{*2}(t_0, \varphi)(t)$$
where $y^{*i}(t_0, \varphi)(t)$, $i = 1, 2$ are strictly increasing and absolutely continuous functions. Obviously, due to linearity, both $y^{*i}(t_0, \varphi)(t)$, $i = 1, 2$ can be used as initial functions, and initial problems
$$y^*_{t_0+r} = y^{*i}_{t_0+r}(t_0, \varphi), \qquad i = 1, 2$$
can be discussed. Both initial problems define strictly increasing solutions $y^{*i}(t_0, \varphi)(t)$, $i = 1, 2$ on I_0 (we will not repeat the details given in the proof of Theorem 2.3). Neither of $y^{*i}(t_0, \varphi)(t)$, $i = 1, 2$ satisfies
$$\lim_{t\to\infty} y^{*i}(t_0, \varphi)(t) = \infty, \qquad i = 1, 2$$
since this is excluded by the assumptions of the theorem. Therefore, since the above limits exists, there are constants M^*_i, $i = 1, 2$ such that
$$\lim_{t\to\infty} y^{*i}(t_0, \varphi)(t) = M^*_i, \qquad i = 1, 2.$$
Finally, since
$$y^*(t_0, \varphi)(t) = y^{*1}(t_0, \varphi)(t) - y^{*2}(t_0, \varphi)(t), \qquad t \in I_0,$$
we have
$$\lim_{t\to\infty} y^*(t_0, \varphi)(t) = \lim_{t\to\infty} (y^{*1}(t_0, \varphi)(t) - y^{*2}(t_0, \varphi)(t)) = M^* := M^*_1 - M^*_2.$$
The proof of the uniqueness of representation (2.20) can be performed in much the same way as the proof of the uniqueness of representation (2.9) in the proof of Theorem 2.3. Assume that a solution $y^*(t)$ admits two different representations of the type (2.20), that is,
$$y^*(t) = M^*_a + \varepsilon_a(t)$$
and
$$y^*(t) = M^*_b + \varepsilon_b(t).$$
Subtracting, we get
$$0 = M^*_a - M^*_b + \varepsilon_a(t) - \varepsilon_b(t).$$
By (2.21), we get $M^*_a = M^*_b$ and, consequently, $\varepsilon_a(t) \equiv \varepsilon_b(t)$. □

Corollary 2.6. *If all assumptions of Theorem 2.5 are valid, then, obviously, the existence of a strictly increasing and convergent solution is equivalent with the convergence of all solutions.*

This statement can be improved irrespective of whether a solution $y = Y(t)$, $t \in I$ of (1.7) satisfying (2.8) exists. Omitting this existence, the following holds:

1) If there exists a strictly increasing and convergent solution of (1.7), then all solutions are convergent.

2) *If all bounded solutions of* (1.7) *are convergent, then all solutions of* (1.7) *are convergent.*

3) *For every two strictly increasing and positive solutions* $y = Y_i(t)$, $i = 1, 2$, $t \in I$ *of* (1.7)*, there exists a constant* $\nu \in (0, \infty)$ *such that*

$$\lim_{t\to\infty} \frac{Y_1(t)}{Y_2(t)} = \nu.$$

Remark 2.7. In part 3 below, we will discuss the existence of eventually positive solutions to equation (1.2). In this connection, the following remark is important. If all assumptions of Theorem 2.5 are valid, then formula (2.20) holds. Let us remark that, in this case, there exists a solution

$$y = \varepsilon_+^*(t), \qquad t \in I$$

of equation (1.7) such that

$$\varepsilon_+^*(t) > 0, \qquad t \in I, \qquad \lim_{t\to\infty} \varepsilon_+^*(t) = 0. \tag{2.22}$$

To explain the existence of such a solution, assume that a function $\varphi \in \mathcal{C}$ satisfies $\varphi(\theta) > \varphi(0)$, $\theta \in [-r, 0)$. Then, property (2.2) implies that the solution $y(t_0, \varphi)(t)$ is strictly decreasing on I_0 and, by Theorem 2.5, there exists a finite limit

$$\lim_{t\to\infty} y(t_0, \varphi)(t) = M^{**}$$

and

$$y(t_0, \varphi)(t) > M^{**}, \qquad t \in I.$$

By property (2.3), the function

$$\varepsilon_+^*(t) := y(t_0, \varphi)(t) - M^{**}, \qquad t \in I$$

is a solution of (1.7) obviously having properties (2.22).

3 Structure of solutions of equation (1.2)

It this part, we use the results derived in part 2 to explain the asymptotic behaviour of solutions to equation (1.2), that is,

$$\dot{x}(t) = -L(t, x_t)$$

for $t \to \infty$ in the so called "non-oscillatory case". This terminology, in contrast to the oscillatory case of all solutions of (1.2) being oscillatory for $t \to \infty$, is used if (1.2) has an eventually positive solution. We recall that $L(t, \cdot)$ is a linear functional with respect to the second argument.

Theorem 3.1. *Let the functional* L*, defined on* $I_0 \times \mathcal{C}$*, be linear and strongly increasing with respect to the second argument. If there exists a positive solution* $x = \varphi(t)$, $t \in I$ *of* (1.2)*, then there exist two positive solutions* $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $t \in I$ *of* (1.2) *such that*

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = 0. \tag{3.1}$$

Proof. Substituting

$$x(t) = \varphi(t)y(t) \tag{3.2}$$

in (1.2), we get

$$\dot{\varphi}(t)y(t) + \varphi(t)\dot{y}(t) = -L(t,(y\varphi)_t). \tag{3.3}$$

or

$$\dot{y}(t) = \frac{1}{\varphi(t)}\left[L(t,\varphi_t)y(t) - L(t,(y\varphi)_t)\right]. \tag{3.4}$$

Because of the linearity of L, the right-hand side of (3.4) equals

$$\begin{aligned}\frac{1}{\varphi(t)}&\left[L(t,\varphi_t)y(t) - L(t,(y\varphi)_t)\right]\\ &= \frac{1}{\varphi(t)}\left[L(t,y(t)\varphi_t) - L(t,(y\varphi)_t)\right] = \frac{1}{\varphi(t)}L(t,y(t)\varphi_t - (y\varphi)_t).\end{aligned} \tag{3.5}$$

Define a functional $L^*(t,y_t)$ on the right-hand side of (1.7) by the formula (suggested by (3.4) and (3.5))

$$L^*(t,\psi) := \frac{1}{\varphi(t)}L(t,\psi(0)\varphi_t - \psi\varphi_t) = \frac{1}{\varphi(t)}L(t,(\psi(0)-\psi)\varphi_t).$$

The functional $L^*(t,\cdot)$ is linear with respect to the second argument since, for arbitrary constants α, β and functions $\xi,\zeta \in \mathcal{C}$, we have

$$\begin{aligned}L^*(t,\alpha\xi+\beta\zeta) &= \frac{1}{\varphi(t)}L(t,((\alpha\xi(0)+\beta\zeta(0)) - (\alpha\xi+\beta\zeta))\varphi_t)\\ &= \frac{1}{\varphi(t)}L(t,(\alpha\xi(0)+\beta\zeta(0))\varphi_t) - \frac{1}{\varphi(t)}L(t,(\alpha\xi+\beta\zeta)\varphi_t)\\ &= \frac{1}{\varphi(t)}L(t,\alpha\xi(0)\varphi_t+\beta\zeta(0)\varphi_t) - \frac{1}{\varphi(t)}L(t,\alpha\xi\varphi_t+\beta\zeta\varphi_t)\\ &= \frac{\alpha}{\varphi(t)}L(t,\xi(0)\varphi_t) + \frac{\beta}{\varphi(t)}L(t,\zeta(0)\varphi_t) - \frac{\alpha}{\varphi(t)}L(t,\xi\varphi_t) - \frac{\beta}{\varphi(t)}L(t,\zeta\varphi_t)\\ &= \frac{\alpha}{\varphi(t)}L(t,(\xi(0)-\xi)\varphi_t) + \frac{\beta}{\varphi(t)}L(t,(\zeta(0)-\zeta)\varphi_t)\\ &= \alpha L^*(t,\xi) + \beta L^*(t,\zeta).\end{aligned}$$

Now, let us verify that inequality (2.1) holds. The functional L is strongly increasing, therefore, if $\psi(0) > \psi(\theta)$, $\theta \in [-r,0)$, we have

$$L^*(t,\psi) = \frac{1}{\varphi(t)}L(t,\psi(0)\varphi_t - \psi\varphi_t) > \frac{1}{\varphi(t)}L(t,\psi(0)\varphi_t - \psi(0)\varphi_t) = \frac{\psi(0)}{\varphi(t)}L(t,\varphi_t-\varphi_t) = 0.$$

Similarly, inequality (2.2) holds since, if $\psi(0) < \psi(\theta)$, $\theta \in [-r,0)$, we get

$$L^*(t,\psi) = \frac{1}{\varphi(t)}L(t,\psi(0)\varphi_t - \psi\varphi_t) < \frac{1}{\varphi(t)}L(t,\psi(0)\varphi_t - \psi(0)\varphi_t) = \frac{\psi(0)}{\varphi(t)}L(t,\varphi_t-\varphi_t) = 0.$$

We conclude that the functional L^* satisfies assumptions (2.1), (2.2). Consequently, either Theorem 2.3 or Theorem 2.5 can be applied.

Assume, at first, that the assumptions of Theorem 2.3 hold. Then, the family of all solutions to equation (1.2), as follows from the substitution (3.2) and formula (2.9), is

$$x(t) = \varphi(t)y(t) = \varphi(t)y^*(t) = \varphi(t)(K^*Y(t)+\delta(t)), \qquad t \in I. \tag{3.6}$$

Set $K^* = 0$, $\delta(t) = 1$ and define

$$\varphi^{**}(t) := \varphi(t).$$

Moreover, let $K^* = 1$, $\delta(t) = 0$ and define

$$\varphi^*(t) := \varphi(t)Y(t).$$

Solutions $\varphi^{**}(t)$, $\varphi^*(t)$ are positive on I and (3.1) holds since

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = \lim_{t\to\infty} \frac{\varphi(t)}{\varphi(t)Y(t)} = \lim_{t\to\infty} \frac{1}{Y(t)} = 0.$$

It remains to consider the case covered by Theorem 2.5. Then, the family of all solutions to equation (1.2), as follows from substitution (3.2) and formula (2.20), is

$$x(t) = \varphi(t)y(t) = \varphi(t)y^*(t) = \varphi(t)(M^* + \varepsilon(t)), \qquad t \in I. \tag{3.7}$$

Let, in (3.7), $M^* = 0$ and $\varepsilon(t) = \varepsilon^*_+(t)$ where $\varepsilon^*_+(t)$ is a positive solution mentioned in Remark 2.7, with properties given by formula (2.22). Then,

$$x(t) = \varphi(t)\varepsilon^*_+(t), \qquad t \in I.$$

Set

$$\varphi^{**}(t) := \varphi(t)\varepsilon^*_+(t).$$

If $M^* = 1$ and $\varepsilon(t) = 0$ in (3.7), then

$$x(t) = \varphi(t)$$

and we define

$$\varphi^*(t) := \varphi(t).$$

The new set of solutions $\varphi^{**}(t)$, $\varphi^*(t)$ positive on I satisfies (3.1) as well since

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = \lim_{t\to\infty} \frac{\varphi(t)\varepsilon^*_+(t)}{\varphi(t)} = \lim_{t\to\infty} \varepsilon^*_+(t) = 0. \qquad \square$$

Theorem 3.2. *Let the functional L, defined on $I_0 \times \mathcal{C}$, be linear and strongly increasing with respect to the second argument. If there exists a positive solution $x = \varphi(t)$ of (1.2), then there exist two positive solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $t \in I$ of (1.2) satisfying (3.1) such that every solution $x = x(t)$, $t \in I$ of (1.2) can be uniquely represented by the formula*

$$x(t) = K^*\varphi^*(t) + \delta^*(t)\varphi^{**}(t), \qquad t \in I \tag{3.8}$$

where $K^* \in \mathbb{R}$ and $\delta^* \colon I \to \mathbb{R}$ is a continuous and bounded function.

Proof. The assumptions of the theorem are the same as those of Theorem 3.1 and, in the proof, we utilize some parts of the proof of this theorem. Then, obviously, a given solution $x(t)$ is represented either by formulas (3.6) and (3.8) where $\varphi^*(t) = \varphi(t)Y(t)$, $\varphi^{**}(t) = \varphi(t)$, or by formulas (3.7) and (3.8) where $\varphi^*(t) = \varphi(t)$, $\varphi^{**}(t) = \varphi(t)\varepsilon^*_+(t)$.

To prove the uniqueness, assume that, except for the formula (3.8), $x(t)$ is represented, by the formula

$$x(t) = K^*_b\varphi^*(t) + \delta^*_b(t)\varphi^{**}(t), \qquad t \in I,$$

where $K^* \neq K_b^*$, $\delta^*(t) \not\equiv \delta_b^*(t)$, as well. Subtracting, we obtain

$$0 = (K^* - K_b^*)\varphi^*(t) + (\delta^*(t) - \delta_b^*(t))\varphi^{**}(t), \qquad t \in I,$$

or

$$0 = (K^* - K_b^*) + (\delta^*(t) - \delta_b^*(t))\frac{\varphi^{**}(t)}{\varphi^*(t)}, \qquad t \in I. \tag{3.9}$$

Passing to the limit in (3.9) as $t \to \infty$, we see that property (3.1) is in contradiction to our assumption. Therefore, $K^* = K_b^*$ and, consequently, $\delta^*(t) \equiv \delta_b^*(t)$. □

Now we show that the representation (3.8) is, in a sense, independent of the choice of solutions $x = \varphi^*(t)$ and $x = \varphi^{**}(t)$.

Theorem 3.3. *Let the functional L, defined on $I_0 \times \mathcal{C}$, be linear and strongly increasing with respect to the second argument. Let $x = \varphi_A^*(t)$, $x = \varphi_A^{**}(t)$, $t \in I$ be two positive solutions of* (1.2) *such that the property*

$$\lim_{t\to\infty} \frac{\varphi_A^{**}(t)}{\varphi_A^*(t)} = 0 \tag{3.10}$$

holds. Then, every solution $x = x(t)$, $t \in I$ of (1.2) *can be uniquely represented by the formula*

$$x(t) = K_A^*\varphi_A^*(t) + \delta_A^*(t)\varphi_A^{**}(t), \qquad t \in I \tag{3.11}$$

where $K_A^* \in \mathbb{R}$ and $\delta_A^* \colon I \to \mathbb{R}$ is a continuous and bounded function.

Proof. In formula (3.2) of the proof of Theorem 3.1, we assume $\varphi(t) := \varphi_A^{**}(t)$. Then, the family of all solutions to equation (1.2) is described by a formula of the type (3.6), that is,

$$x(t) = \varphi_A^{**}(t)(K_A^* Y_A^{**}(t) + \delta_A^{**}(t)), \qquad t \in I \tag{3.12}$$

where $\lim_{t\to\infty} Y_A^{**}(t) = \infty$ and $\delta_A^{**}(t)$ has the same properties as $\delta(t)$ in the original formula (2.9), and K_A^*, $\delta_A^{**}(t)$ are uniquely determined by $x(t)$. For the choice $x(t) := \varphi^{**}(t)$ in (3.12), we have

$$\varphi^{**}(t) = \varphi_A^{**}(t)(K_B^* Y_A^{**}(t) + \delta_B^{**}(t)), \qquad t \in I \tag{3.13}$$

and, for the choice $x(t) := \varphi^*(t)$ in (3.12), we have

$$\varphi^*(t) = \varphi_A^{**}(t)(K_C^* Y_A^{**}(t) + \delta_C^{**}(t)), \qquad t \in I. \tag{3.14}$$

Using representations (3.13), (3.14), consider the limit

$$0 = \lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = \lim_{t\to\infty} \frac{K_B^* Y_A^{**}(t) + \delta_B^{**}(t)}{K_C^* Y_A^{**}(t) + \delta_C^{**}(t)}.$$

From this, we deduce $K_B^* = 0$ and (3.13) yields

$$\varphi^{**}(t) = \varphi_A^{**}(t)\delta_B^{**}(t), \qquad t \in I. \tag{3.15}$$

In view of (3.15), formula (3.8) can be written as

$$x(t) = K^*\varphi^*(t) + \delta^*(t)\delta_B^{**}(t)\varphi_A^{**}(t), \qquad t \in I. \tag{3.16}$$

Since (3.16) is assumed to be valid for an arbitrary solution $x = x(t)$ of (1.2), set, in (3.16), $x(t) := \varphi_A^*(t)$. Then,

$$\varphi_A^*(t) = K_1^*\varphi^*(t) + \delta_1^*(t)\delta_B^{**}(t)\varphi_A^{**}(t), \qquad t \in I \tag{3.17}$$

or, dividing by $\varphi_A^*(t)$,

$$1 = K_1^* \frac{\varphi^*(t)}{\varphi_A^*(t)} + \delta_1^*(t)\delta_B^{**}(t)\frac{\varphi_A^{**}(t)}{\varphi_A^*(t)}, \qquad t \in I.$$

Passing to limit as $t \to \infty$, we have, by (3.10),

$$1 = K_1^* \lim_{t\to\infty} \frac{\varphi^*(t)}{\varphi_A^*(t)},$$

therefore, $K_1^* \neq 0$. From (3.17), we get

$$\varphi^*(t) = \frac{1}{K_1^*}\varphi_A^*(t) - \frac{1}{K_1^*}\delta_1^*(t)\delta_B^{**}(t)\varphi_A^{**}(t), \qquad t \in I. \tag{3.18}$$

Now, apply (3.15) and (3.18) in (3.8). Then,

$$\begin{aligned} x(t) &= K^*\varphi^*(t) + \delta^*(t)\varphi^{**}(t) \\ &= K^*\left(\frac{1}{K_1^*}\varphi_A^*(t) - \frac{1}{K_1^*}\delta_1^*(t)\delta_B^{**}(t)\varphi_A^{**}(t)\right) + \delta^*(t)\varphi_A^{**}(t)\delta_B^{**}(t) \\ &= \frac{K^*}{K_1^*}\varphi_A^*(t) + \left(\delta^*(t) - \frac{K^*}{K_1^*}\delta_1^*(t)\right)\delta_B^{**}(t)\varphi_A^{**}(t), \qquad t \in I. \end{aligned}$$

To end the proof, we set

$$K_A^* := \frac{K^*}{K_1^*}, \qquad \delta_A^*(t) := \left(\delta^*(t) - \frac{K^*}{K_1^*}\delta_1^*(t)\right)\delta_B^{**}(t).$$

Then, (3.11) is valid. The proof of the uniqueness of such a representation can be done in much the same way as that of Theorem 3.2 and, therefore, is omitted. □

We will finish this part with a theorem saying that equation (1.2) cannot have three positive solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$ and $x = \varphi^{***}(t)$ on I such that, simultaneously,

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = 0 \tag{3.19}$$

and

$$\lim_{t\to\infty} \frac{\varphi^{***}(t)}{\varphi^{**}(t)} = 0. \tag{3.20}$$

Theorem 3.4. *Let the functional L, defined on $I_0 \times \mathcal{C}$, be linear and strongly increasing with respect to the second argument. Then, there exist no three solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $x = \varphi^{***}(t)$, to (1.2) positive on I such that the limit properties (3.19), (3.20) hold.*

Proof. Assume that there exists a triplet of solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$ and $x = \varphi^{***}(t)$ to (1.2) positive on I and such that the limit properties (3.19), (3.20) hold. Then, all assumptions of Theorem 3.2 are satisfied. By formula (3.8), every solution $x = x(t)$, $t \in I$ of (1.2) can be uniquely represented in the form

$$x(t) = K\varphi^{**}(t) + \delta(t)\varphi^{***}(t), \qquad t \in I$$

where $K \in \mathbb{R}$ and $\delta\colon I \to \mathbb{R}$ is a continuous and bounded function. Therefore, for a constant $K = K_1$ and a function $\delta(t) = \delta_1(t)$,

$$\varphi^*(t) = K_1\varphi^{**}(t)) + \delta_1(t)\varphi^{***}(t), \qquad t \in I. \tag{3.21}$$

Considering in formula (3.21) divided by $\varphi^{**}(t)$ the limit for $t \to \infty$, we get, with the aid of (3.19), (3.20),

$$\infty = \lim_{t\to\infty} \frac{\varphi^{*}(t)}{\varphi^{**}(t)} = \lim_{t\to\infty} \left(K_1 + \delta_1(t) \frac{\varphi^{***}(t)}{\varphi^{**}(t)} \right) = K_1 + 0 = K_1.$$

We get a contradiction, therefore, the mentioned triplet of solutions does not exist. □

4 Applications

In this part, we give two applications of the results derived. First we consider a linear integro-differential equation. Then a linear differential equation with multiple delays will be investigated.

4.1 Positive solutions to an integro-differential equation

Let us consider a scalar linear integro-differential equation with delay

$$\dot{x}(t) = -\int_{t-\tau(t)}^{t} c(t,\sigma)x(\sigma)\mathrm{d}\sigma, \tag{4.1}$$

where $c\colon I_0 \times I \to \mathbb{R}^+$ is a continuous function, $\tau\colon I_0 \to (0, r]$ and $t - \tau(t)$ is a non-decreasing function on I_0. We will rewrite the equation (4.1) in the form (1.2). Define

$$L(t,\psi) := \int_{-\tau(t)}^{0} c(t, t+s)\psi(s)\mathrm{d}s, \qquad \psi \in \mathcal{C}. \tag{4.2}$$

Then, the relevant form (1.2) for (4.1) is

$$\dot{x}(t) = -L(t, x_t) = -\int_{-\tau(t)}^{0} c(t, t+s)x_t(s)\mathrm{d}s, \qquad t \in I_0$$

because

$$\int_{-\tau(t)}^{0} c(t, t+s)x_t(s)\mathrm{d}s = \int_{-\tau(t)}^{0} c(t, t+s)x(t+s)\mathrm{d}s = \int_{t-\tau(t)}^{t} c(t,\sigma)x(\sigma)\mathrm{d}\sigma.$$

Obviously, functional (4.2) is linear. Moreover, it is strongly increasing with respect to the second argument within the meaning of Definition 1.1 since

$$\int_{-\tau(t)}^{0} c(t, t+s)\psi_1(s)\mathrm{d}s < \int_{-\tau(t)}^{0} c(t, t+s)\psi_2(s)\mathrm{d}s$$

for arbitrary $\psi_1, \psi_2 \in \mathcal{C}$ satisfying (1.3).

Equation (4.1) is a particular case of equation considered in [19, formula (17)]. The following theorem on the existence of a positive solution to equation (4.1) is an adaptation of Theorem 6 in [19].

Theorem 4.1. *Equation (4.1) has a positive solution $x = x(t)$ on I if and only if there exists a continuous function $\lambda\colon I \to \mathbb{R}$ such that $\lambda(t) > 0$ for $t \in I_0$ and*

$$\lambda(t) \geq \int_{t-\tau(t)}^{t} c(t,\sigma)\mathrm{e}^{\int_{\sigma}^{t} \lambda(u)\,\mathrm{d}u}\,\mathrm{d}\sigma \tag{4.3}$$

on the interval I_0.

If Theorem 4.1 holds and a function $\lambda(t)$ satisfying (4.3) exists, then the existence of a positive solution $x = x(t)$ to equation (4.1) defined on I is guaranteed. Under the above assumptions, Theorems 3.1, 3.2, 3.3 hold as well and, therefore, the following result can be formulated.

Theorem 4.2. *Let $c\colon I_0 \times I \to \mathbb{R}$ be a positive continuous function. If there exists a positive solution $x = x(t)$ to equation (4.1) defined on I, then there are two positive solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $t \in I$ of (4.1) such that*

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = 0 \tag{4.4}$$

and, moreover, every solution $x = x(t)$, $t \in I$ of (4.1) can be uniquely represented by the formula

$$x(t) = K^*\varphi^*(t) + \delta^*(t)\varphi^{**}(t), \qquad t \in I \tag{4.5}$$

where $K^* \in \mathbb{R}$ and $\delta^*\colon I \to \mathbb{R}$ is a continuous and bounded function (K^* and δ^* depend on $x(t)$). In (4.5), solutions $\varphi^*(t)$, $\varphi^{**}(t)$ can be replaced by any two arbitrary positive solutions to equation (4.1) defined on I and satisfying a limit property analogous to (4.4).

For an equation

$$\dot{x}(t) = -c(t)\int_{t-r}^{t} x(\sigma)\,\mathrm{d}\sigma \tag{4.6}$$

being a particular case of (4.1) if $c(t,\sigma) := c(t)$ for every $(t,\sigma) \in I_0 \times I$, $c(t)$ is a positive continuous function and $\tau(t) \equiv r$ on I_0, the following is proved in [19, Theorem 8]:

Theorem 4.3. *For the existence of a solution of equation (4.6) positive on I, the inequality*

$$c(t) \le M, \qquad t \in I_0$$

is sufficient for $M = \alpha(2-\alpha)/r^2 = \mathrm{const}$ with α being the positive root of the equation $2 - \alpha = 2\mathrm{e}^{-\alpha}$. (Approximate values are $\alpha \approx 1.5936$ and $M \approx 0.6476/r^2$.)

Example 4.4. Let equation (4.6) be of the form

$$\dot{x}(t) = -\frac{1}{\mathrm{e}-1}\int_{t-1}^{t} x(\sigma)\mathrm{d}\sigma. \tag{4.7}$$

Obviously, $c(t) \equiv 1/(\mathrm{e}-1)$, $t \in I_0$ and $r = 1$. Applying Theorem 4.3, we compute

$$c(t) \equiv \frac{1}{\mathrm{e}-1} \approx 0.58198 < M \approx 0.6476/r^2 = 0.6476$$

and a solution of (4.7) exists positive on I. Consequently, by Theorem 4.2, there exist two positive solutions on I such that (4.4) holds.

Looking for a solution of (4.7) in the form $x = \exp(-\lambda t)$ where λ is a suitable constant, we arrive at the equation

$$\lambda^2(\mathrm{e}-1) - \mathrm{e}^{\lambda} + 1 = 0$$

having two positive roots, one being $\lambda_1 = 1$ and the other (found with the "WolframAlpha" software) $\lambda_2 \approx 2.35151$. So, equation (4.7) has two positive solutions

$$x = \varphi^*(t) = \exp(-t), \qquad x = \varphi^{**}(t) = \exp(-\lambda_2 t)$$

satisfying (4.4) and an arbitrary solution $x = x(t)$ to (4.7) can be uniquely represented by formula (4.5), that is,

$$x(t) = K^*\,\mathrm{e}^{-t} + \delta^*(t)\mathrm{e}^{-\lambda_2 t}, \qquad t \in I$$

where K^* is a constant and $\delta^*(t)$ is a bounded continuous function.

4.2 A linear differential equation with multiple delays

Consider a scalar linear differential equation with multiple delays of the form

$$\dot{x}(t) = -\sum_{i=1}^{n} c_i(t)x(t-\tau_i(t)) \tag{4.8}$$

where $c_i\colon I_0 \to [0,\infty)$, $i = 1,2,\dots,n$ are continuous functions such that $\sum_{i=1}^{n} c_i(t) > 0$, $t \in I_0$ and delays $\tau_i\colon I_0 \to (0,r]$, $i = 1,2,\dots,n$ are continuous.

Let us rewrite the equation (4.8) in the form (1.2). Obviously, in this case,

$$L(t,\psi) := \sum_{i=1}^{n} c_i(t)\psi(-\tau_i(t)), \tag{4.9}$$

and the functional (4.9) is linear and strongly increasing with respect to the second argument within the meaning of Definition 1.1.

Let us recall the following result [7, Theorem 8].

Theorem 4.5. *Equation (4.8) has a positive solution $x = x(t)$ on I if and only if there is a function $\lambda\colon I \to \mathbb{R}$ such that $\lambda(t) > 0$ for $t \in I_0$ and*

$$\lambda(t) \geq \sum_{i=1}^{n} c_i(t) \exp\left[\int_{t-\tau_i(t)}^{t} \lambda(\sigma)\mathrm{d}\sigma\right], \qquad t \in I_0. \tag{4.10}$$

If Theorem 4.5 holds and a function λ, satisfying (4.10) exists, then the existence of a positive solution $x = x(t)$ to equation (4.8), defined on I, is guaranteed. Theorems 3.1, 3.2, 3.3 can be applied to equation (4.8) because all their assumptions hold. We derive the following result.

Theorem 4.6. *Let $c_i\colon I_0 \to [0,\infty)$, $i = 1,2,\dots,n$ be continuous functions such that $\sum_{i=1}^{n} c_i(t) > 0$, $t \in I_0$ and delays $\tau_i\colon I_0 \to (0,r]$, $i = 1,2,\dots,n$ be continuous. If there exists a positive solution $x = x(t)$ to equation (4.8) defined on I, then there are two positive solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $t \in I$ such that*

$$\lim_{t\to\infty} \frac{\varphi^{**}(t)}{\varphi^*(t)} = 0 \tag{4.11}$$

and, moreover, every solution $x = x(t)$, $t \in I$ of (4.8) can be uniquely represented by the formula

$$x(t) = K^*\varphi^*(t) + \delta^*(t)\varphi^{**}(t), \qquad t \in I \tag{4.12}$$

where $K^* \in \mathbb{R}$ and $\delta^*\colon I \to \mathbb{R}$ is a continuous and bounded function (K^* and δ^* depend on $x(t)$). In (4.12), solutions $\varphi^*(t)$, $\varphi^{**}(t)$ can be replaced by arbitrary two positive solutions to equation (4.8) defined on I and satisfying limit property analogous to (4.11).

A sufficient condition for the existence of a positive solution to (4.8) is, for example, the following [7, Theorem 9], [26, Theorem 3.3.1].

Theorem 4.7. *Let $c_i\colon I_0 \to [0,\infty)$, $i = 1,2,\dots,n$ be continuous functions such that $\sum_{i=1}^{n} c_i(t) > 0$, $t \in I_0$ and let delays $\tau_i\colon I_0 \to (0,r]$, $i = 1,2,\dots,n$ be continuous. If, moreover,*

$$\int_{t-\tau(t)}^{t} \sum_{i=1}^{n} c_i(\sigma)\mathrm{d}\sigma \leq \frac{1}{e} \tag{4.13}$$

where $\tau(t) = \max_i\{\tau_i(t)\}$, $i = 1,\dots,n$ and $t \in I_0$, then equation (4.8) has a positive solution on interval I.

Example 4.8. Let equation (4.8) be of the form

$$\dot{x}(t) = -\frac{1}{5\mathrm{e}}x(t-1) - \frac{1}{5\mathrm{e}}x(t-2). \tag{4.14}$$

Here, $n = 2$, $c_1(t) = c_2(t) = 1/5\mathrm{e}$, $\tau_1(t) = 1$, $\tau_2(t) = 2$ and $\tau(t) = 2$. Inequality (4.13) is satisfied since

$$\int_{t-\tau(t)}^{t} \sum_{i=1}^{n} c_i(\sigma)\mathrm{d}\sigma = \int_{t-2}^{t} \frac{2}{5\mathrm{e}}\mathrm{d}\sigma = \frac{4}{5\mathrm{e}} < \frac{1}{\mathrm{e}}.$$

By Theorem 4.7, there exists a solution of (4.14) positive on I. Then, by Theorem 4.6, there exist two positive solutions on I such that (4.12) holds. Assuming a solution of (4.14) in the exponential form $x = \exp(-\lambda t)$ with suitable constant λ, we arrive at the equation

$$\mathrm{e}^{2\lambda} + \mathrm{e}^{\lambda} - 5\mathrm{e}\lambda = 0.$$

This equation has two positive roots $\lambda = \lambda_i$, $i = 1, 2$. By the "WolframAlpha" software, their values are:

$$\lambda_1 \approx 0.199462, \qquad \lambda_2 \approx 1.32996,$$

and equation (4.14) has two positive solutions

$$x = \varphi^*(t) = \exp(-\lambda_1 t), \qquad x = \varphi^{**}(t) = \exp(-\lambda_2 t)$$

satisfying (4.11). An arbitrary solution $x = x(t)$ to (4.14) can be uniquely represented by formula (4.12),

$$x(t) = K_1\mathrm{e}^{-\lambda_1 t} + \delta_1(t)\mathrm{e}^{-\lambda_2 t}, \qquad t \in I$$

where K_1 is a constant and $\delta_1(t)$ is a bounded function.

Remark 4.9. Consider equation (1.4) with $r = 1$, that is, the equation

$$\dot{x}(t) = -\frac{1}{\mathrm{e}}x(t-1). \tag{4.15}$$

Searching for a positive solution in the form $x = \exp(-\lambda t)$ with a suitable constant λ leads to an equation

$$\mathrm{e}^{\lambda} - \mathrm{e}\lambda = 0$$

which has "only" one positive solution $\lambda = 1$ and, by this approach, we detected "only" one positive solution $x(t) = \mathrm{e}^{-t}$ of equation (4.15). Nevertheless, Theorem 4.6 holds and, therefore, there exist two positive solutions (see formulas (1.5) with $r = 1$). The above case is called critical and was investigated in detail, for example, in [12].

Remark 4.10. Note that the criteria for the existence of positive solutions to various classes of linear delayed differential equations can be found, for example in books [1,2,24–26] and papers [3–7,10–12,19–23,31,38,39,43]. Theorems of the type of Theorem 4.5, including generalizations to linear and non-linear delayed systems can be found in [1,2,7,19,24,25] and sharp explicit criteria for detecting positive solutions are presented in [3,4,6,12,20,22,23,26,31,38].

5 Concluding remarks and open problems

Now we will discuss some interesting features of the results derived, formulating some new, not yet solved problems.

5.1 Dominant and subdominant solutions

Two positive solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $t \in I$ of (1.2) satisfying (3.1) always exist if, as stated in Theorem 3.1, equation (1.2) has a positive solution on I. Moreover, by Theorem 3.2, such two solutions characterize all solutions of equation (1.2) as described by formula (3.8). This formula is "invariant" in a sense since, by Theorem 3.3, a given pair of solutions can be replaced by an arbitrary pair of two positive solutions $x = \varphi^*_A(t)$, $x = \varphi^{**}_A(t)$, $t \in I$ of (1.2) satisfying (3.10).

The above properties are the reason for the following terminology introduced in [12]. The solutions $x = \varphi^*(t)$, $x = \varphi^*_A(t)$ are called dominant and the solutions $x = \varphi^{**}(t)$, $x = \varphi^{**}_A(t)$ are called subdominant. For this type of investigation, we refer also to [32–34] where the first results of this type were obtained by E. Kozakiewicz. An overview of a part of these results is given in [35], and in [30, pp. 155–159], a conclusion about structure of the set of solutions of equation $\dot{x}(t) = -a(t)x(t - h(t))$, based on the above results, is demonstrated). The original sources [32–34] are written in German and [35] in Russian, a short remark [30] (where the references to original sources are not cited) is written in English. Comparing our results with those in [32–34], we see that the main difference is that, there, the existence of dominant and subdominant (in our terminology) solutions is assumed (or sufficient conditions for their existence are given). Our results say that their existence is always guaranteed if there exists a positive solution (irrespectively on the fact whether it is dominant or subdominant), therefore, the results proved explain some characteristic properties of the solutions of a given class of equations under very simple assumptions.

Dominant and subdominant solutions to the Dickman equation (having applications in the number theory), that is, to the equation

$$\dot{x}(t) = -\frac{1}{t}\,x(t-1), \tag{5.1}$$

for $t \to \infty$, have recently been investigated in [17, 36]. Dominant solutions to (5.1) are studied in [36], here it is proved, for example, that a positive solution is asymptotically described by the formula $x(t) \sim C/t$ where the constant C is exactly computed by using the initial function, generating the solution x. From the results of the paper, the existence immediately follows of a second, subdominant, positive solution, which is investigated in [17]. A general approach how to an investigation of the asymptotic of dominant solutions of some classes of linear equations is suggested in [28].

The results of the paper also generalize some of the previous results for particular classes of equations (we refer, for example, to [8, 9, 12, 18, 42]). The paper [18] served as a motivation for the present research to extend some statements to general linear functional differential equations. Two classes of positive solutions to nonlinear functional differential equations are analyzed in [13, 14, 16].

5.2 Oscillating solutions in the "non-oscillatory" case

Consider formula (3.8), that is, the formula

$$x(t) = K^*\varphi^*(t) + \delta^*(t)\varphi^{**}(t), \qquad t \in I$$

where solutions $x = \varphi^*(t)$, $x = \varphi^{**}(t)$, $t \in I$ of (1.2) satisfy (3.1). Since every positive solution of (1.2) is strictly decreasing on I_0, formula (3.1) implies that $\lim_{t\to\infty} \varphi^{**}(t) = 0$ and the following is true: if equation (1.2) has a positive solution on I, then it also has a positive solution on I tending to zero.

Moreover, formula (3.8) implies that every eventually oscillating solution must satisfy this formula as well. Due to the positivity of $\varphi^*(t)$, we get $K^* = 0$. Therefore, every oscillating solution $x_o(t)$ satisfies

$$x_o(t) = \delta^*(t)\varphi^{**}(t), \qquad t \in I$$

and tends to zero with a speed not less than $\varphi^{**}(t)$. In such a case, the function $\delta^*(t)$ is oscillating.

5.3 Open problems

In connection with the above investigation, the following open problems are arising.

Open Problem 1. It was stated above that, if equation (1.2) has one positive solution $x = x(t)$ on I, then it has two classes of positive solutions – dominant and subdominant. There is an open question of how to know when a positive solution $x = x(t)$ defined on I is dominant and when it is subdominant.

Open Problem 2. Similar, but different is a question of which initial-value problem

$$x_{t_0} = \psi \in \mathcal{C}$$

generates a positive solution (some results, for some classes of linear equations, can be found in [26]). A more deep problem should be considered as well – if this problem defines a positive solution, is this solution dominant or subdominant? Some results giving particular answers can be found, for example, in [15, 17].

Open Problem 3. The following problems are not solved here either. How could the above results be extended to cover the neutral delayed linear functional differential equations, differential systems of linear functional equations of retarded-type, and higher-order linear delayed differential equations?

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