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A Perron type theorem for positive solutions of functional differential equations

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. A nonlinear perturbation of a linear autonomous retarded functional differential equation is considered. According to a Perron type theorem, with the possible exception of small solutions the Lyapunov exponents of the solutions of the perturbed equation coincide with the real parts of the characteristic roots of the linear part. In this paper, we study those solutions which are positive in the sense that they lie in a given order cone in the phase space. The main result shows that if the Lyapunov exponent of a positive solution of the perturbed equation is finite, then it is a characteristic root of the unperturbed equation with a positive eigenfunction. As a corollary, a necessary and sufficient condition for the existence of a positive solution of a linear autonomous delay differential equation is obtained.

Keywords: functional differential equation, perturbation, Lyapunov exponent, cone, positivity

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1 Introduction and the main results

Given $r > 0$, let $C = C([-r, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions from $[-r, 0]$ into $\mathbb{R}^n$ with the supremum norm $\|\phi\| = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|$ for $\phi \in C$, where $|\cdot|$ is any norm on $\mathbb{R}^n$.

Consider the nonlinear retarded functional differential equation

$$x'(t) = L(x_t) + f(t, x_t) \tag{1.1}$$

as a perturbation of the linear autonomous equation

$$x'(t) = L(x_t), \tag{1.2}$$

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where $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$, $L : C \to \mathbb{R}^n$ is a bounded linear functional and $f : [\sigma_0, \infty) \times C \to \mathbb{R}^n$ is a continuous function which satisfies some smallness condition specified later. According to the Riesz representation theorem, L has the form

$$L(\phi) = \int_{-r}^{0} d[\eta(\theta)]\phi(\theta), \qquad \phi \in C,$$

where $\eta : [-r,0] \to \mathbb{R}^{n\times n}$ is a matrix function of bounded variation normalized so that η is left continuous on $(-r,0)$ and $\eta(0) = 0$.

In [17], we proved the following generalization of a well-known Perron type theorem for ordinary differential equations [6,16] to Eq. (1.1).

Theorem 1.1. [17, Theorem 3.1] *Let x be a solution of* (1.1) *on* $[\sigma_0 - r, \infty)$ *such that*

$$|f(t,x_t)| \le \gamma(t)\|x_t\|, \qquad t \ge \sigma_0, \tag{1.3}$$

where $\gamma : [\sigma_0,\infty) \to [0,\infty)$ is a continuous function satisfying

$$\int_t^{t+1} \gamma(s)\,ds \to 0 \qquad \text{as } t \to \infty. \tag{1.4}$$

Then either

(i) for each $b \in \mathbb{R}$, we have $\lim_{t\to\infty} e^{bt}x(t) = 0$, or

(ii) the limit

$$\mu = \mu(x) = \lim_{t\to\infty} \frac{\log \|x_t\|}{t} \tag{1.5}$$

exists and is equal to the real part of one of the roots of the characteristic equation

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta). \tag{1.6}$$

Solutions which satisfy conclusion (i) of Theorem 1.1 are known as *small solutions*. The quantity $\mu = \mu(x)$ defined by the limit (1.5) (if it exists) is called the *strict Lyapunov exponent* of the solution x.

For generalizations of Theorem 1.1 to other classes of differential equations and further related results, see the papers by Barreira, Dragičević and Valls [1], Barreira and Valls [2–5], Drisi, Es-sebbar and K. Ezzinbi [7], Matsui, Matsunaga and Murakami [14] and the references therein.

In this paper, we will consider those solutions of (1.1) which are positive with respect to the partial ordering induced by a given cone in C. Recall that a subset K of a real Banach space X is a *cone* if all the three conditions below hold.

(c_1) K is a nonempty, convex and closed subset of X,

(c_2) $tK \subset K$ for all $t \ge 0$, where $tK = \{tx \mid x \in K\}$,

(c_3) $K \cap (-K) = \{0\}$, where $-K = \{-x \mid x \in K\}$.

Each cone K induces a partial ordering $\le_K$ in X by $x \le_K y$ if and only if $y - x \in K$. An element $x \in X$ is called *K-positive* if $0 \le_K x$ and $x \ne 0$. Thus, $x \in X$ is K-positive if and only if $x \in K \setminus \{0\}$. Let K be a cone in C. A solution x of Eq. (1.1) is called *K-positive* on $[\sigma_0 - r, \infty)$ if $x_t \in K \setminus \{0\}$ for $t \ge \sigma_0$.

In this paper, we will show that if the solution x in Theorem 1.1 is positive, then conclusion (ii) can be considerably improved. Our main result is the following theorem.

Theorem 1.2. *In addition to hypotheses* (1.3) *and* (1.4), *suppose that the solution x of* (1.1) *is K-positive on $[\sigma_0 - r, \infty)$ for some cone K in C. Then either*

(i) x is a small solution, or

(ii) the strict Lyapunov exponent μ of x given by (1.5) *is a (real) root of the characteristic equation* (1.6) *with a K-positive eigenfunction, i.e. there exists a nonzero vector $v_\mu \in \mathbb{R}^n$ such that*

$$\Delta(\mu)v_\mu = 0 \tag{1.7}$$

and

$$\phi_\mu \in K, \qquad \textit{where} \quad \phi_\mu(\theta) = e^{\mu\theta}v_\mu \quad \textit{for } \theta \in [-r, 0]. \tag{1.8}$$

The proof of Theorem 1.2 will be given in Section 3.

Note that under the hypotheses of Theorem 1.2 positive small solutions may exist. Indeed, if $n = 1$ and $K = C([-r, 0], [0, \infty))$, then $x(t) = e^{-t^2}$ is a K-positive solution of the equation [11, p. 67]

$$x'(t) = -2te^{1-2t}x(t-1),$$

a perturbation of the linear autonomous equation (1.2) with $L \equiv 0$. For conditions under which a nonautonomous linear scalar delay differential equation has no positive small solutions, see [18, Proposition 4.2].

According to a result due to Henry [13] any small solution of the linear autonomous equation (1.2) must be identically zero after some finite time. Therefore Eq. (1.2) has no K-positive small solution for any cone K in C. It follows from the cone property (c_2) that if μ is a real root of the characteristic equation (1.6) with a K-positive eigenfunction (1.8), then $x(t) = e^{\mu t}v_\mu$ is a K-positive solution of (1.2). This, combined with Theorem 1.2, yields the following necessary and sufficient condition for the existence of a positive solution of Eq. (1.2).

Theorem 1.3. *Let K be a cone in C. Then Eq.* (1.2) *has a K-positive solution on $[-r, \infty)$ if and only if the characteristic equation* (1.6) *has a real root with a K-positive eigenfunction.*

Theorem 1.3 may be viewed as a generalization of a well-known oscillation criterion for linear autonomous delay differential equations [9, 10].

We can interpret the solutions of (1.1) in $\mathbb{R}^n$ and positivity can be defined with respect to a cone in $\mathbb{R}^n$. Given a cone $\tilde{K}$ in $\mathbb{R}^n$, a solution x of Eq. (1.1) is called *$\tilde{K}$-positive* on $[\sigma_0 - r, \infty)$ if $x(t) \in \tilde{K} \setminus \{0\}$ for $t \geq \sigma_0 - r$. It is easily seen that each cone $\tilde{K}$ in $\mathbb{R}^n$ induces a cone in C by $K = \{\phi \in C \mid \phi(\theta) \in \tilde{K} \text{ for } \theta \in [-r, 0]\}$ and every solution of (1.1) which is $\tilde{K}$-positive on $[\sigma_0 - r, \infty)$ is K-positive there. Consequently, from Theorems 1.2 and 1.3, we obtain the following results.

Theorem 1.4. *In addition to hypotheses* (1.3) *and* (1.4), *suppose that the solution x of* (1.1) *is $\tilde{K}$-positive on $[\sigma_0 - r, \infty)$ for some cone $\tilde{K}$ in $\mathbb{R}^n$. Then either*

(i) x is a small solution, or

(ii) the strict Lyapunov exponent μ of x given by (1.5) *is a (real) root of the characteristic equation* (1.6) *and there exists a $\tilde{K}$-positive vector v_μ satisfying* (1.7).

Theorem 1.5. *Let $\tilde{K}$ be a cone in $\mathbb{R}^n$. Then Eq.* (1.2) *has a $\tilde{K}$-positive solution on $[-r, \infty)$ if and only if the characteristic equation* (1.6) *has a real root μ with a $\tilde{K}$-positive vector v_μ satisfying* (1.7).

Theorem 1.5 confirms the conjecture formulated in [19, Sec. 3].

2 Preliminaries

In this section, we introduce the notations and formulate some known results which will be used in the proof of Theorem 1.2.

The linear autonomous equation (1.2) generates in C a strongly continuous semigroup $(T(t))_{t\geq 0}$, where $T(t)$ is the *solution operator* defined by $T(t)\phi = x_t(\phi)$ for $t \geq 0$ and $\phi \in C$, $x_t(\phi)$ being the unique solution of (1.2) with initial value ϕ at zero [11,12]. The infinitesimal generator $A : \mathcal{D}(A) \to C$ of this semigroup is defined by

$$A\phi = \lim_{t\to 0+} \frac{1}{t}[T(t)\phi - \phi] \tag{2.1}$$

whenever the limit exists in C. It is known [11,12] that

$$\mathcal{D}(A) = \{\, \phi \in C \mid \phi' \in C, \phi'(0) = L(\phi) \,\} \qquad \text{and} \qquad A\phi = \phi'. \tag{2.2}$$

The spectrum $\sigma(A)$ of the linear operator $A : \mathcal{D}(A) \to C$ is a point spectrum and it consists of the roots of the characteristic equation (1.6). The *stability modulus* of (1.2) defined by

$$d = \sup\{\, \operatorname{Re}\lambda \mid \det\Delta(\lambda) = 0 \,\} \tag{2.3}$$

is finite. If Λ is a finite set of characteristic roots of (1.2), then C is decomposed by Λ into a direct sum

$$C = P_\Lambda \oplus Q_\Lambda, \tag{2.4}$$

where P_Λ is the *generalized eigenspace of* (1.2) *associated with* Λ (see [12, Sec. 7.5] for the definition) and Q_Λ is the complementary subspace of C such that $T(t)Q_\Lambda \subset Q_\Lambda$ for $t \geq 0$. The corresponding projections of an element $\phi \in C$ in P_Λ and Q_Λ will be denoted by ϕ^{P_Λ} and ϕ^{Q_Λ}, respectively.

In the following proposition, we summarize some important properties of the solutions of the perturbed equation (1.1) with finite Lyapunov exponents [17].

Proposition 2.1. [17, Theorem 3.4 and Proposition 3.5] *Let x be a solution of* (1.1) *satisfying the hypotheses of Theorem 1.1 with a finite strict Lyapunov exponent $\mu(x) = \mu$. Let $P_0 = P_{\Lambda_0}$, $P_1 = P_{\Lambda_1}$ and $Q = Q_\Lambda$, where the spectral sets Λ_0, Λ_1 and Λ are defined by*

$$\Lambda_0 = \Lambda_0(\mu) = \{\lambda \mid \det\Delta(\lambda) = 0,\ \operatorname{Re}\lambda = \mu\}, \tag{2.5}$$

$$\Lambda_1 = \Lambda_1(\mu) = \{\lambda \mid \det\Delta(\lambda) = 0,\ \operatorname{Re}\lambda > \mu\}, \tag{2.6}$$

and

$$\Lambda = \Lambda(\mu) = \{\lambda \mid \det\Delta(\lambda) = 0,\ \operatorname{Re}\lambda \geq \mu\}, \tag{2.7}$$

respectively. Then

$$x_t = x_t^{P_0} + x_t^{P_1} + x_t^{Q}, \qquad t \geq \sigma_0, \tag{2.8}$$

$$x_t^{P_1} = o(\|x_t^{P_0}\|) \qquad \text{as } t \to \infty \tag{2.9}$$

and

$$x_t^{Q} = o(\|x_t^{P_0}\|) \qquad \text{as } t \to \infty. \tag{2.10}$$

Furthermore, there exists $\delta > 0$ such that

$$\frac{\|x_{t+r}\|}{\|x_t\|} \geq \delta, \qquad t \geq \sigma_0. \tag{2.11}$$

The next result gives an estimate on the growth of the solutions of (1.1).

Proposition 2.2. [17, Lemma 3.2] *Let x be a solution of (1.1) satisfying the hypotheses of Theorem 1.1. Then for every $\epsilon > 0$ there exist constants $C_1, C_2 > 0$ such that for all $t \geq \sigma_1 \geq \sigma_0$,*

$$\|x_t\| \leq C_1 \|x_{\sigma_1}\| e^{(d+\epsilon)(t-\sigma_1)} \exp\left(C_2 \int_{\sigma_1}^{t} \gamma(s)\, ds\right), \tag{2.12}$$

where d is the stability modulus of (1.2) given by (2.3).

Finally, we will need a result from [19] which gives a necessary and sufficient condition for the existence of a positive orbit of a linear invertible operator in a finite dimensional real Banach space.

Proposition 2.3. [19, Theorem 3] *Let K_0 be a cone in a finite dimensional real Banach space X_0. Suppose that $S : X_0 \to X_0$ is a linear invertible operator. Then the following statements are equivalent.*

(i) S has a K_0-positive orbit, i.e. there exists $x_0 \in K_0 \setminus \{0\}$ such that $S^m x_0 \in K_0 \setminus \{0\}$ for $m = 1, 2, \ldots$

(ii) S has a positive eigenvalue with a K_0-positive eigenvector.

Note that in [19] Proposition 2.3 is proved for $X_0 = \mathbb{R}^n$, but the same argument is valid in an arbitrary finite dimensional real Banach space.

3 Proof of the main theorem

Before we give a proof of Theorem 1.2, we establish an important lemma. It says that if the perturbed equation (1.1) has a K-positive solution with a finite Lyapunov exponent μ, then the unperturbed equation (1.2) has a K-positive solution which lies in the finite dimensional space $P_{\Lambda_0(\mu)}$ with $\Lambda_0(\mu)$ as in (2.5).

Lemma 3.1. *Let K be a cone in C. Suppose that x is a K-positive solution of (1.1) on $[\sigma_0 - r, \infty)$ satisfying conditions (1.3) and (1.4). Assume also that the strict Lyapunov exponent $\mu(x) = \mu$ is finite so that the spectral set $\Lambda_0(\mu)$ defined by (2.5) is nonempty (see Theorem 1.1). Let $P_0 = P_{\Lambda_0(\mu)}$, the generalized eigenspace of (1.2) associated with $\Lambda_0(\mu)$. Then there exists a nonzero $\phi \in K \cap P_0$ such that*

$$T(t)\phi \in K \cap P_0, \qquad t \geq 0, \tag{3.1}$$

where $(T(t))_{t\geq 0}$ is the solution semigroup of Eq. (1.2).

Proof. Let x be a K-positive solution of (1.1) on $[\sigma_0 - r, \infty)$ satisfying conditions (1.3) and (1.4). For every integer $k \geq \sigma_0/r$ there exists $t_k \in [kr, (k+1)r]$ such that $|x(t_k)| = \|x_{(k+1)r}\|$. For $t \geq 0$ and $k \geq \sigma_0/r$, define

$$y_k(t) = |x(t_k)|^{-1} x(t_k + t) \tag{3.2}$$

so that for $t \geq r$ and $k \geq \sigma_0/r$,

$$(y_k)_t = |x(t_k)|^{-1} x_{t_k + t}. \tag{3.3}$$

We will show that an appropriate subsequence of $\{y_k\}$ converges locally uniformly on $[0, \infty)$ to a continuous limit function y which is a K-positive solution of the unperturbed equation (1.2) on $[0, \infty)$.

For $k \geq \sigma_0/r$, we have

$$|x(t_k)| \leq \|x_{t_k}\| \leq \|x_{kr}\| + \|x_{(k+1)r}\| = \|x_{(k+1)r}\| \left(1 + \frac{\|x_{kr}\|}{\|x_{(k+1)r}\|}\right) = |x(t_k)| \left(1 + \frac{\|x_{kr}\|}{\|x_{(k+1)r}\|}\right).$$

This, together with inequality (2.11) of Proposition 2.1, yields for $k \geq \sigma_0/r$,

$$|x(t_k)| \leq \|x_{t_k}\| \leq |x(t_k)|(1+\delta^{-1}). \tag{3.4}$$

By Proposition 2.2, we have for $t \geq 0$ and $k \geq \sigma_0/r$,

$$|x(t_k+t)| \leq \|x_{t_k+t}\| \leq C_1\|x_{t_k}\|e^{(d+\epsilon)t}\exp\left(C_2\int_{t_k}^{t_k+t}\gamma(s)\,ds\right). \tag{3.5}$$

From this and (3.4), we find for $t \geq 0$ and $k \geq \sigma_0/r$,

$$|y_k(t)| \leq C_1(1+\delta^{-1})e^{(d+\epsilon)t}\exp\left(C_2\int_{t_k}^{t_k+t}\gamma(s)\,ds\right). \tag{3.6}$$

From (1.1) and (1.3), we obtain for $t \geq 0$ and $k \geq \sigma_0/r$,

$$|y_k'(t)| = |x(t_k)|^{-1}|x'(t_k+t)| \leq |x(t_k)|^{-1}(\|L\|+\gamma(t_k+t))\|x_{t_k+t}\|,$$

where $\|L\|$ denotes the operator norm of L. From the last inequality, (3.4) and (3.5), we find for $t \geq 0$ and $k \geq \sigma_0/r$,

$$|y_k'(t)| \leq (\|L\|+\gamma(t_k+t))C_1(1+\delta^{-1})e^{(d+\epsilon)t}\exp\left(C_2\int_{t_k}^{t_k+t}\gamma(s)\,ds\right). \tag{3.7}$$

We will show that the functions y_k, $k \geq \sigma_0/r$, are uniformly bounded and equicontinuous on every compact interval $[0,\tau]$, $\tau > 0$. Let $\tau > 0$ be fixed. By virtue of (1.4), we have

$$\int_t^{t+\tau}\gamma(s)\,ds \longrightarrow 0, \qquad t \to \infty \tag{3.8}$$

and hence

$$M = M(\tau) = \sup_{t\geq\sigma_0}\int_t^{t+\tau}\gamma(s)\,ds < \infty. \tag{3.9}$$

From (3.6) and (3.9), we obtain for $t \in [0,\tau]$ and $k \geq \sigma_0/r$,

$$|y_k(t)| \leq C_1(1+\delta^{-1})e^{(|d|+\epsilon)\tau+C_2M} \equiv C_3 \tag{3.10}$$

which proves the uniform boundedness of the functions y_k, $k \geq \sigma_0/r$, on $[0,\tau]$.

From (3.7) and (3.9), we obtain for $t \in [0,\tau]$ and $k \geq \sigma_0/r$,

$$|y_k'(t)| \leq C_3(\|L\|+\gamma(t_k+t))$$

with C_3 as in (3.10). From this, we find for $0 \leq \tau_1 < \tau_2 \leq \tau$ and $k \geq \sigma_0/r$,

$$\begin{aligned}|y_k(\tau_2)-y_k(\tau_1)| &\leq \int_{\tau_1}^{\tau_2}|y_k'(t)|\,dt \leq C_3\|L\|(\tau_2-\tau_1)+C_3\int_{\tau_1}^{\tau_2}\gamma(t_k+t)\,dt\\ &= C_3\|L\|(\tau_2-\tau_1)+C_3\int_{t_k+\tau_1}^{t_k+\tau_2}\gamma(s)\,ds\end{aligned}$$

and hence

$$|y_k(\tau_2)-y_k(\tau_1)| \leq C_3\|L\|(\tau_2-\tau_1)+C_3\int_{t_k}^{t_k+\tau}\gamma(s)\,ds. \tag{3.11}$$

Let $\eta > 0$. By virtue of (3.8), there exists $\sigma > \sigma_0$ such that

$$\int_t^{t+\tau} \gamma(s)\, ds < \frac{\eta}{2C_3} \qquad \text{whenever } t > \sigma. \tag{3.12}$$

From (3.11) and (3.12), we see that if $k > \sigma/r$ so that $t_k \geq kr > \sigma$, then

$$|y_k(\tau_2) - y_k(\tau_1)| < \eta \qquad \text{whenever } 0 \leq \tau_1 < \tau_2 \leq \tau \text{ and } \tau_2 - \tau_1 < \frac{\eta}{2C_3\|L\|}. \tag{3.13}$$

The uniform continuity of the functions y_k, $k \geq \sigma_0/r$, on $[0,\tau]$ implies the existence of $\rho_k > 0$ such that

$$|y_k(\tau_2) - y_k(\tau_1)| < \eta \qquad \text{whenever } 0 \leq \tau_1 < \tau_2 \leq \tau \text{ and } \tau_2 - \tau_1 < \rho_k. \tag{3.14}$$

Define

$$\rho = \min\left\{ \frac{\eta}{2C_3\|L\|}, \min_{\sigma_0/r \leq k \leq \sigma/r} \rho_k \right\}.$$

Then (3.13) and (3.14) imply that for all $k \geq \sigma_0/r$,

$$|y_k(\tau_2) - y_k(\tau_1)| < \eta \qquad \text{whenever } 0 \leq \tau_1 < \tau_2 \leq \tau \text{ and } \tau_2 - \tau_1 < \rho. \tag{3.15}$$

Since $\eta > 0$ was arbitrary, this proves the equicontinuity of the functions y_k, $k \geq \sigma_0/r$, on $[0,\tau]$. By the application of the Arzèla–Ascoli theorem, we conclude that there exists a subsequence $\{y_{k_j}\}$ of $\{y_k\}$ such that the limit

$$y(t) = \lim_{j\to\infty} y_{k_j}(t), \qquad t \geq 0, \tag{3.16}$$

exists and the convergence is uniform on every compact subinterval of $[0,\infty)$.

Next we show that y is a solution of the unperturbed equation (1.2) on $[0,\infty)$. From (1.1), we obtain for $\tau \geq r$ and $k \geq \sigma_0/r$,

$$x(t_k+\tau) - x(t_k+r) = \int_r^\tau x'(t_k+t)\, dt = \int_r^\tau \big(L(x_{t_k+t}) + f(t_k+t, x_{t_k+t})\big)\, dt$$

and hence

$$\begin{aligned} |x(t_k)|^{-1}x(t_k+\tau) = |x(t_k)|^{-1}x(t_k+r) + \int_r^\tau L(|x(t_k)|^{-1}x_{t_k+t})\, dt \\ + |x(t_k)|^{-1} \int_r^\tau f(t_k+t, x_{t_k+t})\, dt. \end{aligned} \tag{3.17}$$

By virtue of (1.3) and (3.3), we have for $t \in [r,\tau]$ and $k \geq \sigma_0/r$,

$$|x(t_k)|^{-1}|f(t_k+t, x_{t_k+t})| \leq \gamma(t_k+t)|x(t_k)|^{-1}\|x_{t_k+t}\| = \gamma(t_k+t)\|(y_k)_t\| \leq C_3\gamma(t_k+t),$$

where the last inequality follows from (3.10). This implies for $\tau \geq r$ and $k \geq \sigma_0/r$,

$$|x(t_k)|^{-1} \int_r^\tau |f(t_k+t, x_{t_k+t})|\, dt \leq C_3 \int_r^\tau \gamma(t_k+t)\, dt = C_3 \int_{t_k+r}^{t_k+\tau} \gamma(s)\, ds \longrightarrow 0, \qquad k \to \infty,$$

as a consequence of (3.8). By passing to a limit in (3.17) and using (3.2), (3.3) and (3.16), we obtain for $\tau \geq r$,

$$y(\tau) = y(r) + \int_r^\tau L(y_t)\, dt.$$

Thus, y is a solution of the linear autonomous equation (1.2) on $[0,\infty)$. We will show that $\phi = y_r$ has the desired properties. Clearly, $y_t = T(t-r)\phi$ for $t \geq r$. Since $|y_k(0)| = 1$ for $k \geq \sigma_0/r$, by passing to a limit $k = k_j \to \infty$, we have $|\phi(-r)| = |y(0)| = 1$. Thus, ϕ is a nonzero element of C. From (3.3), the K-positivity of x and the cone property (c_2), we see that $(y_k)_t \in K$ for $t \geq r$. By passing to a limit and using the closedness of K, we find that $y_t = T(t-r)\phi \in K$ for $t \geq r$ and hence $T(t)\phi \in K$ for $t \geq 0$. Finally, we show that $\phi \in P_0$. Since P_0 is invariant under the solution semigroup $(T(t))_{t\geq 0}$ [11, 12], this will complete the proof of (3.1). Applying the spectral projection of (1.2) associated with the set $\Lambda_1 = \Lambda_1(\mu)$ given by (2.6) to Eq. (3.16), we find that

$$\phi^{P_1} = y_r^{P_1} = \lim_{j\to\infty} (y_{k_j})_r^{P_1}. \tag{3.18}$$

From (3.3), we obtain for $k \geq \sigma_0/r$,

$$(y_k)_r^{P_1} = |x(t_k)|^{-1} x_{t_k+r}^{P_1}. \tag{3.19}$$

The boundedness of the spectral projections implies that

$$x_t^{P_0} = O(\|x_t\|), \qquad t \to \infty,$$

which, together with conclusion (2.9) of Proposition 2.1, yields

$$x_t^{P_1} = o(\|x_t\|), \qquad t \to \infty. \tag{3.20}$$

From (3.3) and (3.19), we find for $k \geq \sigma_0/r$,

$$\|(y_k)_r^{P_1}\| = |x(t_k)|^{-1} \|x_{t_k+r}\| \frac{\|x_{t_k+r}^{P_1}\|}{\|x_{t_k+r}\|} = \|(y_k)_r\| \frac{\|x_{t_k+r}^{P_1}\|}{\|x_{t_k+r}\|}. \tag{3.21}$$

By similar estimates as in the proof of (3.10), we obtain for $k \geq \sigma_0/r$,

$$\|(y_k)_r\| \leq C_1(1+\delta^{-1}) e^{(|d|+\epsilon)r + C_2 M(r)}.$$

This, together with (3.20) and (3.21), implies that $(y_k)_r^{P_1} \longrightarrow 0$ as $k \to \infty$. From this and (3.18), we find that $\phi^{P_1} = y_r^{P_1} = 0$. It can be shown in a similar manner that $\phi^Q = 0$. Therefore

$$\phi = \phi^{P_0} + \phi^{P_1} + \phi^Q = \phi^{P_0} \in P_0$$

as desired. □

Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let x be a K-positive solution of (1.1) on $[\sigma_0 - r, \infty)$. Suppose that x is not a small solution. By Theorem 1.1, the strict Lyapunov exponent $\mu(x) = \mu$ is finite and the spectral set $\Lambda_0 = \Lambda_0(\mu)$ given by (2.5) is nonempty. It is known [11, 12] that the generalized eigenspace $P_0 = P_{\Lambda_0}$ of (1.2) associated with Λ_0 is finite dimensional and invariant under the solution semigroup $(T(t))_{t\geq 0}$ of (1.2) with infinitesimal generator given by (2.2). Since $P_0 \subset \mathcal{D}(A)$ is finite dimensional, it is a closed subspace of C and therefore we can define the subspace semigroup $(T_0(t))_{t\geq 0}$ on P_0 by $T_0(t) = T(t)|_{P_0}$, the restriction of $T(t)$ to P_0 [8, Paragraph I.5.12]. Its generator is $A_0 = A|_{P_0}$ with domain $\mathcal{D}(A_0) = P_0$ [8, Paragraph II.2.3]. It is known [11, 12]) that $\sigma(A_0) = \Lambda_0$. Since $\dim P_0 < \infty$, the generator $A_0 : P_0 \to P_0$ is bounded and therefore $T_0(t) = e^{tA_0}$ for $t \geq 0$ [15, Chap. I, Sec. 1.1]. By the spectral mapping theorem [15], we have

$$\sigma(T_0(t)) = e^{t\sigma(A_0)} = \{\, e^{\lambda t} \mid \lambda \in \Lambda_0 \,\}, \qquad t \geq 0. \tag{3.22}$$

Define $K_0 = K \cap P_0$. By the application of Lemma 3.1, we conclude that there exists a nonzero $\phi \in K_0$ such that $T_0(t)\phi \in K_0$ for $t \geq 0$. Since $T_0(t) : P_0 \to P_0$ is invertible for $t \geq 0$, we have

$$T_0(t)\phi \in K_0 \setminus \{0\}, \qquad t \geq 0. \tag{3.23}$$

It is easily seen that K_0 is a cone in P_0. Choose a sequence of positive numbers $t_k \to 0$ and for each k consider the linear operator $S = T_0(t_k)$ in P_0. By the semigroup property, we have $S^m = T_0(mt_k)$ for $m = 1, 2, \ldots$ This, together with (3.23), implies that the orbit of S starting from ϕ is K_0-positive. According to Proposition 2.3, this implies the existence of a positive eigenvalue ρ_k of $S = T_0(t_k)$ with a K_0-positive eigenvector ψ_k. Without loss of generality, we may assume that $\|\psi_k\| = 1$. Otherwise, we replace ψ_k with $\|\psi_k\|^{-1}\psi_k$ which remains in K_0 by the cone property (c_2). By virtue of (3.22), $\rho_k = e^{\lambda t_k}$ for some characteristic root of (1.2) with $\operatorname{Re}\lambda = \mu$. From this, using the positivity of ρ_k, we find that

$$\rho_k = |\rho_k| = |e^{\lambda t_k}| = e^{t_k \operatorname{Re}\lambda} = e^{\mu t_k}.$$

Thus,

$$T_0(t_k)\psi_k = e^{\mu t_k}\psi_k, \qquad \psi_k \in K_0, \quad \|\psi_k\| = 1 \tag{3.24}$$

for $k = 1, 2, \ldots$ By passing to a subsequence, we can ensure that the limit

$$\phi_\mu = \lim_{k\to\infty} \psi_k \tag{3.25}$$

exists in P_0. Evidently $\|\phi_\mu\| = 1$. The set K_0 is closed in P_0, therefore $\phi_\mu \in K_0$. From (3.24), we find that

$$\frac{T_0(t_k)\psi_k - \psi_k}{t_k} = \frac{e^{\mu t_k} - 1}{t_k}\psi_k.$$

for $k = 1, 2, \ldots$ From this, letting $k \to \infty$, using (3.25) and the fact that

$$\left.\frac{d^+}{dt}\right|_{t=0} T_0(t) = \left.\frac{d^+}{dt}\right|_{t=0} e^{tA_0} = A_0,$$

we obtain

$$\phi_\mu' = A_0\phi_\mu = \mu\phi_\mu$$

which, together with $\phi_\mu \in \mathcal{D}(A)$, implies that ϕ_μ has the form as in (1.8). □

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