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Parametrisation for boundary value problems with transcendental non-linearities using polynomial interpolation

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. A constructive technique of analysis involving parametrisation and polynomial interpolation is suggested for general non-local problems for ordinary differential systems with locally Lipschitzian transcendental non-linearities. The practical application of the approach is shown on a numerical example.

Keywords: boundary value problem, transcendental non-linearity, parametrisation, successive approximations, Chebyshev nodes, interpolation

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1 Introduction

The present note deals with parametrisation techniques for constructive investigation of boundary value problems and its purpose is to provide a justification of the *polynomial* version of the method suggested in [14].

We consider the non-local boundary value problem

$$u'(t) = f(t, u(t)), \qquad t \in [a, b], \tag{1.1}$$

$$\phi(u) = \gamma, \tag{1.2}$$

where $\phi : C([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a non-linear vector functional, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in a certain bounded set, and $\gamma \in \mathbb{R}^n$ is a given vector.

By a solution of the problem (1.1), (1.2) we understand a continuously differentiable vector function with property (1.2) satisfying (1.1) everywhere on $[a, b]$.

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The idea of our approach (see, e. g., [14, 16, 17]) is based on the reduction (1.1), (1.2) to a family of simpler auxiliary problems with two-point linear separated conditions at a and b:

$$u(a) = \xi, \qquad u(b) = \eta, \tag{1.3}$$

where ξ and η are unknown parameters. By doing so, one can use in the non-local case the techniques adopted to two-point problems [14].

2 Notation and preliminary results

In order to use the reduction to two-point problems (1.1), (1.3), we need some results from [14]. The study of problems (1.1), (1.3) in [14] is based on properties of the iteration sequence $\{u_m(\cdot,\xi,\eta) : m \geq 0\}$ defined as follows:

$$u_0\,(t,\xi,\eta) := \left(1 - \frac{t-a}{b-a}\right)\xi + \frac{t-a}{b-a}\eta, \tag{2.1}$$

$$\begin{aligned} u_m\,(t,\xi,\eta) &:= u_0(t,\xi,\eta) + \int_a^t f\,(s, u_{m-1}\,(s,\xi,\eta))\,ds \\ &\quad - \frac{t-a}{b-a}\int_a^b f\,(s, u_{m-1}\,(s,\xi,\eta))\,ds, \qquad t \in [a,b],\ m = 1,2,\ldots \end{aligned} \tag{2.2}$$

Fix certain closed bounded sets D_0, D_1 in $\mathbb{R}^n$ and assume that we are looking for solutions u of problem (1.1), (1.3) with $u(a) \in D_0$ and $u(b) \in D_1$. Put

$$\Omega := \{(1-\theta)\xi + \theta\eta : \xi \in D_0,\ \eta \in D_1,\ \theta \in [0,1]\} \tag{2.3}$$

and, for any $\varrho \in \mathbb{R}^n_+$, define the set

$$\Omega_\varrho := O_\varrho(\Omega), \tag{2.4}$$

where $O_\varrho(\Omega) := \bigcup_{z\in\Omega} O_\varrho(z)$ and $O_\varrho(\xi) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \varrho\}$ for any ξ. Here and below, the operations $\leq$ and $|\cdot|$ are understood componentwise. Set (2.4) is a *componentwise ϱ-neighbourhood* of Ω.

Introduce some notation. Given a domain $D \subset \mathbb{R}^n$, we write $f \in \mathrm{Lip}_K(D)$ if K is an $n \times n$ matrix with non-negative entries and the inequality

$$|f(t,u) - f(t,v)| \leq K\,|u - v| \tag{2.5}$$

holds for all $\{u,v\} \subset D$ and $t \in [a,b]$. We also put

$$\delta_{[a,b],D}(f) := \sup_{(t,x)\in[a,b]\times D} f(t,x) - \inf_{(t,x)\in[a,b]\times D} f(t,x). \tag{2.6}$$

The computation of the greatest and least lower bounds for vector functions is understood in the componentwise sense.

The following statement is a combination of Proposition 1 and Theorem 3 from [14].

Theorem 2.1 ([14]). *Let there exist a non-negative vector ϱ satisfying the inequality*

$$\varrho \geq \frac{b-a}{4}\,\delta_{[a,b],\Omega_\varrho}(f). \tag{2.7}$$

Assume, furthermore, that there exists a non-negative matrix K such that

$$r(K) < \frac{10}{3(b-a)} \tag{2.8}$$

and $f \in \mathrm{Lip}_K(\Omega_\varrho)$. Then, for all fixed $(\xi,\eta) \in D_0 \times D_1$:

1. For every m, the function $u_m(\cdot,\xi,\eta)$ satisfies the two-point separated boundary conditions (1.3) and

$$\{u_m(t,\xi,\eta) : t \in [a,b]\} \subset \Omega_\varrho.$$

2. The limit

$$u_\infty(t,\xi,\eta) = \lim_{m\to\infty} u_m(t,\xi,\eta) \tag{2.9}$$

exists uniformly in $t \in [a,b]$. The function $u_\infty(\cdot,\xi,\eta)$ satisfies the two-point conditions (1.3).

3. The function $u_\infty(\cdot,\xi,\eta)$ is a unique solution of the integral equation

$$u(t) = \xi + \int_a^t f(s,u(s))ds - \frac{t-a}{b-a}\int_a^b f(s,u(s))ds + \frac{t-a}{b-a}(\eta-\xi), \qquad t \in [a,b], \tag{2.10}$$

or, equivalently, of the Cauchy problem

$$\begin{aligned} u'(t) &= f(t,u(t)) + \frac{1}{b-a}\Delta(\xi,\eta), \qquad t \in [a,b],\\ u(a) &= \xi, \end{aligned} \tag{2.11}$$

where $\Delta : D_0 \times D_1 \to \mathbb{R}^n$ is a mapping given by the formula

$$\Delta(\xi,\eta) := \eta - \xi - \int_a^b f(s,u_\infty(s,\xi,\eta))ds. \tag{2.12}$$

4. The following error estimate holds:

$$|u_\infty(t,\xi,\eta) - u_m(t,\xi,\eta)| \le \frac{10}{9}\alpha_1(t)K_*^m(1_n - K_*)^{-1}\delta_{[a,b],\Omega_\varrho}(f), \tag{2.13}$$

for any $t \in [a,b]$ and $m \ge 0$, where

$$K_* := \frac{3}{10}(b-a)K \tag{2.14}$$

and

$$\alpha_1(t) := 2(t-a)\left(1 - \frac{t-a}{b-a}\right), \qquad t \in [a,b]. \tag{2.15}$$

In (2.13) and everywhere below, the symbol 1_n stands for the unit matrix of dimension n.

Theorem 2.2 ([14, Proposition 8])**.** *Under the assumption of Theorem 2.1, the function $u_\infty(\cdot,\xi,\eta) : [a,b] \times D_0 \times D_1 \to \mathbb{R}^n$ defined by (2.9) is a solution of problem (1.1), (1.2) if and only if the pair of vectors (ξ,η) satisfies the system of $2n$ equations*

$$\Delta(\xi,\eta) = 0, \tag{2.16}$$

$$\phi(u_\infty(\cdot,\xi,\eta)) = \gamma, \tag{2.17}$$

where Δ is given by (2.12).

Equations (2.16), (2.17) are usually referred to as *determining* equations because their roots determine solutions of the original problem. This system, in fact, determines all possible solutions of the original boundary value problem the graphs of which are contained in the region under consideration.

Theorem 2.3 ([14, Theorem 9]). *Let $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with a certain ϱ satisfying* (2.7) *and K such that* (2.8) *holds. Then:*

1. if there exists a pair of vectors $(\xi, \eta) \in D_0 \times D_1$ satisfying (2.16), (2.17)*, then the non-local problem* (1.1), (1.2) *has a solution $u(\cdot)$ such that*

$$\{u(t) : t \in [a,b]\} \subset \Omega_\varrho \tag{2.18}$$

and $u(a) = \xi$, $u(b) = \eta$;

2. if problem (1.1), (1.2) *has a solution $u(\cdot)$ such that* (2.18) *holds and $u(a) \in D_0$, $u(b) \in D_1$, then the pair $(u(a), u(b))$ is a solution of system* (2.16), (2.17).

The solvability of the determining system (2.16), (2.17) can be analysed by using its mth approximate version

$$\eta - \xi - \int_a^b f(s, u_m(s, \xi, \eta))ds = 0, \tag{2.19}$$

$$\phi(u_m(\cdot, \xi, \eta)) = \gamma, \tag{2.20}$$

where m is fixed, similarly to [10,12,13,15]. Equations (2.19), (2.20), in contrast to (2.16), (2.17), involve only terms which are obtained in a finite number of steps.

The explicit computation of functions (2.2) (and, as a consequence of this, the construction of equations (2.19), (2.20)) may be difficult or impossible if the expression for f involves complicated non-linearities with respect to the space variable, which causes problems with symbolic integration. In order to facilitate the computation of $u_m(\cdot, \xi, \eta)$, $m \geq 0$, one can use a *polynomial* version of the iterative scheme (2.2), in which the results of iteration are replaced by suitable interpolation polynomials before passing to the next step. This scheme is described below.

3 Some results from interpolation theory

Recall some results of the theory of approximations [2,3,6]. In a similar situation, we have used these facts in [11].

Denote by $\mathcal{P}_q$ a set of all polynomials of degree not higher than q, $q \geq 1$, on $[a,b]$. For any continuous function $y : [a,b] \to \mathbb{R}$, there exists a unique polynomial $p_q^* \in \mathcal{P}_q$, for which $\max_{t\in[a,b]} |y(t) - p_q^*(t)| = E_q(y)$, where

$$E_q(y) := \inf_{p \in \mathcal{P}_q} \max_{t\in[a,b]} |y(t) - p(t)|. \tag{3.1}$$

This p_q^* is the polynomial of the *best uniform approximation* of y in $\mathcal{P}_q$ and the number $E_q(y)$ is called the *error of the best uniform approximation*.

For a given continuous function $y : [a,b] \to \mathbb{R}$ and a natural number q, denote by $L_q y$ the Lagrange interpolation polynomial of degree q such that

$$(L_q y)(t_i) = y(t_i), \qquad i = 1, 2, \ldots, q+1, \tag{3.2}$$

where

$$t_i = \frac{b-a}{2} \cos \frac{(2i-1)\pi}{2(q+1)} + \frac{a+b}{2}, \qquad i = 1, 2, \ldots, q+1, \tag{3.3}$$

are the Chebyshev nodes translated from $(-1, 1)$ to the interval (a, b) (see, e. g., [7]).

Proposition 3.1 ([7, p. 18]). *For any $q \geq 1$ and a continuous function $y : [a, b] \to \mathbb{R}$, the corresponding interpolation polynomial* (3.2) *constructed with the Chebyshev nodes* (3.3) *admits the estimate*

$$\left|y(t) - (L_q y)(t)\right| \leq \left(\frac{2}{\pi} \ln q + 1\right) E_q(y), \qquad t \in [a, b]. \tag{3.4}$$

Recall that the *modulus of continuity* [5, p. 116] of a continuous function $y : [a, b] \to \mathbb{R}$ is the function $\delta \mapsto \omega(y; \delta)$, where

$$\omega(y; \delta) := \sup\{|y(t) - y(s)| : \{t, s\} \subset [a, b], \ |t - s| \leq \delta\} \tag{3.5}$$

for all positive δ. Note that $\omega(y; \cdot)$ is a continuous non-decreasing function on $(0, \infty)$. A function y is uniformly continuous if and only if $\lim_{\delta \to 0} \omega(y; \delta) = 0$ [5, p. 131].

Proposition 3.2 (Jackson's theorem; [6, p. 22]). *If $y \in C([a, b], \mathbb{R}), q \geq 1$, then*

$$E_q(y) \leq 6\, \omega\Big(y; \frac{b - a}{2q}\Big). \tag{3.6}$$

A function $y : [a, b] \to \mathbb{R}$ is said to satisfy the *Dini–Lipschitz condition* (see, e. g., [3, p. 50]) if its modulus of continuity has the property

$$\lim_{\delta \to 0} \omega(y; \delta) \ln \delta = 0.$$

It follows from (3.6) that

$$\lim_{q \to \infty} E_q(y) \ln q = 0 \tag{3.7}$$

for any y satisfying the Dini–Lipschitz condition. In view of (3.4), equality (3.7) ensures the uniform convergence of Lagrange interpolation polynomials at Chebyshev nodes for this class of functions. In particular, every α-Hölder continuous function $[a, b] \to \mathbb{R}$ with $\alpha > 0$ satisfies the Dini–Lipschitz condition.

4 Polynomial successive approximations

Rewrite (2.2) in the form

$$u_m\,(t, \xi, \eta) = u_0(t, \xi, \eta) + (\Lambda N_f u_{m-1}(\cdot, \xi, \eta)])(t), \qquad t \in [a, b], \ m = 1, 2, \ldots, \tag{4.1}$$

where Λ is the linear operator in the space of continuous functions defined by the formula

$$(\Lambda y)\,(t) := \int_a^t y(s) ds - \frac{t - a}{b - a} \int_a^b y(s) ds, \qquad t \in [a, b], \tag{4.2}$$

and N_f is the Nemytskii operator generated by the non-linearity from (1.1),

$$(N_f y)\,(t) := f(t, y(t)), \qquad t \in [a, b], \tag{4.3}$$

for any continuous $y : [a, b] \to \mathbb{R}^n$.

Fix a natural number q and extend the notation $L_q y$ to vector functions by putting

$$L_q y := \mathrm{col}(L_q y_1, L_q y_2, \ldots, L_q y_n) \tag{4.4}$$

for any continuous $y : [a,b] \to \mathbb{R}^n$. In (4.4), $L_q y_i$ is the qth degree interpolation polynomial for y_i at the Chebyshev nodes (3.3). By analogy to (4.4), put

$$E_q y = \operatorname{col}(E_q y_1, E_q y_2, \ldots, E_q y_n). \tag{4.5}$$

If $D \subset \mathbb{R}^n$ is a closed domain and $f : [a,b] \times D \to \mathbb{R}^n$, put

$$l_{q,D}(f) := \left(\frac{2}{\pi}\ln q + 1\right) \sup_{p \in \mathcal{P}_{q+1,D}} E_q(N_f p), \tag{4.6}$$

where

$$\mathcal{P}_{q,D} := \{u : u \in \mathcal{P}_q^n,\ u([a,b]) \subset D\} \tag{4.7}$$

with $\mathcal{P}_q^n := \mathcal{P}_q \times \cdots \times \mathcal{P}_q$. The second multiplier in (4.6) is the least upper bound of errors of best uniform approximations of the functions obtained by substitution into the right-hand side of equation (1.1) of vector polynomials of degree $\leq q+1$ with values in D.

Introduce now a modified iteration process keeping formula (2.1) for $u_0(\cdot,\xi,\eta)$:

$$v_0^q(\cdot,\xi,\eta) := u_0(\cdot,\xi,\eta) \tag{4.8}$$

and replacing (4.1) by the formula

$$v_m^q(t,\xi,\eta) := u_0(t,\xi,\eta) + (\Lambda L_q N_f v_{m-1}^q(\cdot,\xi,\eta))(t), \qquad t \in [a,b],\ m = 1,2,\ldots \tag{4.9}$$

For any $q \geq 1$, formula (4.9) defines a vector polynomial $v_m^q(\cdot,\xi,\eta)$ of degree $\leq q+1$ (in particular, all these functions are continuously differentiable), which, moreover, satisfies the two-point boundary conditions (1.3). The coefficients of the interpolation polynomials depend on the parameters ξ and η.

Similarly to (4.1), functions (4.9) can also be used to study the auxiliary problems (1.1), (1.3).

Let H_k^β, where $k \in \mathbb{R}_+^n$, $k_i \geq 0$, $0 < \beta_i \leq 1$, $i = 1,2,\ldots,n$, be the set of vector functions $y : [a,b] \to \mathbb{R}^n$ satisfying the Hölder conditions

$$|y_i(t) - y_i(s)| \leq k_i |t-s|^{\beta_i} \tag{4.10}$$

for all $\{t,s\} \subset [a,b]$, $i = 1,2,\ldots,n$. Now we can state the "polynomial" version of Theorem 2.1.

Theorem 4.1. *Let there exist a non-negative vector ϱ such that*

$$\varrho \geq \frac{b-a}{4}\left(\delta_{[a,b],\Omega_\varrho}(f) + 2 l_{q,\Omega_\varrho}(f)\right) \tag{4.11}$$

and $f \in \operatorname{Lip}_K(\Omega_\varrho)$ with a certain matrix K satisfying (2.8). Furthermore, let there exist vectors c and β with $c_i \geq 0$, $0 < \beta_i \leq 1$, $i = 1,2,\ldots,n$, such that

$$f(\cdot,\xi) \in H_c^\beta \tag{4.12}$$

for all fixed $\xi \in \Omega_\varrho$. Then, for all fixed $(\xi,\eta) \in D_0 \times D_1$:

1. *For any $m \geq 0$, $q \geq 1$, the function $v_m^q(\cdot,\xi,\eta)$ is a vector polynomial of degree $q+1$ having values in Ω_ϱ and satisfying the two-point conditions (1.3).*

2. *The limits*

$$v^q_\infty(\cdot,\xi,\eta) := \lim_{m\to\infty} v^q_m(\cdot,\xi,\eta), \qquad v_\infty(\cdot,\xi,\eta) := \lim_{q\to\infty} v^q_\infty(\cdot,\xi,\eta) \tag{4.13}$$

exist uniformly on $[a,b]$. *Functions* (4.13) *satisfy conditions* (1.3).

3. *The estimate*

$$\left|u_\infty(t,\xi,\eta) - v^q_m(\cdot,\xi,\eta)\right| \le \frac{10}{9}\alpha_1(t)K^m_*(1_n - K_*)^{-1}\left(\delta_{[a,b],\Omega_\varrho}(f) + l_{q,\Omega_\varrho}(f)\right) \tag{4.14}$$

holds for any $t\in[a,b]$, $m\ge 0$, $q\ge 1$, *where* K_* *and* α_1 *are given by* (2.14), (2.15).

The proof of this theorem is given in Section 5.1. Note that v_∞ coincides with u_∞ appearing in Theorems 2.1 and 2.2.

Similarly to (2.19), (2.20), in order to study the solvability of the determining system (2.16), (2.17), one can use its mth approximate polynomial version

$$\eta - \xi = \int_a^b (L_q N_f v^q_m(\cdot,\xi,\eta))(s)ds, \tag{4.15}$$

$$\phi(v^q_m(\cdot,\xi,\eta)) = \gamma, \tag{4.16}$$

which can be regarded as an approximate version of (2.19), (2.20). If $(\hat\xi,\hat\eta)$ is a root of (4.15), (4.16) in a particular region, then the function

$$U^q_m(t) := v^q_m\left(t,\hat\xi,\hat\eta\right), \qquad t\in[a,b], \tag{4.17}$$

provides the mth *polynomial approximation* to a solution of the original problem with the corresponding localisation of initial data. Of course, system (2.19), (2.20) may have multiple roots; in such cases, these roots determine different solutions.

It should be noted that, under conditions of Theorem 4.1, the function $N_f v^q_{m-1}(\cdot,\xi,\eta))$ appearing in (4.9) always satisfies the Dini–Lipschitz condition and, therefore, the corresponding interpolation polynomials at Chebyshev nodes uniformly converge to it as q grows to ∞. This follows from Lemma 5.1 of the next section.

Condition (4.11) on ϱ assumed in Theorem 4.1 is stronger than (2.7) of Theorem 2.1 due to the presence of an additional positive term on the right-hand side. A stronger version of (2.7) is needed in order to ensure that the values of iterations do not escape from the set where the Lipschitz condition on f is assumed, for which purpose (2.7) is sufficient in the case of iterations (2.1), (2.2).

The value $E_q(N_f p)$, where $p\in\mathcal{P}_{q+1}$, appearing in (4.6) essentially depends on the character of the non-linearity f. In particular, if f is linear, then $E_q(N_f p)$ is the error of the best uniform approximation of a polynomial of degree $\le q+1$ by polynomials of degree $\le q$.

In spite of the presence of an additional expression in (4.11), for which the theorem does not provide explicit estimates, one may however say that, technically, it is (2.7) that plays the most important role here because the extra term is due to the polynomial approximation, the quality of which grows with q. One can treat this in a different way as follows. Instead of assuming condition (4.11), let us suppose that there exists a non-negative vector ϱ such that

$$\varrho \ge \frac{b-a}{4}\left(\delta_{[a,b],\Omega_\varrho}(f) + r\right) \tag{4.18}$$

with a certain strictly positive vector r. Put

$$w_0(\cdot,\xi,\eta) := u_0(\cdot,\xi,\eta), \tag{4.19}$$
$$w_m(t,\xi,\eta) := u_0(t,\xi,\eta) + (\Lambda L_{q_m} N_f w_{m-1}(\cdot,\xi,\eta))(t), \qquad t \in [a,b],\ m = 1,2,\dots \tag{4.20}$$

where $\{q_m : m \geq 1\} \subset \mathbb{N}$; the choice of this sequence will be discussed below. The condition (2.8) on the maximal in modulus eigenvalue of the Lipschitz matrix K for f in (1.1) is left intact.

Repeating almost word for word the argument from the proof of Theorem 4.1 (see Section 5.1), we find that the sequence $\{w_m(\cdot,\xi,\eta) : m \geq 0\}$ defined according to (4.19), (4.20) converges to the same limit as $\{u_m(\cdot,\xi,\eta) : m \geq 0\}$ given by (4.1) provided that

$$\sup_{\xi\in D_0,\,\eta\in D_1}\ \sup_{m\geq 1}\left(\frac{2}{\pi}\ln q_m + 1\right) E_{q_m}(N_f w_{m-1}^{q_m}(\cdot,\xi,\eta)) \leq \frac{1}{2} r, \tag{4.21}$$

where r is the vector appearing in (4.18). Although (4.21) involves the members of sequence (4.19), (4.20), other assumptions on f (namely, (4.12) and the Lipschitz condition in the space variable) and Jackson's theorem (Proposition 3.2) guarantee that, given any value of r in (4.18), the corresponding condition (4.21) can always be satisfied by choosing $q_1, q_2, \dots$ appropriately. This means that the following is true.

Proposition 4.2. *Under conditions (2.8), (4.12), and (4.18), sequence (4.19), (4.20) uniformly converges provided that q_m is chosen large enough at every step m.*

In that case, sequence (4.19), (4.20) will serve the same purpose as sequence (4.8), (4.9) under the assumptions of Theorem 4.1.

The argument above relies on the knowledge of smallness of the related term appearing on the left-hand side of (4.21). It is however natural to expect that such quantities should diminish if the number of nodes gets larger. To see this, let us now assume conditions somewhat stronger than those of Theorem 4.1.

Assume that, instead of (2.8), the matrix K appearing in the inclusion $f \in \mathrm{Lip}_K(\Omega_\varrho)$ satisfies the condition

$$r(K) < \frac{2}{b-a}. \tag{4.22}$$

Theorem 4.3. *Let there exist a non-negative vector ϱ and positive vector r such that (4.18) holds and $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with K satisfying (4.22). Assume that $f(\cdot,\xi)$ is Lipschitzian with some constant vector c for all fixed $\xi \in \Omega_\varrho$. Then the iteration process (4.19), (4.20) can be made convergent by choosing $q_m = q$, $r = 1,2,\dots$, with q sufficiently large.*

In other words, under conditions of Theorem 4.3, the iteration process (4.19), (4.20) reduces to (4.8), (4.9) with q large enough.

5 Proofs

5.1 Proof of Theorem 4.1

We shall use several auxiliary statements formulated below.

Lemma 5.1. *Let $D \subset \mathbb{R}^n$ and $f : [a,b] \times D \to \mathbb{R}^n$ be a function satisfying condition (4.12) on D with certain vectors c and $\beta = (\beta_i)_{i=1}^n$, $0 < \beta_i \le 1$, $i = 1,2,\dots,n$. Let $f \in \mathrm{Lip}_K(D)$ with a certain $n \times n$ matrix K with non-negative entries. If $u \in H_{\tilde{c}}^{\tilde{\beta}}$ with $\tilde{\beta} = (\tilde{\beta}_i)_{i=1}^n$, $0 < \tilde{\beta}_i \le 1$, $i = 1,2,\dots,n$, then*

$$N_f u \in H^{\mu}_{K\tilde{c}+c}, \tag{5.1}$$

where $\mu := \min\{\beta, \tilde{\beta}\}$.

Proof. Assume that $u \in H_{\tilde{c}}^{\tilde{\beta}}$ and the values of u lie in D. For the sake of brevity, introduce the notation $t^{\beta} := \mathrm{col}(t^{\beta_1}, t^{\beta_2}, \dots, t^{\beta_n})$ for any $t \in [a,b]$. Using (4.12) and the Lipschitz condition for f, we obtain

$$\begin{aligned}
\left|(N_f u)(t) - (N_f u)(s)\right| &= |f(t,u(t)) - f(t,u(s)) + f(t,u(s)) - f(s,u(s))| \\
&\le K\,|u(t) - u(s)| + c\,|t-s|^{\beta} \\
&\le K\tilde{c}\,|t-s|^{\tilde{\beta}} + c\,|t-s|^{\beta} \\
&\le (K\tilde{c} + c)\,|t-s|^{\mu}
\end{aligned}$$

with $\mu = \min\{\beta, \tilde{\beta}\}$, i.e., the function $N_f u$ satisfies a condition of form (4.10), which proves relation (5.1). □

Let the functions $\alpha_m : [a,b] \to \mathbb{R}_+$, $m \ge 0$, be defined by the recurrence relation

$$\alpha_0(t) := 1, \tag{5.2}$$

$$\alpha_{m+1}(t) := \left(1 - \frac{t-a}{b-a}\right) \int_a^t \alpha_m(s)ds + \frac{t-a}{b-a} \int_t^b \alpha_m(s)ds, \qquad m = 0,1,2,\dots \tag{5.3}$$

For $m = 0$, formula (5.3) reduces to (2.15).

Lemma 5.2 ([8, Lemma 3])**.** *For any continuous function $y : [a,b] \to \mathbb{R}$, the estimate*

$$\left|\int_a^t \left(y(\tau) - \frac{1}{b-a}\int_a^b y(s)ds\right)d\tau\right| \le \frac{1}{2}\alpha_1(t)\left(\max_{s\in[a,b]} f(s) - \min_{s\in[a,b]} f(s)\right), \qquad t \in [a,b], \tag{5.4}$$

holds, where $\alpha_1(\cdot)$ is given by (2.15).

Lemma 5.3 ([9, Lemma 3.16])**.** *The following estimates hold for all $t \in [a,b]$:*

$$\begin{aligned}
\alpha_{m+1}(t) &\le \frac{10}{9}\left(\frac{3(b-a)}{10}\right)^m \alpha_1(t), \qquad && m \ge 0, \\
\alpha_{m+1}(t) &\le \frac{3}{10}(b-a)\,\alpha_m(t), && m \ge 2.
\end{aligned} \tag{5.5}$$

Let us now turn to the *proof* of Theorem 4.1. Fix $\xi \in D_0$, $\eta \in D_1$, $q \ge 1$, and put

$$y_m^q := N_f v_m^q(\cdot, \xi, \eta) \tag{5.6}$$

for $m \ge 0$. We need to show that

$$\{v_m^q(t,\xi,\eta) : t \in [a,b]\} \subset \Omega_{\varrho} \tag{5.7}$$

for any m. Obviously, (5.7) holds if $m = 0$.

For $m \geq 1$, in view of (2.6), (4.2) and (5.6), Lemma 5.2 yields the componentwise estimates

$$\begin{aligned}
|(\Lambda y_m^q)(t)| &\leq \frac{1}{2}\alpha_1(t)\Big(\max_{s\in[a,b]} y_m^q(s) - \min_{s\in[a,b]} y_m^q(s)\Big) \\
&= \frac{1}{2}\alpha_1(t)\Big(\max_{s\in[a,b]} f(s, v_m^q(s,\xi,\eta)) - \min_{s\in[a,b]} f(s, v_m^q(\cdot,\xi,\eta))\Big) \\
&\leq \frac{1}{2}\alpha_1(t)\delta_{[a,b],\Omega_\varrho}(f) \\
&\leq \frac{1}{4}(b-a)\delta_{[a,b],\Omega_\varrho}(f)
\end{aligned} \tag{5.8}$$

for all $t \in [a,b]$. In (5.8), we have used the equality

$$\max_{t\in[a,b]} \alpha_1(t) = \frac{1}{2}(b-a) \tag{5.9}$$

which follows directly from (2.15). Furthermore, using relations (5.4), (5.9) and estimate (3.4) of Proposition 3.1, we obtain

$$\begin{aligned}
|(\Lambda(L_q y_{m-1}^q - y_{m-1}^q))(t)| &\leq \frac{1}{2}\alpha_1(t)\Big(\max_{s\in[a,b]}(L_q y_{m-1}^q(s) - y_{m-1}^q(s)) - \min_{s\in[a,b]}(L_q y_{m-1}^q(s) - y_{m-1}^q(s))\Big) \\
&\leq \alpha_1(t)\max_{s\in[a,b]} |L_q y_{m-1}^q(s) - y_{m-1}^q(s)| \\
&\leq \frac{1}{2}(b-a)\left(\frac{2}{\pi}\ln q + 1\right) E_q(y_{m-1}^q).
\end{aligned} \tag{5.10}$$

Combining (5.8) with (5.10) and recalling (4.9), we find

$$\begin{aligned}
\left|v_m^q(t,\xi,\eta) - v_0^q(t,\xi,\eta)\right| &= (\Lambda L_q y_{m-1}^q)(t) \\
&= (\Lambda y_{m-1}^q)(t) + (\Lambda(L_q y_{m-1}^q - y_{m-1}^q))(t) \\
&\leq \frac{1}{4}(b-a)\left(\delta_{[a,b],\Omega_\varrho}(f) + 2\left(\frac{2}{\pi}\ln q + 1\right)E_q(y_{m-1}^q)\right).
\end{aligned} \tag{5.11}$$

For $m = 1$, (5.11) and condition (4.11) yield

$$\begin{aligned}
\left|v_1^q(t,\xi,\eta) - v_0^q(t,\xi,\eta)\right| &\leq \frac{1}{4}(b-a)\left(\delta_{[a,b],\Omega_\varrho}(f) + 2\left(\frac{2}{\pi}\ln q + 1\right)E_q(N_f u_0(\cdot,\xi,\eta))\right) \\
&\leq \frac{1}{4}(b-a)\left(\delta_{[a,b],\Omega_\varrho}(f) + 2l_{q,\Omega_\varrho}(f)\right) \\
&\leq \varrho,
\end{aligned}$$

which, by virtue of (2.4), shows that (5.7) holds with $m = 1$. Arguing by induction, we show that (5.7) holds for any m. The values of every function of sequence (4.9) are thus contained in Ω_ϱ. Using the Lipschitz condition on f and Proposition 3.1, we get

$$\begin{aligned}
&|(N_f u_m(\cdot,\xi,\eta))(t) - (L_q N_f v_m^q(\cdot,\xi,\eta))(t)| \\
&\quad\leq |(N_f u_m(\cdot,\xi,\eta))(t) - (N_f v_m^q(\cdot,\xi,\eta))(t)| + |(N_f v_m^q(\cdot,\xi,\eta))(t) - (L_q N_f v_m^q(\cdot,\xi,\eta))(t)| \\
&\quad\leq K|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)| + \left(\frac{2}{\pi}\ln q + 1\right)E_q(N_f v_m^q(\cdot,\xi,\eta))
\end{aligned} \tag{5.12}$$

for all t and m.

Let us put

$$(My)(t) := \left(1 - \frac{t-a}{b-a}\right) \int_a^t y(s)ds + \frac{t-a}{b-a} \int_t^b y(s)ds, \qquad t \in [a,b], \tag{5.13}$$

for any continuous vector function y. Then, according to (4.1), (4.9), (4.6), and (5.12), we obtain

$$\begin{aligned}
|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)| &= |(\Lambda[N_f u_{m-1}(\cdot,\xi,\eta) - L_q N_f v_{m-1}^q(\cdot,\xi,\eta)])(t)| \\
&\leq (M\,|N_f u_{m-1}(\cdot,\xi,\eta) - L_q N_f v_{m-1}^q(\cdot,\xi,\eta)\,|)(t) \\
&\leq (M\,K|u_{m-1}(\cdot,\xi,\eta) - v_{m-1}^q(\cdot,\xi,\eta)\,|)(t) \\
&\quad + \left(\frac{2}{\pi}\ln q + 1\right) E_q(N_f v_m^q(\cdot,\xi,\eta))(Me)(t) \\
&\leq (M\,K|u_{m-1}(\cdot,\xi,\eta) - v_{m-1}^q(\cdot,\xi,\eta)\,|)(t) + l_{q,\Omega_\varrho}(f)\,(Me)(t)
\end{aligned}$$

for $t \in [a,b]$, $m \geq 1$, where $e = \mathrm{col}(1,1,\ldots,1)$. In particular,

$$\begin{aligned}
|u_1(t,\xi,\eta) - v_1^q(t,\xi,\eta)| &\leq l_{q,\Omega_\varrho}(f)\,(Me)(t) \\
&= l_{q,\Omega_\varrho}(f)\alpha_1(t), \\
|u_2(t,\xi,\eta) - v_1^q(t,\xi,\eta)| &\leq (M\,K|u_1(\cdot,\xi,\eta) - v_1^q(\cdot,\xi,\eta)\,|)(t) + l_{q,\Omega_\varrho}(f)\,(Me)(t) \\
&\leq K(Ml_{q,\Omega_\varrho}\alpha_1 e)(t) + l_{q,\Omega_\varrho}(f)\alpha_1(t) \\
&= (K\alpha_2(t) + 1_n\alpha_1(t))\,l_{q,\Omega_\varrho}(f).
\end{aligned}$$

Arguing by induction, we obtain

$$\left|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| \leq (\alpha_m(t)K^{m-1} + \alpha_{m-1}(t)K^{m-2} + \cdots + 1_n\alpha_1(t))\,l_{q,\Omega_\varrho}(f),$$

where α_k, $k = 1,2,\ldots$, are given by (5.2), (5.3). Estimate (5.5) of Lemma 5.3 now yields

$$\left|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| \leq \frac{10}{9}[1_n + K_* + K_*^2 + \ldots K_*^{m-1}]\,\alpha_1(t)\,l_{q,\Omega_\varrho}(f)$$

with K_* as in (2.14), whence, due to assumption (2.8),

$$\left|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| \leq \frac{10}{9}(1_n - K_*)^{-1}\alpha_1(t)\,l_{q,\Omega_\varrho}(f). \tag{5.14}$$

Using (5.14) and estimate (2.13) of Theorem 2.1, we get

$$\begin{aligned}
\left|u_\infty(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| &\leq |(u_\infty(t,\xi,\eta) - u_m(t,\xi,\eta))| + |(u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta))| \\
&\leq \frac{10}{9}\alpha_1(t)\,K_*^m\,(1_n - K_*)^{-1}\,\delta_{[a,b],\Omega_\varrho}(f) \\
&\quad + \frac{10}{9}\,(1_n - K_*)^{-1}\,\alpha_1(t)\,l_{q,\Omega_\varrho}(f) \\
&= \frac{10}{9}\alpha_1(t)K_*^m(1_n - K_*)^{-1}(\delta_{[a,b],\Omega_\varrho}(f) + l_{q,\Omega_\varrho}(f)),
\end{aligned} \tag{5.15}$$

where $u_\infty(\cdot,\xi,\eta)$ is a limit function (2.9) of sequence (2.2) (the limit exists by Theorem 2.1). In view of (2.8) and (2.14), estimate (5.15) shows that sequence (4.8), (4.9) converges to the same limit.

5.2 Proof of Theorem 4.3

We shall use the following Ostrowski inequality [4] for Lipschitz continuous functions [1].

Lemma 5.4 ([1]). *If $y : [a,b] \to \mathbb{R}$, $y \in H^1_c$, then*

$$\left| y(t) - \frac{1}{b-a}\int_a^b y(s)ds \right| \le \left(\frac{1}{4} + \left(\frac{t - \frac{1}{2}(a+b)}{b-a} \right)^2 \right) c(b-a) \tag{5.16}$$

for all $t \in [a,b]$.

If $y \in H^1_c$, $y : [a,b] \to \mathbb{R}^n$, then c in (5.16) is a vector and the inequality is understood componentwise. Recall that H^1_c is the class of functions y satisfying (4.10) with $k_i = 1$, $i = 1, 2, \dots, n$, i. e., y is Lipschitzian with the vector c.

In view of the observation made after the formulation of Theorem 4.3, we shall consider sequence (4.8), (4.9).

Fix $\xi \in D_0$ and $\eta \in D_1$ and write $v^q_m(t) = v^q_m(t, \xi, \eta)$ for the sake of brevity. Let us put

$$c^q_m := \max_{t\in[a,b]} |\dot{v}^q_m(t)|, \qquad m \ge 0,\ q \ge 1, \tag{5.17}$$

where $\dot{} = d/dt$. In other words, c^q_m is the Lipschitz constant of the polynomial v^q_m (we know from (4.9) that v^q_m is a polynomial of degree $\le q+1$, i. e., $v^q_m \in \mathcal{P}_{q+1}$). Thus,

$$v^q_m \in H^1_{c^q_m}. \tag{5.18}$$

According to (4.2), (4.9), we have

$$\dot{v}^q_{m-1}(t) = \dot{u}_0(t) + (L_q N_f v^q_{m-2})(t) - \frac{1}{b-a}\int_a^b (L_q N_f v^q_{m-2})(s)ds. \tag{5.19}$$

Since, by (2.1),

$$\dot{u}_0(t) = \frac{1}{b-a}(\eta - \xi), \tag{5.20}$$

it follows from (5.19) and Lemma 5.4 that

$$|\dot{v}^q_{m-1}(t)| \le \frac{1}{b-a}|\eta - \xi| + \left(\frac{1}{4} + \left(\frac{t - \frac{1}{2}(a+b)}{b-a} \right)^2 \right)(b-a)\lambda^q_{m-2}, \tag{5.21}$$

where λ^q_{m-2} is the Lipschitz constant (actually, vector) of the vector function $N_f v^q_{m-2}$.

By assumption, f satisfies condition (4.12) with $\beta = 1$. Therefore, by virtue of equality (5.17) and Lemma 5.1,

$$N_f v^q_{m-2} \in H^1_{Kc^q_{m-2}+c} \tag{5.22}$$

and, hence,

$$\lambda^q_{m-2} \le Kc^q_{m-2} + c. \tag{5.23}$$

It is easy to check that

$$\max_{t\in[a,b]} (2t - a - b)^2 = (b-a)^2$$

and, therefore, combining (5.21) and (5.23), we obtain

$$\begin{aligned}|\dot{v}^q_{m-1}(t)| &\le \frac{1}{b-a}|\eta-\xi| + \frac{1}{4}\left(1+\left(\frac{2t-a-b}{b-a}\right)^2\right)(b-a)\lambda^q_{m-2}\\ &\le \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)\lambda^q_{m-2}\\ &\le \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)(Kc^q_{m-2}+c),\end{aligned} \tag{5.24}$$

whence, due to (5.17),

$$c^q_{m-1} \le \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)(Kc^q_{m-2}+c). \tag{5.25}$$

Using (5.25) and arguing by induction, we get

$$\begin{aligned}c^q_{m-1} \le h &+ \frac{1}{2}(b-a)Kh + \frac{1}{4}(b-a)^2K^2h + \frac{1}{8}(b-a)^3K^3h\\ &+ \dots + \frac{1}{2^{m-2}}(b-a)^{m-2}K^{m-2}h + \frac{1}{2^{m-1}}(b-a)^{m-1}K^{m-1}c^q_0,\end{aligned} \tag{5.26}$$

where

$$h := \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)c. \tag{5.27}$$

By (5.20), we have

$$c^q_0 = \frac{1}{2}|\eta-\xi|$$

and, therefore, (5.26) implies that

$$\begin{aligned}c^q_{m-1} &\le (1-K_0)^{-1}\left(\frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)c\right) + \frac{1}{b-a}K_0^{m-1}|\eta-\xi|\\ &\le (1-K_0)^{-1}\left(\frac{1}{b-a}d + \frac{1}{2}(b-a)c\right) + \frac{1}{b-a}K_0^{m-1}d\\ &\le (1-K_0)^{-1}\left(\frac{1}{b-a}d + \frac{1}{2}(b-a)c\right) + \frac{1}{b-a}d,\end{aligned} \tag{5.28}$$

where

$$K_0 := \frac{1}{2}(b-a)K$$

and d is the vector defined componentwise as follows:

$$d := \operatorname{col}\left(\sup_{\xi\in D_0,\,\eta\in D_1}|\eta_1-\xi_1|, \sup_{\xi\in D_0,\,\eta\in D_1}|\eta_2-\xi_2|, \dots, \sup_{\xi\in D_0,\,\eta\in D_1}|\eta_n-\xi_n|\right).$$

Note that the term at the right-hand side of (5.28) depends neither on m nor on q.

Since λ^q_{m-1} denotes the Lipschitz constant of $N_f v^q_{m-1}$, it follows from Jackson's theorem (see [6, Corollary 1.4.2]) and inequality (5.23) that

$$\begin{aligned}E_q(N_f v^q_{m-1}) &\le \frac{6}{q}\lambda^q_{m-1}(b-a)\\ &\le \frac{6}{q}(Kc^q_{m-1}+c)(b-a),\end{aligned} \tag{5.29}$$

whence, using (5.28), we obtain

$$
\begin{aligned}
E_q(N_f v_{m-1}^q) &\leq \frac{6}{q}\Big(K(1-K_0)^{-1}\Big(\frac{1}{b-a}d+\frac{1}{2}(b-a)c\Big)+\frac{1}{b-a}d\Big)(b-a) \\
&= \frac{6}{q}\Big(K(1-K_0)^{-1}\Big(d+\frac{1}{2}(b-a)^2c\Big)+d\Big).
\end{aligned}
\tag{5.30}
$$

Recall that we use notation (4.5) for vector functions and the inequalities in (5.29), (5.30) are componentwise.

Estimate (5.30) implies that, by choosing $q_m = q$, $m \geq 1$, with q large enough, we guarantee the fulfilment of condition (4.21), which, as have already been said, ensures the converegence of sequence (4.19), (4.20), or, which is the same in this case, of sequence (4.8), (4.9).

6 A numerical example

Let us apply the approach described above to the system of differential equations with transcendental non-linearities

$$
\begin{aligned}
&u_1'(t) = u_1(t)u_2(t), \\
&u_2'(t) = -\ln(2u_1(t)), \qquad t \in [0, \pi/4],
\end{aligned}
\tag{6.1}
$$

considered under the non-linear two-point boundary conditions

$$
(u_1(a))^2 + (u_2(b))^2 = \frac{3}{8}, \qquad u_1(a)u_2(b) = \frac{\sqrt{2}}{8}. \tag{6.2}
$$

We have $a = 0$, $b = \pi/4$, $f = \operatorname{col}(f_1, f_2)$,

$$
f_1(t, u_1, u_2) = u_1 u_2, \qquad f_2(t, u_1, u_2) = -\ln(2u_1) \tag{6.3}
$$

and $\phi(u) = \operatorname{col}((u_1(a))^2 + (u_2(b))^2 - 3/8,\ u_1(a)u_2(b) - \sqrt{2}/8)$ in this case.

Introduce the vectors of parameters $\xi = \operatorname{col}(\xi_1, \xi_2)$, $\eta = \operatorname{col}(\eta_1, \eta_2)$ and, instead of problem (6.1), (6.2), consider (6.1) under the parametrised boundary conditions (1.3).

Let us choose the sets D_0 and D_1, where one looks the values $u(a)$ and $u(b)$, e.g., as follows:

$$
D_0 = \{(u_1, u_2) : 0.35 \leq u_1 \leq 0.75,\ 0.35 \leq u_2 \leq 0.55\}, \qquad D_1 = D_0. \tag{6.4}
$$

Note that this choice of sets is motivated by the results of computation (it is always useful to start the computation before trying to check the conditions in order to avoid unnecessary computations, see Section 6.1).

According to (2.3), it follows from (6.4) that $\Omega = D_0$. For $\varrho = \operatorname{col}(\varrho_1, \varrho_2)$, we choose the value

$$
\varrho = \operatorname{col}(0.2, 0.4). \tag{6.5}
$$

Then, in view of (6.4), (6.5), set (2.4) has the form

$$
\Omega_\varrho = \{(u_1, u_2) : 0.15 \leq u_1 \leq 0.95,\ -0.05 \leq u_2 \leq 0.95\}. \tag{6.6}
$$

According to (2.6), (6.3), and (6.6),

$$\begin{aligned}\frac{b-a}{4}\delta_{[a,b],\Omega_\varrho}(f) &= \frac{\pi}{8}\Big(\max_{(t,u)\in[a,b]\times\Omega_\varrho} f(t,u) - \inf_{(t,u)\in[a,b]\times\Omega_\varrho} f(t,u)\Big)\\ &\approx \frac{\pi}{8}\begin{pmatrix}0.95\\1.845826690\end{pmatrix} \approx \begin{pmatrix}0.1865320638\\0.3624272230\end{pmatrix} < \begin{pmatrix}0.2\\0.4\end{pmatrix} = \varrho,\end{aligned} \tag{6.7}$$

which means that, for ϱ given by (6.5), condition (4.18) holds with $r_1 < 0.013$, $r_2 < 0.037$. Then, by Proposition 4.2, the scheme (4.19), (4.20) is applicable for sufficiently large numbers of nodes if f is Lipschitzian on Ω_ϱ with a matrix K satisfying condition (2.8). However, a direct computation shows that $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with

$$K = \begin{pmatrix}0.95 & 0.95\\ 6.7 & 0\end{pmatrix}, \tag{6.8}$$

whence, after determining the eigenvalues, we find that (2.8) is satisfied:

$$r(K) \approx 3.04222 < 4.24413 \approx \frac{40}{3\pi} = \frac{10}{3(b-a)}.$$

We can now proceed to the construction of approximations. The question on choosing a suitable value of q we will treat in a heuristic manner and select a certain value according to the practical experience; for larger, "guaranteed" values of q, the quality of results still increases.

We thus use the iteration process $\{v_m^q(\cdot,\xi,\eta) : m \geq 0\}$ defined according to equalities (4.8), (4.9). Using Maple 17, we carry out computations for several values of m at different numbers of Chebyshev nodes on the interval $[a,b]$.

6.1 Approximations of the first solution

It is easy to verify by substitution that

$$u_1^*(t) = \frac{1}{2}\exp\Big(\frac{1}{2}\sin t\Big), \qquad u_2^*(t) = \frac{1}{2}\cos t \tag{6.9}$$

is a solution of problem (6.1), (6.2). Let us show how the corresponding approximate solutions are constructed according to the method indicated above.

Putting, e. g., $q = 4$, we get the corresponding five Chebyshev nodes (3.3) transformed from $(-1,1)$ into interval (a,b):

$$t_1 = 0.7661781024, \qquad t_2 = 0.6235218106, \qquad t_3 = 0.3926990817,$$
$$t_4 = 0.1618763528, \qquad t_5 = 0.0192200611.$$

The approximate determining system (4.15), (4.16), by solving which the numerical values of the parameters determining the approximate solutions are obtained, for this example is constituted by four scalar non-linear equations with respect to ξ_1, ξ_2, η_1, η_2. For $m = 0$, it has the form

$$\begin{aligned}
\eta_1 - \xi_1 = {} & 0.2617993878\,\eta_1\eta_2 + 0.1308996940\,\eta_1\xi_2 + 0.1308996940\,\xi_1\eta_2 \\
& + 0.2617993878\,\xi_1\xi_2, \\
\eta_2 - \xi_2 = {} & -0.20638381\,\ln\left(0.4122147477\,\eta_1 + 1.587785252\,\xi_1\right) \\
& - 0.20638383\,\ln\left(1.587785252\,\eta_1 + 0.4122147477\,\xi_1\right) \\
& - 0.065887535\,\ln\left(0.0489434837\,\eta_1 + 1.951056516\,\xi_1\right) \\
& - 0.065887536\,\ln\left(1.951056516\,\eta_1 + 0.0489434837\,\xi_1\right) \\
& - 0.24085543\,\ln\left(\xi_1 + \eta_1\right), \\
\xi_1\eta_2 = {} & 0.1767766952, \\
\eta_2^2 + \xi_1^2 = {} & 0.375.
\end{aligned} \tag{6.10}$$

Solving (6.10) for $\xi_1 \in (0.45, 0.55)$, we get the root

$$\xi_1 = 0.5000000003, \quad \xi_2 = 0.4910030682, \quad \eta_1 = 0.6966729228, \quad \eta_2 = 0.3535533902, \tag{6.11}$$

by substituting which into formula (4.8) the *zeroth approximation* $U_0 = \mathrm{col}(U_{01}, U_{02})$ (i. e., function (4.17) for $m = 0$) is obtained:

$$U_{01}(t) = 0.5000000003 + 0.2504117432t, \qquad U_{02}(t) = 0.4910030705 - 0.1750063683t. \tag{6.12}$$

This initial approximation is obtained before any iteration is carried out and is useful as a source of preliminary information on the localisation of solutions (in particular, the graph of function (6.12) is a motivation to choose D_0, D_1 in form (6.4)).

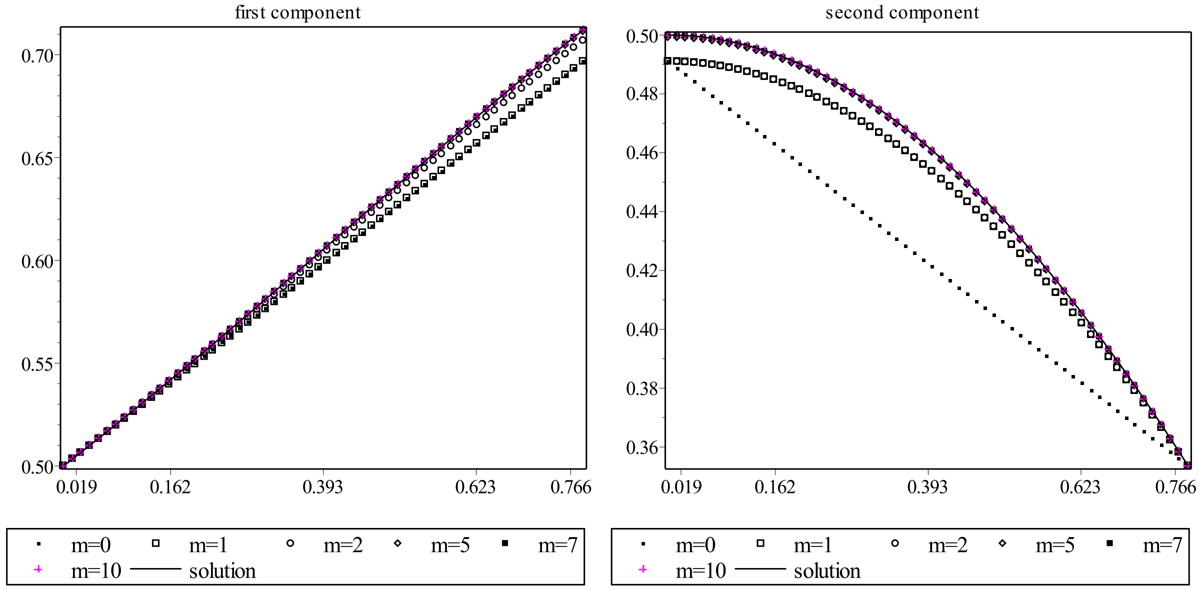
In order to construct higher approximations, we use the *frozen parameters* simplification [14], i. e., before passing from step m to step $m+1$, we substitute the roots of the mth approximate determining equation into the formula obtained on step m. In this way, at the expense of some extra error which tends to zero as m grows, the construction of determining equations is considerably simplified. Note also that, at every step of iteration carried out according to (4.8), (4.9), we obtain a polynomial of degree $\leq q+1$.

Constructing the functions $v_m^4(\cdot, \xi, \eta)$ for several values of m and solving the corresponding approximate determining systems (4.15), (4.16), we obtain the numerical values of the parameters presented in Table 6.1. The last row of the table contains the exact values corresponding to solution (6.9). Since $q = 4$, all these approximations are polynomials of degree 5; e. g., for $m = 7$, it has the form

$$U_{71}^4(t) \approx 0.00456\,t^5 - 0.02668\,t^4 - 0.02838\,t^3 + 0.06195\,t^2 + 0.24987\,t + 0.5, \tag{6.13}$$

$$U_{72}^4(t) \approx 0.49982 - 0.0017\,t^5 + 0.02231\,t^4 - 0.00062\,t^3 - 0.24956\,t^2 + 0.49982. \tag{6.14}$$

The graphs of the seventh approximation (6.13), (6.14) and of the exact solution (6.9) are shown on Figure 6.1.



(a) (b)

Figure 6.1: First solution: $q = 4$, $m = 0, 1, 2, 5, 7, 10$.

m	ξ_1	ξ_2	η_1	η_2
0	0.5000000003	0.4910030682	0.6966729228	0.3535533902
1	0.5000000003	0.4910030705	0.6966729234	0.3535533902
2	0.5000000003	0.4909073352	0.7067944705	0.3535533902
5	0.5000000003	0.4990243859	0.7110836712	0.3535533902
7	0.5000000003	0.4997040346	0.7117894333	0.3535533902
10	0.5000000003	0.4999499916	0.7120202126	0.3535533902
16	0.5000000003	0.4999983385	0.7120583725	0.3535533902
20	0.5000000003	0.4999993608	0.7120592079	0.3535533902
	....	....	....	....
∞	$\frac{1}{2}$	$\frac{1}{2}$	0.7120595095	0.3535533905

Table 6.1: First solution: values of parameters for $q = 4$.

m	ξ_1	ξ_2	η_1	η_2
0	0.5000000003	0.4910340532	0.696681237	0.3535533902
1	0.5000000003	0.4909136731	0.7068092824	0.3535533902
2	0.5000000003	0.4969678528	0.7084223215,	0.3535533902
3	0.5000000003	0.4975642896	0.7104804038	0.3535533902
4	0.5000000003	0.4990270554	0.7110851380	0..3535533902
5	0.50000000039	0.4993503524	0.711592909	0.3535533902
6	0.5000000003	0.4997051246	0.7117900289	0.3535533902
7	0.5000000003	0.4998223937	0.7119239307	0.3535533902
	....	....	....	....
∞	$\frac{1}{2}$	$\frac{1}{2}$	0.7120595095	0.3535533905

Table 6.2: First solution: values of parameters for $q = 11$.

m	ξ_1	ξ_2	η_1	η_2
11	0.4999999999	0.4999103564	0.7120195453	0.3535533905
12	0.4999999999	0.4999501433	0.7120195453	0.3535533905

Table 6.3: First solution: values of parameters for $q = 17$.

For $q = 11$, the Chebyshev nodes (3.3) on (a, b) have the form

$$t_1 = 0.7820385685, \quad t_2 = 0.7555057258, \quad t_3 = 0.70424821007, \quad t_4 = 0.6317591359,$$
$$t_5 = 0.5429785144, \quad t_6 = 0.4439565976, \quad t_7 = 0.3414415658, \quad t_8 = 0.242419649,$$
$$t_9 = 0.1536390274, \quad t_{10} = 0.0811499534, \quad t_{11} = 0.0298924377, \quad t_{12} = 0.0033595951.$$

Computing several approximations, we get from (4.15), (4.16) the numerical values for the parameters presented in Table 6.2. Table 6.3 contains the approximate values of parameters for $q = 17$ and $m \in \{11, 12\}$.

6.2 Approximations of the second solution

Choosing different constraints when solving the approximate determining system (4.15), (4.16), we find that, along with the root from Table 6.1, it has also another root presented in Table 6.4. It is quite evident from the results of computation that this indicates the existence of another solution of the boundary value problem (6.1), (6.2), which is different from (6.9).

On Figure 6.2, one can see the graph of approximations to the second solution, while Figure 6.3 shows the residuals obtained by substituting these approximations into the given differential system (i. e., the functions $t \mapsto U'_{mk}(t) - f_k(t, U_m(t))$, $k = 1, 2$). We see that, e. g., at $m = 10$, we get a residual of order about 10^{-5}. The computation of 20 approximations with $q = 4$ on a standard portable computer with Intel® Core i3-2310M CPU @ 2.10 GHz takes about 130 seconds.

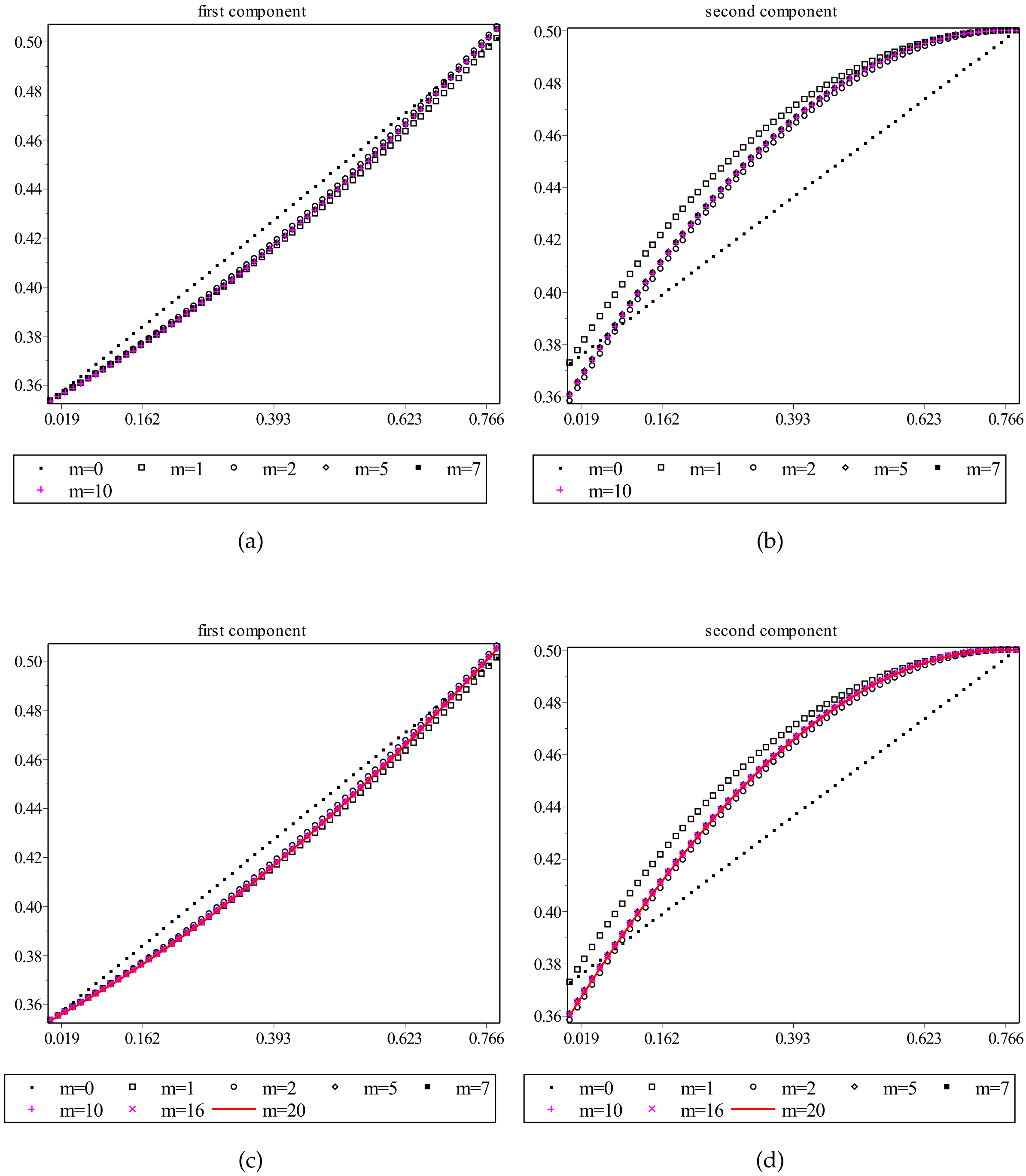


Figure 6.2: Second solution: $q = 4$, $m = 0, 1, 2, 5, 7, 10, 16, 20$.

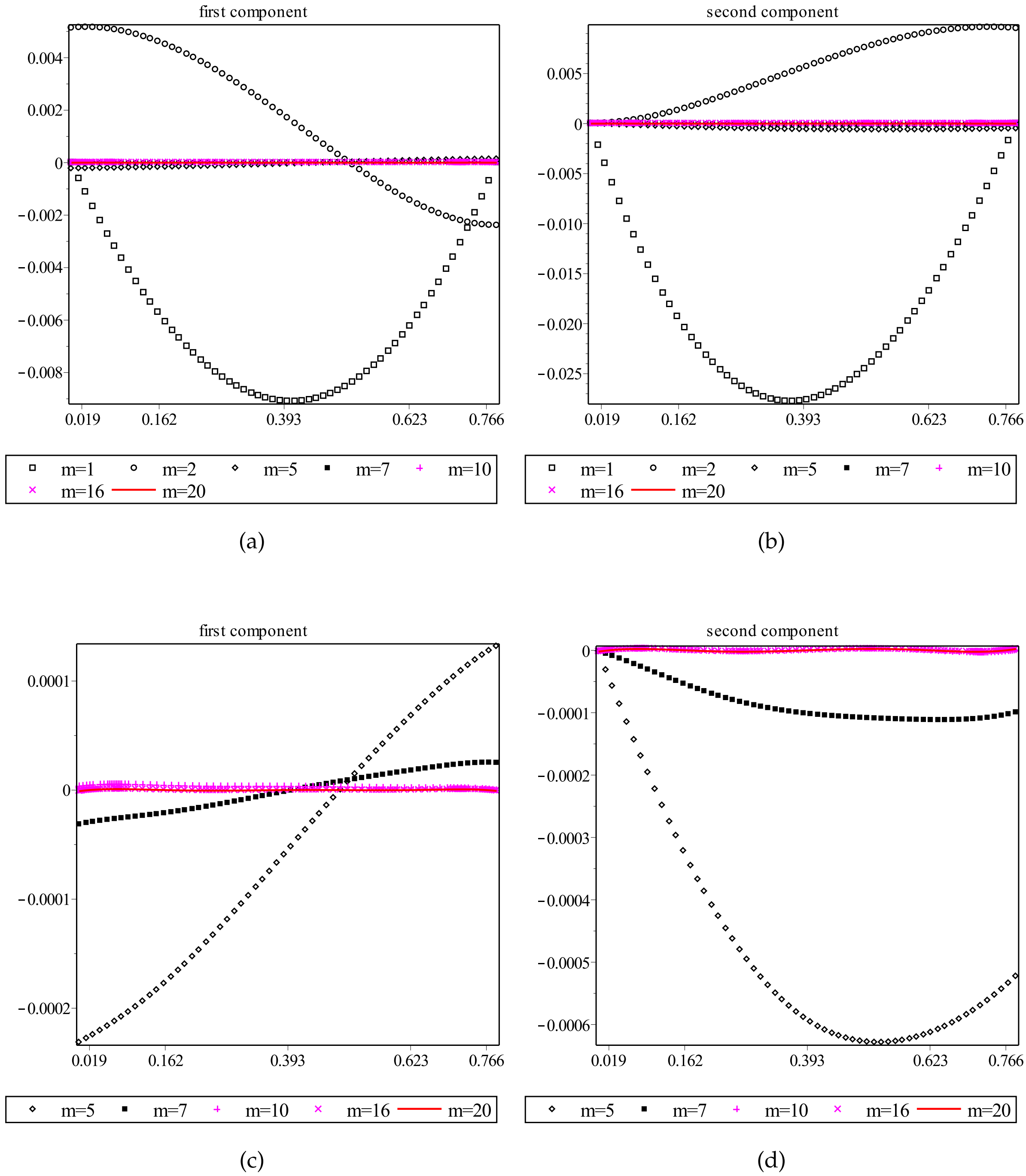


Figure 6.3: The residuals of approximations to the second solution: $q = 4$, $m = 1, 2, 5, 7, 10, 16, 20$.

m	ξ_1	ξ_2	η_1	η_2
0	0.3535533902	0.372879209	0.5012944951	0.5000000003
1	0.3535533902	0.372879209	0.5012944951	0.5000000003
2	0.3535533902	0.3583701009	0.5060832907	0.5000000003
5	0.3535533902	0.360895369	0.5049836277	0.5000000002
7	0.3535533902	0.3606070436	0.5049746944	0.5000000003
10	0.3535533902	0.3605371997	0.504964082	0.5000000003
16	0.3535533902	0.3605333927	0.5049600567	0.5000000003
20	0.3535533902	0.3605332714	0.5049599787	0.5000000003

Table 6.4: Second solution: values of parameters for $q = 4$.

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