Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **40**, 1–16; https://doi.org/10.14232/ejqtde.2018.1.40 www.math.u-szeged.hu/ejqtde/

Global stability in a system using echo for position control

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

Ferenc A. Bartha and **Tibor Krisztin** ✉

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, H–6720, Hungary

Received 27 January 2018, appeared 26 June 2018
Communicated by Gergely Röst

Abstract. We consider a system of equations describing automatic position control by echo. The system can be reduced to a single differential equation with state-dependent delay. The delayed terms come from the control mechanism and the reaction time. H.-O. Walther [*Differ. Integral Equ.* **15**(2002), No. 8, 923–944] proved that stable periodic motion is possible for large enough reaction time. We show that, for sufficiently small reaction lag, the control is perfect, i.e., the preferred position of the system is globally asymptotically stable.

Keywords: state-dependent delay, reaction lag, functional differential equation, asymptotic stability.

2010 Mathematics Subject Classification: 34K20, 34K25, 34K35.

1 Introduction

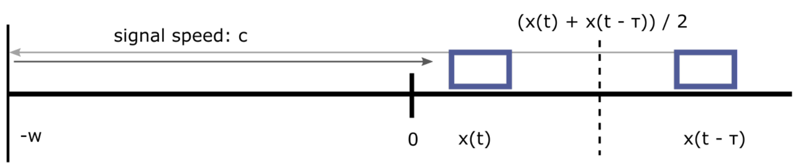

Figure 1.1: A control system

H.-O. Walther [10] considered the following idealized model of a control system depicted on Figure 1.1. An object moves along a line and attempts to control its position relative to an obstacle by approximating its position through sending and receiving reflected signals. The obstacle is positioned at $x = -w < 0$, and the goal of the mechanism is to achieve (asymptotically as time goes to infinity) the ideal position $x = 0$ while avoiding collision with

✉ Corresponding author. Email: krisztin@math.u-szeged.hu

the obstacle. The signals travel at speed $c > 0$. The object is able to measure the time $\tau(t)$ between emission of a signal at time $t - \tau(t)$ and detection of the reflected signal at time t as

$$\tau(t) = \frac{1}{c}\left(|x(t-\tau(t)) + w| + |x(t) + w|\right).$$

The object approximates its current position based on the measured time $\tau(t)$ by

$$d(t) = \frac{c}{2}\tau(t) - w,$$

which is the true position at least when $x(t-\tau(t)) = x(t) > -w$. The object adjusts its velocity, after a constant reaction lag $r \geq 0$, that is,

$$\dot{x}(t) = v\left(d(t-r) - w\right),$$

where v is a response function. Thus, for given constants $w > 0$, $c > 0$, $r \geq 0$, and a *response function* $v\colon \mathbb{R} \to \mathbb{R}$, we obtain the following system of equations

$$\dot{x}(t) = v\left(\frac{c}{2}\tau(t-r) - w\right), \tag{1.1}$$

$$\tau(t) = \frac{1}{c}\left(|x(t-\tau(t)) + w| + |x(t) + w|\right). \tag{1.2}$$

Furthermore, assume that there is a constant b such that

$$0 < b < \frac{c}{4}, \tag{1.3}$$

and the response function $v\colon \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

$$\max_{\xi \in [-w,w]} |v(\xi)| \leq b, \tag{1.4}$$

$$v \text{ is Lipschitz continuous,} \tag{1.5}$$

$$\xi v(\xi) < 0 \quad \text{for all } \xi \in [-w,w] \setminus \{0\}, \tag{1.6}$$

$$v \text{ is differentiable at 0 and } v'(0) < 0. \tag{1.7}$$

Note that (1.3) and (1.4) imply that the signals travel faster than the object itself. The negative feedback (1.6) is a natural condition, and (1.5), (1.6) imply

$$v(0) = 0. \tag{1.8}$$

Therefore, $x(t) \equiv 0$ and $\tau(t) \equiv \frac{2w}{c}$ satisfy (1.1) and (1.2) for all $t \in \mathbb{R}$.

Walther [10] showed that the above hypotheses with certain additional conditions imply that system (1.1) and (1.2) has a stable periodic orbit. The result of [10] is valid for more general response functions as well. However, it is a general additional condition of [10] that the reaction lag is large enough: $r > \frac{8w}{c}$.

Our main result is that, for sufficiently small reaction lag r, the automatic control described by system (1.1) and (1.2) is perfect: the zero solution is globally asymptotically stable.

This paper is organized as follows. In Section 2, following [10], we define the appropriate phase space, which is an open subset of a compact metric space. On this phase space the solutions generate a continuous semiflow. The semiflow continuously depends on the reaction lag r as well. On the phase space, and in particular for possible solution segments, from equation (1.2) the delay $\tau(t)$ can be expressed uniquely as a functional of the solution segment. A consequence is that system (1.1) and (1.2) is equivalent to the single differential equation

$$\dot{x}(t) = v\left(\frac{x(t-r) + x(t-r-\sigma(t))}{2}\right), \tag{1.9}$$

where $\sigma(t)$ is a functional defined on the solution segment. In Section 3, we show that, in case $r = 0$, all solutions of (1.9) approach zero as $t \to \infty$. The proof uses an idea of Nussbaum [9] (see also [7, 8]) which enables us to reduce the delay differential equation to an ordinary differential equation in the Banach space l^∞. We utilize the associated linearized equation in Section 4 to prove local asymptotic stability of $x = 0$ for small r. Finally, Section 5 establishes our main result, that is, global exponential stability of the zero solution for small reaction lags. The fact that the phase space is an open subset of a compact metric space and the continuity of the semiflow in the reaction lag r together with the results of Sections 3, 4 are applied in the proof.

Our result shows that periodic solutions may appear in system (1.1) and (1.2) only if the reaction lag is sufficiently large. In fact, the linearization in Section 4 shows that $x = 0$ is unstable if $r > r^*$ for some $r^* > 0$. It is expected that a Hopf bifurcation occurs at $r = r^*$. It would be of interest to estimate the optional region for the reaction lag so that the global attractivity of $x = 0$ persists. The result of [10] is different. It works for response functions which are, in some sense, close to a step function. On the other hand, [10] gives stable periodic orbits. We mention that a more realistic model, for automatic position control by echo, was studied in [11] and [13] by Walther based on Newton's law. In [11] and [13] the reaction lag was assumed to be zero. It is also an interesting problem to understand the effect of a reaction lag on the results of [11] and [13].

2 The phase space and solutions

In the sequel, we shall only consider solutions of (1.1) and (1.2) such that $x(t) \in (-w, w)$ for all t in the domain of x. For such x, we have from (1.2) that $0 \le \tau(t) \le \frac{4w}{c}$ for all t. In addition, (1.1) becomes

$$\dot{x}(t) = v\left(\frac{x(t-r) + x(t-r-\tau(t-r))}{2}\right). \tag{2.1}$$

Now, assume that the reaction lag r is *small*, that is,

$$0 \le r \le r_0 < \left(\frac{1}{b} - \frac{4}{c}\right) w, \tag{2.2}$$

with some positive constant r_0. Setting $h = \frac{4w}{c}$, the delays appearing in (2.1) have the upper bound $R = r_0 + h$. Thus, we may work in the Banach space $C = C([-R, 0], \mathbb{R})$ with the norm $\|\varphi\| = \max_{s\in[-R,0]} |\varphi(s)|$ for $\varphi \in C$.

We need some notation. For a map $G\colon A \to F$, $A \subset E$, E and F Banach spaces, we set

$$\mathrm{Lip}(G) = \sup_{x,y\in A,\ x\neq y} \frac{|G(x) - G(y)|}{|x-y|} \le \infty.$$

If $t_1 < t_2$ and $u\colon [t_1 - R, t_2) \to \mathbb{R}$ is continuous, then, for $t \in [t_1, t_2)$, $u_t \in C$ is defined by $u_t(s) = u(t+s)$, $-R \le s \le 0$.

Consider $X = \{\varphi \in C\colon \|\varphi\| \leq w,\ \mathrm{Lip}(\varphi) \leq b\}$. X is a compact subset of C by the Arzelà–Ascoli theorem. The open subset $Y = \{\varphi \in X\colon \|\varphi\| < w\}$ of X will be the phase space like in [10]. We remark that it would be possible to apply the approach of Walther [12] and work on a manifold, provided v is C^1-smooth.

2.1 The delay τ

Now we show that equation (1.2) defines $\tau(t)$ uniquely provided that the segment x_t of x is in Y. Note that when analyzing τ, it is sufficient to consider the Banach space $C_0 = C([-h,0],\mathbb{R})$ with the norm $\|\varphi\|_0 = \max_{s\in[-h,0]} |\varphi(s)|$ for $\varphi \in C_0$. The set $X_0 = \{\varphi \in C_0 : \|\varphi\|_0 \leq w,\ \mathrm{Lip}(\varphi) \leq b\}$ is compact in C_0.

Proposition 2.1. *For each $\varphi \in X_0$ there is a unique $\sigma^*(\varphi) \in [0,h]$ such that*

$$\sigma^*(\varphi) = \frac{1}{c}\left[\varphi(0) + \varphi(-\sigma^*(\varphi)) + 2w\right].$$

The map $\sigma^*\colon X_0 \to [0,h]$ is Lipschitz continuous:

$$|\sigma^*(\varphi) - \sigma^*(\psi)| \leq \frac{2}{c-b}\|\varphi - \psi\|_0 \quad \text{for all } \varphi, \psi \in X_0;$$

moreover, if $\|\varphi\|_0 < w$, then $\sigma^*(\varphi) \in (0,h)$.

Proof. For given $\varphi \in X_0$ and $s \in [0,h]$ define

$$\sigma(\varphi)(s) = \frac{1}{c}\left[\varphi(0) + \varphi(-s) + 2w\right].$$

Then $\sigma(\varphi)(s) \in [0,h]$, and, for $s,t \in [0,h]$ we have

$$|\sigma(\varphi)(s) - \sigma(\varphi)(t)| \leq \frac{1}{c}|\varphi(-s) - \varphi(-t)| \leq \frac{1}{c}\mathrm{Lip}(\varphi)\,|s-t| \leq \frac{b}{c}|s-t| \leq \frac{1}{4}|s-t|.$$

This implies that $\sigma(\varphi)\colon [0,h] \to [0,h]$ is a contraction for all $\varphi \in X_0$. Thus, $\sigma(\varphi)\colon [0,h] \to [0,h]$ has a unique fixed point denoted by $\sigma^*(\varphi)$.

If $\varphi, \psi \in X_0$, then

$$\begin{aligned}
|\sigma^*(\varphi) - \sigma^*(\psi)| &= \frac{1}{c}\left|\varphi(0) - \psi(0) + \varphi(-\sigma^*(\varphi)) - \psi(-\sigma^*(\psi))\right| \\
&\leq \frac{1}{c}|\varphi(0) - \psi(0)| + \frac{1}{c}|\varphi(-\sigma^*(\varphi)) - \psi(-\sigma^*(\varphi))| \\
&\quad + \frac{1}{c}|\psi(-\sigma^*(\varphi)) - \psi(-\sigma^*(\psi))| \\
&\leq \frac{2}{c}\|\varphi - \psi\|_0 + \frac{1}{c}\mathrm{Lip}(\psi)\,|\sigma^*(\varphi) - \sigma^*(\psi)| \\
&\leq \frac{2}{c}\|\varphi - \psi\|_0 + \frac{b}{c}|\sigma^*(\varphi) - \sigma^*(\psi)|
\end{aligned}$$

holds. This inequality clearly gives Lipschitz continuity with $\mathrm{Lip}(\sigma^*) \leq \frac{2}{c-b}$ since $b < c$.

Finally, for $\|\varphi\|_0 < w$ it is obvious that $\sigma^*(\varphi) \in (0,h)$. □

For $\rho \in [0, r_0]$, define $\Pi_\rho\colon X \to X_0$ as $(\Pi_\rho\varphi)(s) = \varphi(s-\rho)$ for $s \in [-h,0]$.

2.2 Solutions

A *solution* of system (1.1), (1.2) on $[-R, t_*)$ is a pair of continuous functions $x\colon [-R, t_*) \to (-w, w)$ and $\tau\colon [-r, t_*) \to (0, h)$ such that x is differentiable on $(0, t_*)$, and equation (1.1) holds on $(0, t_*)$, equation (1.2) is satisfied on $(-r, t_* - r)$. If (x, τ) is a solution on $[-R, t_*)$, and, in addition, $x_0 \in Y$, then from equation (1.1) and condition (1.4) it follows clearly that $x_t \in Y$ for all $t \in [0, t_*)$. Then, for each $t \in [0, t_*)$ we have $\Pi_r x_t \in X_0$, and by Proposition 2.1 with $\varphi = \Pi_r x_t$ it follows that $\sigma^*(\Pi_r x_t) \in [0, h]$ is unique with

$$\sigma^*(\Pi_r x_t) = \frac{1}{c}\left[x(t-r) + x(t - r - \sigma^*(\Pi_r x_t)) + 2w\right].$$

Therefore,

$$\tau(t-r) = \sigma^*(\Pi_r x_t).$$

Consequently, a pair (x, τ) is a solution of system (1.1), (1.2) on $[-R, t_*)$ with $x_0 \in Y$ if and only if $x\colon [-R, t_*) \to (-w, w)$ is continuous, it is differentiable on $(0, t_*)$, $x_0 \in Y$, and x satisfies

$$\dot{x}(t) = v\left(\frac{1}{2}x(t-r) + \frac{1}{2}x(t - r - \sigma^*(\Pi_r x_t))\right) \tag{2.3}$$

for all $t \in (0, t^*)$.

Define $f\colon Y \times [0, r_0] \to \mathbb{R}$ by

$$f(\varphi, r) = v\left(\frac{1}{2}\varphi(-r) + \frac{1}{2}\varphi(-r - \sigma^*(\Pi_r \varphi))\right).$$

Then, considering solutions with $|x(t)| < w$, an initial value problem for system (1.1), (1.2) with initial segments in Y is equivalent with the initial value problem

$$\begin{cases} \dot{x}(t) = f(x_t, r), \\ x_0 = \varphi \in Y. \end{cases} \tag{2.4}$$

A *solution* of (2.4) on $[-R, t_*)$ is a continuous function $x\colon [-R, t_*) \to (-w, w)$ such that x is differentiable on $(0, t_*)$, $x_t \in Y$ for all $t \in [0, t_*)$, $x_0 = \varphi$, and the differential equation in (2.4) holds for all $t \in (0, t_*)$. A solution of $\dot{x}(t) = f(x_t, r)$ on $\mathbb{R}$ is a differentiable function $x\colon \mathbb{R} \to (-w, w)$ such that it satisfies the equation for all $t \in \mathbb{R}$.

In order to show that the solutions of (2.4) generate a continuous semiflow we need to show the Lipschitz continuity of f. This is a standard result. We sketch a proof only for completeness, and to emphasize the smooth dependence on r.

Proposition 2.2. *f is Lipschitz continuous in φ and r.*

Proof. We have

$$\begin{aligned}
|f(\varphi, r) - f(\psi, r)| &\le \mathrm{L}_v \frac{1}{2}\left|\varphi(-r) - \psi(-r)\right| + \mathrm{L}_v \frac{1}{2}\left|\varphi(-r - \sigma^*(\Pi_r\varphi)) - \psi(-r - \sigma^*(\Pi_r\psi))\right| \\
&\le \frac{1}{2}\mathrm{L}_v\|\varphi - \psi\| + \frac{1}{2}\mathrm{L}_v\left|\varphi(-r - \sigma^*(\Pi_r\varphi) - \psi(-r - \sigma^*(\Pi_r\varphi))\right| + \\
&\quad + \frac{1}{2}\mathrm{L}_v\left|\psi(-r - \sigma^*(\Pi_r\varphi)) - \psi(-r - \sigma^*(\Pi_r\psi))\right| \\
&\le 2 \cdot \frac{1}{2}\mathrm{L}_v\|\varphi - \psi\| + \frac{1}{2}\mathrm{L}_v \mathrm{Lip}(\psi)\mathrm{Lip}(\sigma^*)\|\Pi_r\varphi - \Pi_r\psi\|_0 \\
&\le \mathrm{L}_v\|\varphi - \psi\| + \frac{1}{2}\mathrm{L}_v b\frac{2}{c-b}\|\varphi - \psi\| \\
&\le \mathrm{L}_v\left(1 + \frac{b}{c-b}\right)\|\varphi - \psi\| = \mathrm{L}_v\frac{c}{c-b}\|\varphi - \psi\|
\end{aligned}$$

and

$$\begin{aligned}
|f(\psi,r)-f(\psi,s)| &\le \mathrm{L}_v\frac{1}{2}\,|\psi(-r)-\psi(-s)| + \mathrm{L}_v\frac{1}{2}\,|\psi(-r-\sigma^*(\Pi_r\psi))-\psi(-s-\sigma^*(\Pi_s\psi))| \\
&\le \frac{1}{2}\mathrm{L}_v\mathrm{Lip}(\psi)\,|r-s| + \frac{1}{2}\mathrm{L}_v\,|\psi(-r-\sigma^*(\Pi_r\psi)-\psi(-r-\sigma^*(\Pi_s\psi))| \\
&\quad + \frac{1}{2}\mathrm{L}_v\,|\psi(-r-\sigma^*(\Pi_s\psi))-\psi(-s-\sigma^*(\Pi_s\psi))| \\
&\le 2\cdot\frac{1}{2}\mathrm{L}_v\mathrm{Lip}(\psi)\,|r-s| + \frac{1}{2}\mathrm{L}_v\mathrm{Lip}(\psi)\mathrm{Lip}(\sigma^*)\|\Pi_r\psi-\Pi_s\psi\|_0 \\
&\le \mathrm{L}_v\mathrm{Lip}(\psi)\,|r-s| + \frac{1}{2}\mathrm{L}_v b\frac{2}{c-b}\mathrm{Lip}(\psi)\,|r-s| \\
&\le \mathrm{L}_v\mathrm{Lip}(\psi)\left(1+\frac{b}{c-b}\right)|r-s| = \mathrm{L}_v\frac{bc}{c-b}\,|r-s|\,.
\end{aligned}$$

The Lipschitz continuity of f follows from the above inequalities and

$$|f(\varphi,r)-f(\psi,s)| \le |f(\varphi,r)-f(\psi,r)| + |f(\psi,r)-f(\psi,s)|\,. \qquad \square$$

The Lipschitz continuity of f allows us to apply standard techniques for existence, uniqueness of solutions and continuous dependence on initial data and parameters, see, e.g., [2, 5]. We state the result without proof.

Proposition 2.3. *For all $\varphi \in Y$ and for all $r \in [0, r_0]$ the initial value problem (2.4) has a solution $x^{\varphi,r}$ on some interval $[-R, t_*)$, $t_* \le \infty$, which is unique and maximal in the sense that for any other solution $\tilde{x}$ on $[-R, T)$ the relations $T \le t_*$ and $x(t) = \tilde{x}(t)$ on $[-R, T)$ hold.*

2.3 Boundedness of solutions

In order to guarantee existence of solutions on $[-R, \infty)$, a boundedness result will be shown. We remark that condition (1.3) allows us to choose r_0 such that (2.2) holds. Then it follows that

$$bR = br_0 + \frac{4bw}{c} < w.$$

Then, by (1.5) and (1.6), we can define the positive constant

$$m = \min_{bR\le|\xi|\le w} |v(\xi)|.$$

Proposition 2.4. *For all $\varphi \in Y$ and $r \in [0, r_0]$ the solution $x^{\varphi,r}\colon [-R, t_*) \to (-w, w)$ satisfies*

$$|x^{\varphi,r}(t)| \le \max\{\|\varphi\|, bR\} \quad \text{for all } t \in [0, t_*),$$

and

$$|x^{\varphi,r}(t)| \le bR \quad \text{for all } t \in [0, t_*) \quad \text{with } t \ge R + \frac{w-bR}{m}.$$

Proof. Let $\varphi \in Y$ and $r \in [0, r_0]$ be fixed. For simplicity, we omit the dependence of the solution on φ and r.

1. We claim that if $t_0 \in (0, t_*)$ and $x(t_0) \ge bR$ then $\dot{x}(t_0) < 0$. Indeed, suppose $x(t_0) \ge bR$ and $\dot{x}(t_0) \ge 0$. Then, (1.6) and (2.3) imply

$$x(t_0 - r) + x(t_0 - r - \sigma^*(\Pi_r x_{t_0})) \le 0.$$

Since $x(t_0) > 0$, $\sigma^*(\Pi_r x_{t_0}) \in (0, h)$, and $r + h \le R$, it follows that there exists $z \in (t_0 - R, t_0)$ such that $x(z) = 0$. Recall that $\mathrm{Lip}(\varphi) \le b$ for all $\varphi \in Y$. Then,

$$x(t_0) = x(t_0) - x(z) \le b(t_0 - z) < bR,$$

a contradiction. Similarly, if $t_0 \in (0, t_*)$ and $x(t_0) \le -bR$ then $\dot{x}(t_0) > 0$.

Hence the first inequality of the proposition easily follows. In addition, the interval $[-bR, bR]$ is positively invariant in the sense that if $t_1 \in [0, t_*)$ with $|x(t_1)| \le bR$ then $|x(t)| \le bR$ for all $t \in [t_1, t_*)$.

2. In order to show the second inequality of the proposition, suppose that $R + \frac{w-bR}{m} < t_*$ and there exists $t_2 \in (0, t_*)$ with $t_2 > R + \frac{w-bR}{m}$ and $|x(t_2)| > bR$. Then, by the positive invariance of $[-bR, bR]$, we have either $x(t) \in (bR, w)$ for all $t \in [0, t_2]$, or $x(t) \in (-w, -bR)$ for all $t \in [0, t_2]$. First, assume $x(t) \in (bR, w)$ for all $t \in [0, t_2]$. Then, by using the negative feedback property (1.6) of v and the definition of m, we obtain that

$$w - bR > x(R) - x(t_2) = -\int_R^{t_2} v\left(\frac{1}{2}x(t-r) + \frac{1}{2}x(t - r - \sigma^*(\Pi_r x_t))\right) \mathrm{d}t \ge (t_2 - R)m.$$

Hence

$$t_2 < R + \frac{w - bR}{m},$$

a contradiction. The case $x(t) \in (-w, -bR)$ on $[0, t_2]$ leads to a contradiction analogously. This completes the proof. □

A consequence of Proposition 2.4 is that a standard continuation result can be applied to conclude that all solutions of the initial value problem (2.4) exist on $[-R, \infty)$, see [2, Chapter VII, Proposition 2.2] or [5, Chapter 2, Theorem 3.1]. We summarize the properties of the solutions obtained so far.

Proposition 2.5. *For all $r \in [0, r_0]$ and for all $\varphi \in Y$, (2.4) has a unique solution $x^{\varphi,r}\colon [-R, \infty) \to \mathbb{R}$ such that*

$$|x^{\varphi,r}(t)| \le \max\{\|\varphi\|, bR\} \quad \text{for all } t \ge 0, \quad |x^{\varphi,r}(t)| \le bR \quad \text{for all } t \ge R + \frac{w - bR}{m},$$

and

$$F\colon [0, \infty) \times Y \times [0, r_0] \ni (t, \varphi, r) \mapsto x_t^{\varphi,r} \in Y$$

is a continuous semiflow.

3 Global attractivity of $x = 0$ in case $r = 0$

In this section we assume $r = 0$. Then equation (2.3) contains only one delay. According to an idea of Nussbaum [9] (see also [7,8]) it is possible to reduce the problem to an equation in l^∞, and this helps to conclude global attractivity of the zero solution.

Theorem 3.1. *For any $\varphi \in Y$*

$$F(t, \varphi, 0) \to 0 \quad \text{as } t \to \infty.$$

Proof. As $r = 0$ throughout the proof, we omit the dependence of the solutions on r. Let $\varphi \in Y$ be given and define $\alpha = \limsup_{t\to\infty} |x^\varphi(t)|$. By Proposition 2.4, the positive semi-orbit $\left\{x_t^\varphi \colon t \geq 0\right\}$ is in the compact subset $\{\psi \in X \colon \|\psi\| \leq \max\{\|\varphi\|, bR\}\}$. Therefore, the ω–limit set $\omega(x^\varphi)$ is nonempty, compact, and invariant in Y, see [3]. By the invariance, for all $\psi \in \omega(x^\varphi)$ there exists a solution $y = y^\psi \colon \mathbb{R} \to \mathbb{R}$ of (2.3) such that $\left|y^\psi(t)\right| \leq \alpha$ for all $t \in \mathbb{R}$. Furthermore, there is a $\psi \in \omega(x^\varphi)$ such that $\left|y^\psi(0)\right| = |\psi(0)| = \alpha$.

We may assume that $\psi(0) = \alpha$ (the case $\psi(0) = -\alpha$ is analogous). Let $y = y^\psi$ and define $\eta(t) = t - \sigma^*(\Pi_0 y_t)$ for $t \in \mathbb{R}$. As $y(0) = \alpha$ and α is a maximum of y, we have that $\dot{y}(0) = 0$. Now, define the sequence $\left(t_j\right)_{j=0}^{\infty}$ by $t_0 = 0$ and $t_j = \eta(t_{j-1})$ for $j \in \mathbb{N}$. Note that

$$\dot{y}(t_j) = v\left(\frac{y(t_j) + y(t_{j+1})}{2}\right) \quad \text{for } j \in \mathbb{N} \cup \{0\}.$$

Then, the negative feedback condition (1.6) and $\dot{y}(t_0) = 0$ imply that $y(t_0) + y(t_1) = 0$. Combining this with $y(t_0) = \alpha$, we obtain $y(t_1) = -\alpha$, that, in turn, implies $\dot{y}(t_1) = 0$. Clearly, by induction, we have

$$\dot{y}(t_j) = 0 \quad \text{and} \quad y(t_j) = (-1)^j \alpha \quad \text{for } j \in \mathbb{N} \cup \{0\}.$$

Let $z_j(t) = y(t_j + t) - y(t_j)$. Then, $z_j(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$. Analyzing the derivative yields

$$\begin{aligned} \left|\dot{z}_j(t)\right| &= \left|v\left(\frac{y(t_j+t) + y(\eta(t_j+t))}{2}\right) - v\left(\frac{y(t_j) + y(\eta(t_j))}{2}\right)\right| \\ &\leq \frac{1}{2}\,\mathrm{L}_v\left(\left|y(t_j+t) - y(t_j)\right| + \left|y(\eta(t_j+t)) - y(\eta(t_j))\right|\right). \end{aligned} \tag{3.1}$$

Note that $t_{j+1} = \eta(t_j) = t_j - \sigma^*(\Pi_0 y_{t_j})$ implies $t_j = t_{j+1} + \sigma^*(\Pi_0 y_{t_j})$. Therefore,

$$\begin{aligned} &\left|y\left(\eta(t_j+t)\right) - y\left(\eta(t_j)\right)\right| \\ &\quad= \left|y\left(t_j + t - \sigma^*(\Pi_0 y_{t_j+t})\right) - y\left(t_{j+1}\right)\right| \\ &\quad= \left|y\left(t_{j+1} + t - \left[\sigma^*(\Pi_0 y_{t_j+t}) - \sigma^*(\Pi_0 y_{t_j})\right]\right) - y\left(t_{j+1}\right)\right| \\ &\quad\leq \left|y\left(t_{j+1}+t\right) - y\left(t_{j+1}\right)\right| + \left|y\left(t_{j+1} + t - \left[\sigma^*(\Pi_0 y_{t_j+t}) - \sigma^*(\Pi_0 y_{t_j})\right]\right) - y\left(t_{j+1}+t\right)\right| \\ &\quad\leq \left|z_{j+1}(t)\right| + \mathrm{Lip}(y)\left|\sigma^*(\Pi_0 y_{t_j+t}) - \sigma^*(\Pi_0 y_{t_j})\right| \\ &\quad= \left|z_{j+1}(t)\right| + b\left|\sigma^*(\Pi_0 y_{t_j+t}) - \sigma^*(\Pi_0 y_{t_j})\right|. \end{aligned} \tag{3.2}$$

Observe that

$$\begin{aligned} &\left|\sigma^*(\Pi_0 y_{t_j+t}) - \sigma^*(\Pi_0 y_{t_j})\right| \\ &\quad= \frac{1}{c}\left|y(t_j+t) - y(t_j) + y(t_j + t - \sigma^*(\Pi_0 y_{t_j+t})) - y(t_j - \sigma^*(\Pi_0 y_{t_j}))\right| \\ &\quad\leq \frac{1}{c}\left|z_j(t)\right| + \frac{1}{c}\left|y(\eta(t_j+t)) - y(\eta(t_j))\right|. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), it follows that

$$\left|y\left(\eta(t_j+t)\right) - y\left(\eta(t_j)\right)\right| \leq \left|z_{j+1}(t)\right| + \frac{b}{c}\left|z_j(t)\right| + \frac{b}{c}\left|y(\eta(t_j+t)) - y(\eta(t_j))\right|.$$

Therefore,

$$\begin{aligned}\left|y\left(\eta(t_j+t)\right)-y\left(\eta(t_j)\right)\right| &\leq \frac{1}{1-\frac{b}{c}}\left(\left|z_{j+1}(t)\right|+\frac{b}{c}\left|z_j(t)\right|\right)\\ &=\frac{c}{c-b}\left|z_{j+1}(t)\right|+\frac{b}{c-b}\left|z_j(t)\right|.\end{aligned} \tag{3.4}$$

Now, (3.1) and (3.4) imply that

$$\begin{aligned}\left|\dot{z}_j(t)\right| &\leq \frac{1}{2}\,\mathrm{L}_v\left(\left|z_j(t)\right|+\frac{b}{c-b}\left|z_j(t)\right|+\frac{c}{c-b}\left|z_{j+1}(t)\right|\right)\\ &=\frac{1}{2}\,\mathrm{L}_v\frac{c}{c-b}\left(\left|z_j(t)\right|+\left|z_{j+1}(t)\right|\right)\end{aligned} \tag{3.5}$$

for all $t\in\mathbb{R}$ and $j\in\mathbb{N}\cup\{0\}$. To analyze z_j, we define $Z\colon\mathbb{R}\ni t\mapsto\left(z_j(t)\right)_{j=0}^{\infty}\in l^{\infty}$. Clearly, $\|Z(t)\|_\infty\leq 2w$, and

$$\|Z(t)-Z(s)\|_\infty=\sup_{j\in\mathbb{N}\cup\{0\}}\left|z_j(t)-z_j(s)\right|=\sup_{j\in\mathbb{N}\cup\{0\}}\left|y(t_j+t)-y(t_j+s)\right|\leq b\left|t-s\right|$$

for all $s,t\in\mathbb{R}$. Therefore, Z is Lipschitz continuous. Using (3.5) and $z_j(0)=0$, we get for $t\geq 0$ that

$$\begin{aligned}\left|z_j(t)\right|=\left|z_j(t)-z_j(0)\right|&=\left|\int_0^t\dot{z}_j(s)\mathrm{d}s\right|\leq\int_0^t\left|\dot{z}_j(s)\right|\mathrm{d}s\\ &\leq\frac{1}{2}\,\mathrm{L}_v\frac{c}{c-b}\int_0^t\left(\left|z_j(s)\right|+\left|z_{j+1}(s)\right|\right)\mathrm{d}s.\end{aligned}$$

Hence,

$$\|Z(t)\|_\infty\leq\mathrm{L}_v\frac{c}{c-b}\int_0^t\|Z(s)\|_\infty\,\mathrm{d}s$$

for $t\geq 0$. Then, Gronwall's inequality implies $Z(t)=0$ for $t\geq 0$. Finally, as

$$t_0-t_1=-\eta(0)=\sigma^*(\Pi_0 y_0)>0,$$

from

$$2\alpha=y(t_0)-y(t_1)=y(t_1+t_0-t_1)-y(t_1)=z_1(t_0-t_1)=0$$

we conclude that $\alpha=0$. The proof is complete. □

4 Local asymptotic stability for small reaction lags

In this section we show that the zero solution of (2.4) is locally asymptotically stable if r is sufficiently small. Namely, we prove

Theorem 4.1. *There exist $M>0$, $\beta>0$, $\delta>0$, and $r_1\in(0,r_0]$ such that for each $r\in[0,r_1]$ and for each $\varphi\in Y$ with $\|\varphi\|\leq\delta$ the inequality*

$$\|F(t,\varphi,r)\|\leq M\|\varphi\|e^{-\beta t}\quad\textit{for all } t\geq 0$$

holds.

Note that r_0 was chosen so that (2.2) holds.

The well known heuristic linearization technique of Cooke and Huang [1] is applied: we freeze the delay in equation (2.3) at $x = 0$, and linearize the obtained equation with constant delay. Then we get the linear equation

$$\dot{y}(t) = -ay(t-r) - ay\left(t - r - \frac{2w}{c}\right), \tag{4.1}$$

where

$$a = -\frac{1}{2}v'(0) > 0.$$

The characteristic function for (4.1) is $\Delta\colon \mathbb{C} \to \mathbb{C}$ given by

$$\Delta(\lambda) = \lambda + ae^{-\lambda r} + ae^{-\lambda r - \lambda\frac{2w}{c}}.$$

From a result of Hale and Huang [4] it follows that $\operatorname{Re}(\lambda) < 0$ holds for all zeros of Δ provided

$$0 \le r \le \frac{1}{2a}.$$

Moreover, by [4], for each fixed positive constants a, w, c, there exists $r^* > \frac{1}{2a}$ so that $\operatorname{Re}(\lambda) < 0$ for all zeros of Δ provided $0 \le r < r^*$, and a pair of complex conjugate zeros cross the imaginary axis at $r = r^*$. It is expected that a Hopf bifurcation takes place at $r = r^*$ for equation (2.3).

Here we consider only the case $r = 0$ where a direct elementary proof can also be given to show that $\operatorname{Re}(\lambda) < 0$ for all zeros of Δ. It would be possible to apply the linearization result of [1] for each $r \in [0, \frac{1}{2a}]$, and then to study the dependence on r, since we need an estimation which is uniform in r. This approach certainly would give a larger r_1 comparing to that of Theorem 4.1. However, as we can show global attractivity only when $r = 0$, this still would not lead to an explicit region for r where global stability of $x = 0$ is valid. Although the technique below to prove Theorem 4.1 is very close to that of [1], we give the proof here as the dependence on the parameter r requires minor modifications.

Consider the associated linear equation to (2.3), when $r = 0$,

$$\dot{y}(t) = -ay(t) - ay\left(t - \frac{2w}{c}\right). \tag{4.2}$$

Now, introducing $h\colon [0,\infty) \to \mathbb{R}$ by

$$h(t) = v\left(\frac{1}{2}x(t-r) + \frac{1}{2}x\left(t - r - \sigma^*(\Pi_r x_t)\right)\right) + a\left[x(t) + x\left(t - \frac{2w}{c}\right)\right] \tag{4.3}$$

problem (2.4) can be written as

$$\begin{cases} \dot{x}(t) = -a\left[x(t) + x\left(t - \frac{2w}{c}\right)\right] + h(t) & \text{for } t > 0, \\ x_0 = \varphi \in Y. \end{cases} \tag{4.4}$$

Proposition 4.2. *There exist $K \ge 1$ and $\alpha > 0$ such that*

$$e^{\alpha t}\left|x^{\varphi,r}(t)\right| \le K\|\varphi\| + K\int_0^t e^{\alpha s}\left|h(s)\right| \mathrm{d}s$$

for all $t \ge 0$, $r \in [0, r_0]$, and $\varphi \in Y$.

Proof. Since $\mathrm{Re}(\lambda) < 0$ holds for all zeros of the characteristic function of (4.2), it follows from [5] that there exist $K \geq 1$ and $\alpha > 0$ such that, for the fundamental solution $\Phi\colon [-R, \infty) \to \mathbb{R}$ of (4.2)

$$|\Phi(t)| \leq Ke^{-\alpha t} \quad \text{for all } t \geq 0 \tag{4.5}$$

holds, and for the solution $y^\varphi\colon [-R, \infty) \to \mathbb{R}$ of (4.2) with $y_0^\varphi = \varphi \in C$ we have

$$\|y_t^\varphi\| \leq Ke^{-\alpha t}\|\varphi\| \quad \text{for all } t \geq 0. \tag{4.6}$$

For given $r \in [0, r_0]$ and $\varphi \in Y$ let $x = x^{\varphi,r}$ be the unique solution of (2.4) on $[-R, \infty)$. By the variation of constants formula for (4.4) (see [5])

$$x(t) = y^\varphi(t) + \int_0^t \Phi(t-s)h(s)\,\mathrm{d}s \quad \text{for } t \geq 0,$$

which together with (4.5) and (4.6) implies

$$|x(t)| \leq Ke^{-\alpha t}\|\varphi\| + K\int_0^t e^{-\alpha(t-s)}\,|h(s)|\,\mathrm{d}s \quad \text{for } \ t \geq 0.$$

Thus, we obtain

$$e^{\alpha t}\,|x(t)| \leq K\|\varphi\| + K\int_0^t e^{\alpha s}\,|h(s)|\,\mathrm{d}s \quad \text{for } \ t \geq 0.$$

□

Proposition 4.3. *For all $t \geq 0$, $r \in [0, r_0]$, and $\varphi \in Y$ we have*

$$\|x_t^{\varphi,r}\| \leq \|\varphi\|e^{\mathrm{L}_v t}.$$

Proof. For given $r \in [0, r_0]$ and $\varphi \in Y$ let $x = x^{\varphi,r}$ be the unique solution of (2.4) on $[-R, \infty)$. From (2.4), we have

$$x(t) = x(0) + \int_0^t v\left(\frac{x(s-r) + x\left(s - r - \sigma^*(\Pi_r x_s)\right)}{2}\right)\mathrm{d}s \quad \text{for } t \geq 0,$$

which implies using equations (1.5) and (1.8) that

$$\begin{aligned}\|x_t\| &\leq \|\varphi\| + \int_0^t \mathrm{L}_v\left|\frac{x(s-r) + x\left(s - r - \sigma^*(\Pi_r x_s)\right)}{2} - 0\right|\mathrm{d}s \\ &\leq \int_0^t \mathrm{L}_v \frac{\|x_s\| + \|x_s\|}{2}\mathrm{d}s = \int_0^t \mathrm{L}_v\|x_s\|\mathrm{d}s \quad \text{for } t \geq 0.\end{aligned}$$

□

Then, by Gronwall's inequality, we obtain

$$\|x_t\| \leq \|\varphi\|e^{\mathrm{L}_v t} \quad \text{for } t \geq 0.$$

We need the following notation. For $x\colon [-R, \infty) \to \mathbb{R}$ and $t \geq R$ let

$$\|x(t+\cdot)\|_{2R} = \max_{\theta \in [-2R, 0]} |x(t+\theta)|\,.$$

Property (1.7) implies that for all $\rho > 0$ there exists $\mu(\rho) > 0$ such that

$$\left|v(u) - v'(0)u\right| \leq \rho\,|u| \quad \text{for all } |u| < \mu. \tag{4.7}$$

For $r \geq 0$, $\rho > 0$, and $\nu \in (0, \mu(\rho)]$, define $\kappa = \kappa(r, \nu, \rho)$ by

$$\kappa(r, \nu, \rho) = |v'(0)|\mathrm{L}_v\left(r + \frac{1}{c}\nu\right) + \rho.$$

Proposition 4.4. *If $r \in [0, r_0]$, $\rho > 0$, $\nu \in (0, \mu(\rho)]$, $\varphi \in Y$, and $T \in (R, \infty)$ are such that $|x^{\varphi,r}(t)| < \nu$ for all $t \in [-R, T]$, then*

$$|h(t)| \leq \kappa(r, \nu, \rho) \|x^{\varphi,r}(t + \cdot)\|_{2R} \quad \text{for all } t \in [R, T].$$

Proof. For given $r \in [0, r_0]$ and $\varphi \in Y$ let $x = x^{\varphi,r}$ be the unique solution of (2.4) on $[-R, \infty)$. We add and subtract

$$a \left[x(t-r) + x\left(t - r - \frac{2w}{c}\right) + x\left(t - r - \sigma^*(\Pi_r x_t)\right) \right]$$

to the right-hand side of (4.3) and regroup the terms to obtain

$$\begin{aligned} h(t) = & \frac{1}{2} v'(0) \Big[x(t-r) - x(t) \Big] + \frac{1}{2} v'(0) \left[x\left(t - r - \frac{2w}{c}\right) - x\left(t - \frac{2w}{c}\right) \right] \\ & + \frac{1}{2} v'(0) \left[x(t - r - \sigma^*(\Pi_r x_t)) - x\left(t - r - \frac{2w}{c}\right) \right] \\ & + v \left(\frac{x(t-r) + x\left(t - r - \sigma^*(\Pi_r x_t)\right)}{2} \right) \\ & - v'(0) \left[\frac{x(t-r) + x\left(t - r - \sigma^*(\Pi_r x_t)\right)}{2} \right]. \end{aligned} \tag{4.8}$$

Equations (1.8) and (2.3) imply $|\dot{x}(s)| \leq \mathrm{L}_v \|x_s\|$ for $s > 0$. Thus, we get the following local upper bound for the Lipschitz constant of x

$$|x(s) - x(s')| \leq \mathrm{L}_v \|x(t + \cdot)\|_{2R} \, |s - s'| \quad \text{for } s, s' \in [t - R, t], \tag{4.9}$$

where $t \in [R, T]$. Note that

$$\left| \sigma^*(\Pi_r x_t) - \frac{2w}{c} \right| = \left| \frac{x(t-r) + x\left(t - r - \sigma^*(\Pi_r x_t)\right)}{c} \right| \leq \frac{2}{c} \|x_t\|. \tag{4.10}$$

As $|x^{\varphi,r}(t)| < \nu$ for all $t \in [-R, T]$, equations (4.7), (4.8), (4.9), and (4.10) imply

$$\begin{aligned} |h(t)| & \leq \frac{1}{2} |v'(0)| \mathrm{L}_v \|x(t + \cdot)\|_{2R} \, r + \frac{1}{2} |v'(0)| \mathrm{L}_v \|x(t + \cdot)\|_{2R} \, r \\ & \quad + \frac{1}{2} |v'(0)| \mathrm{L}_v \|x(t + \cdot)\|_{2R} \frac{2}{c} \|x_t\| + \rho \|x_t\| \\ & \leq \left[|v'(0)| \mathrm{L}_v \left(r + \frac{1}{c} \nu \right) + \rho \right] \|x(t + \cdot)\|_{2R} \end{aligned}$$

for all $t \in [R, T]$. □

Now, we are ready to prove the main result of this section.

Proof of Theorem 4.1. First, choose $r_1 \in (0, r_0]$, $\rho > 0$, $\nu \in (0, \mu(\rho)]$ such that

$$\kappa(r, \nu, \rho) < \frac{\alpha}{2Ke^{2\alpha R}} \quad \text{for all } r \in [0, r_1].$$

Let $\kappa = \kappa(r, \nu, \rho)$. Define

$$M = e^{2\alpha R} \cdot \max \left\{ K + KR(\mathrm{L}_v + 2a) e^{(\alpha + \mathrm{L}_v)R}, \; e^{\mathrm{L}_v R} \right\}.$$

Choose $T_0 > R$ such that

$$e^{(\mathrm{L}_v+\alpha/2)T_0} \geq M.$$

Finally, set

$$\delta = \frac{\nu}{2} e^{-\mathrm{L}_v T_0}.$$

Fix any $\varphi \in Y$ such that $\|\varphi\| \leq \delta$, and fix $r \in [0, r_1]$. Let $x = x^{\varphi,r}$ be the unique solution of (2.4) on $[-R, \infty)$. From Proposition 4.3 we have

$$\|x_t\| \leq \delta e^{\mathrm{L}_v T_0} = \frac{\nu}{2} < \nu$$

for all $t \in [0, T_0]$. Define

$$T_* = \sup \left\{ t \geq 0 \ : \ |x(s)| < \nu \ \text{ for all } s \in [-R, t] \right\}.$$

Clearly, $T_* > T_0$, and $|x(t)| < \nu$ for all $t \in [-R, T_*)$. Thus, Proposition 4.4 establishes

$$|h(t)| \leq \kappa \|x(t+\cdot)\|_{2R} \quad \text{for all } t \in [R, T_*).$$

In addition, (1.8), (4.3), and Proposition 4.3 imply

$$|h(t)| \leq \mathrm{L}_v \|x_t\| + 2a\|x_t\| \leq (\mathrm{L}_v + 2a) e^{\mathrm{L}_v R} \|\varphi\| \quad \text{for all } t \in [0, R].$$

Using Proposition 4.2, the above bounds on $|h(t)|$, and the choices of κ, M, we obtain

$$\begin{aligned} e^{\alpha t} |x(t)| &\leq K\|\varphi\| + K \int_0^R e^{\alpha s} |h(s)| \mathrm{d}s + K \int_R^t e^{\alpha s} |h(s)| \mathrm{d}s \\ &\leq \left[K + KR(\mathrm{L}_v + 2a) e^{(\alpha+\mathrm{L}_v)R} \right] \|\varphi\| + K \int_R^t e^{\alpha s} \kappa \|x(s+\cdot)\|_{2R} \, \mathrm{d}s \\ &\leq e^{-2\alpha R} M \|\varphi\| + e^{-2\alpha R} \frac{\alpha}{2} \int_R^t e^{\alpha s} \|x(s+\cdot)\|_{2R} \, \mathrm{d}s \quad \text{for all } t \in [R, T_*). \end{aligned}$$

On the other hand, it is obvious that

$$e^{\alpha t} |x(t)| \leq e^{-2\alpha R} M \|\varphi\| \quad \text{for all } t \in [-R, R]$$

since $\|x_0\| = \|\varphi\|$ and $\|x_R\| \leq e^{\mathrm{L}_v R} \|\varphi\|$. From the above two estimations for $e^{\alpha t} |x(t)|$, it follows that

$$\begin{aligned} e^{\alpha t} \|x(t+\cdot)\|_{2R} &= \max_{-2R \leq \theta \leq 0} e^{-\alpha\theta} e^{\alpha(t+\theta)} |x(t+\theta)| \\ &\leq e^{2\alpha R} \max_{-2R \leq \theta \leq 0} e^{\alpha(t+\theta)} |x(t+\theta)| \\ &\leq \max_{R \leq V \leq t} \left(M\|\varphi\| + \frac{\alpha}{2} \int_R^V e^{\alpha s} \|x(s+\cdot)\|_{2R} \, \mathrm{d}s \right) \\ &\leq M\|\varphi\| + \frac{\alpha}{2} \int_R^t e^{\alpha s} \|x(s+\cdot)\|_{2R} \, \mathrm{d}s \end{aligned}$$

for all t in $[R, T_*)$. Applying Gronwall's lemma, we obtain

$$e^{\alpha t} \|x(t+\cdot)\|_{2R} \leq M\|\varphi\| e^{(\alpha/2)(t-R)} \leq M\|\varphi\| e^{(\alpha/2)t} \quad \text{for all } t \in [R, T_*).$$

Multiplying by $e^{-\alpha t}$ results in

$$\|x(t+\cdot)\|_{2R} \leq M\|\varphi\| e^{-(\alpha/2)t} \quad \text{for all } t \in [R, T_*),$$

which implies

$$\|x_t\| \le M\|\varphi\|e^{-(\alpha/2)t} \quad \text{for all } t \in [0, T_*).$$

If $T_* < \infty$ then, from the definition of T_* and by continuity, $|x(T_*)| = \nu$ follows. On the other hand, by $T_* > T_0$ and the choice of M and δ,

$$|x(T_*)| \le \|x(T_* + \cdot)\|_{2R} \le M\|\varphi\|e^{-(\alpha/2)T_*} \le M\delta e^{-(\alpha/2)T_0} = M\frac{\nu}{2}e^{-(L_v+(\alpha/2))T_0} \le \frac{\nu}{2},$$

a contradiction. Therefore, $T_* = \infty$, and the proof is complete with $\beta = \frac{\alpha}{2}$. □

5 Global exponential stability for small reaction lags

By Proposition 2.5, we may apply previous results [3, Theorem 3.5.2] that have established that the global attractor of the semiflow F is upper semicontinuous in r. Theorem 3.1 shows that 0 attracts all solutions when $r = 0$. Moreover, Theorem 4.1 establishes that 0 attracts a fixed neighbourhood of itself for all $r \in [0, r_1]$. It follows that the fixed point 0 is the global attractor for all sufficiently small $r \ge 0$. Nevertheless, we provide an elementary proof for the special case (2.4) as we believe it demonstrates some useful, albeit standard, techniques.

Theorem 5.1. *There exist $N > 0$, $\beta > 0$, and $r_2 \in (0, r_1]$ such that for each $r \in [0, r_2]$ and for each $\varphi \in Y$*

$$\|F(t, \varphi, r)\| \le N\|\varphi\|e^{-\beta t} \quad \textit{for all } t \ge 0.$$

Proof. Step 1. Proposition 2.5 implies that if we define $T_1 = 2R + \frac{w-bR}{m}$ and

$$\hat{Y} = \{\varphi \in Y \;:\; \|\varphi\| \le bR\},$$

then for all $\varphi \in Y$ and $r \in [0, r_0]$ the relation

$$F(T_1, \varphi, r) \in \hat{Y}$$

holds.

Step 2. We claim that there exist $T_2 > 0$ and $r_2 \in (0, r_1]$ such that for all $\varphi \in \hat{Y}$ and for all $r \in [0, r_2]$ the inequality

$$\|F(T_2, \varphi, r)\| \le \delta$$

is satisfied, where constant $\delta > 0$ is given in Theorem 4.1.

By Theorem 3.1, for all $\varphi \in Y$ there exists $T_\varphi > 0$ such that

$$\|F(T_\varphi, \varphi, 0)\| < \frac{\delta}{2M},$$

where $M > 1$ is also given in Theorem 4.1. The continuity of F implies that there exist an open neighbourhood V_φ of φ in Y and $r_\varphi \in (0, r_1]$ such that for all $\psi \in V_\varphi$ and for all $r \in [0, r_\varphi]$ we have

$$\|F(T_\varphi, \psi, r)\| < \frac{\delta}{M}.$$

Then, as $M > 1$, Theorem 4.1 implies that

$$\|F(t + T_\varphi, \psi, r)\| \le M\|F(T_\varphi, \psi, r)\|e^{-\beta t} < M\frac{\delta}{M}e^{-\beta t} \le \delta$$

for all $\psi \in V_\varphi$, for all $r \in [0, r_\varphi]$, and for all $t \geq 0$.

The compactness of $\hat{Y}$ implies that there exists a finite number of points $\varphi_1, \dots, \varphi_k$ in $\hat{Y}$ such that

$$\hat{Y} \subseteq \bigcup_{i=1}^{k} V_{\varphi_i}.$$

Then, by setting

$$r_2 = \min \left\{ r_{\varphi_1}, \dots, r_{\varphi_k} \right\}$$

and

$$T_2 = \max \left\{ T_{\varphi_1}, \dots, T_{\varphi_k} \right\},$$

the proof of the claim is completed.

Step 3. With $\beta > 0$ and $M > 1$ given in Theorem 4.1, define

$$N = Me^{(\mathrm{L}_v + \beta)(T_1 + T_2)}.$$

Let $\varphi \in Y$ and $r \in [0, r_2]$.

If $t \in [0, T_1 + T_2]$ then, by Proposition 4.3,

$$\|F(t, \varphi, r)\| \leq e^{\mathrm{L}_v t} \|\varphi\| \leq e^{\mathrm{L}_v (T_1 + T_2)} \|\varphi\|,$$

and

$$\begin{aligned} \|F(t, \varphi, r)\| &\leq e^{\mathrm{L}_v t} \|\varphi\| \\ &= e^{(\mathrm{L}_v + \beta) t} \|\varphi\| e^{-\beta t} \\ &\leq e^{(\mathrm{L}_v + \beta)(T_1 + T_2)} \|\varphi\| e^{-\beta t} \\ &\leq N \|\varphi\| e^{-\beta t}. \end{aligned}$$

By Step 1 we have $F(T_1, \varphi, r) \in \hat{Y}$, and then, by Step 2,

$$\|F(T_1 + T_2, \varphi, r)\| = \|F(T_2, F(T_1, \varphi, r), r)\| < \delta.$$

If $t > T_1 + T_2$ then, by Theorem 4.1 and the above estimations,

$$\begin{aligned} \|F(t, \varphi, r)\| &= \|F(t - (T_1 + T_2), F(T_1 + T_2, \varphi, r), r)\| \\ &\leq M \|F(T_1 + T_2, \varphi, r)\| e^{-\beta(t - (T_1 + T_2))} \\ &\leq M e^{\mathrm{L}_v (T_1 + T_2)} \|\varphi\| e^{-\beta(t - (T_1 + T_2))} \\ &= M e^{(\mathrm{L}_v + \beta)(T_1 + T_2)} \|\varphi\| e^{-\beta t} \\ &= N \|\varphi\| e^{-\beta t}. \end{aligned}$$

This completes the proof. □

Acknowledgments

The authors were supported by EFOP-3.6.2-16-2017-00015. T. K. was also supported by the Hungarian Scientific Research Fund (NKFIH-OTKA), Grant No. K109782.

References

[1] K. L. Cooke, W. Z. Huang, On the problem of linearization for state-dependent delay differential equations, *Proc. Amer. Math. Soc.* **124**(1996), No. 5, 1417–1426. https://doi.org/10.1090/S0002-9939-96-03437-5; MR1340381

[2] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, H.-O. Walther, *Delay equations. Functional, complex, and nonlinear analysis*, Springer-Verlag, New York, 1995. https://doi.org/10.1007/978-1-4612-4206-2; MR1345150

[3] J. K. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs, Vol. 25, American Mathematical Society, Providence, 1988. MR0941371

[4] J. K. Hale, W. Z. Huang, Global geometry of the stable regions for two delay differential equations, *J. Math. Anal. Appl.* **178**(1993), No. 2, 344–362. https://doi.org/10.1006/jmaa.1993.1312; MR1238882

[5] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7; MR1243878

[6] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays, in: *Handbook of differential equations: ordinary differential equations. Vol. III*, Elsevier/North-Holland, Amsterdam, pp. 435–545. https://doi.org/10.1016/S1874-5725(06)80009-X; MR2457636

[7] T. Krisztin, Analyticity of solutions of differential equations with a threshold delay, *Springer Proc. Math. Stat.* **94**(2014), 173–180. https://doi.org/10.1007/978-3-319-08251-6_7; MR3280191

[8] J. Mallet-Paret, R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags. I, *Arch. Rational Mech. Anal.* **120**(1992), No. 2, 99–146. https://doi.org/10.1007/BF00418497; MR1182861

[9] R. D. Nussbaum, Periodic solutions of analytic functional differential equations are analytic, *Michigan Math. J.* **20**(1973), 249–255. https://doi.org/10.1307/mmj/1029001104; MR0322315

[10] H.-O. Walther, Stable periodic motion of a system with state dependent delay, *Differ. Integral Equ.* **15**(2002), No. 8, 923–944. MR1895573

[11] H.-O. Walther, Stable periodic motion of a system using echo for position control, *J. Dynam. Differential Equations* **15**(2003), No. 1, 143–223. https://doi.org/10.1023/A:1026161513363; MR2016913

[12] H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, *J. Differential Equations* **195**(2003), No. 1, 46–65. https://doi.org/10.1016/j.jde.2003.07.001; MR2019242

[13] H.-O. Walther, On a model for soft landing with state-dependent delay, *J. Dynam. Differential Equations* **19**(2007), No. 3, 593–622. https://doi.org/10.1007/s10884-006-9064-8; MR2350240