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Two solutions for a nonhomogeneous Klein–Gordon–Maxwell system

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Abstract. In this paper, we consider the following nonhomogeneous Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u) + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, the primitive of the nonlinearity f is of 2-superlinear growth at infinity. The nonlinearity considered here is weaker than the local (AR) condition and the (Je) condition of Jeanjean. The existence of two solutions is proved by the Mountain Pass Theorem and Ekeland's variational principle.

Keywords: Klein–Gordon–Maxwell system, nonhomogeneous, Mountain Pass Theorem, Ekeland's variational principle.

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1 Introduction and main results

In this paper we consider the following nonhomogeneous Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u) + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{KGM}$$

where $\omega > 0$ is a constant. We are interested in the existence of two nontrivial solutions for system (KGM) under more general nonlinearity f, which doesn't satisfy the (local) (AR) condition or the (Je) condition of Jeanjean.

It is well known that such system has been firstly studied by Benci and Fortunato [5] as a model which describes nonlinear Klein–Gordon fields in three dimensional space interacting with the electrostatic field. For more details on the physical aspects of the problem we refer the readers to see [6] and the references therein. The case of $h \equiv 0$, that is the homogeneous

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case, has been widely studied in recent years. In 2002, Benci and Fortunato [6] considered for the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

for the pure power of nonlinearity, i.e., $f(x,u) = |u|^{q-2}u$, where ω and m are constants. By using a version of the mountain pass theorem, they proved that (1.1) has infinitely many radially symmetric solutions under $|m| > |\omega|$ and $4 < q < 6$. In [16], D'Aprile and Mugnai covered the case $2 < q < 4$ assuming $\sqrt{\frac{q-2}{2}}m > \omega > 0$. Later, the authors in [3] gave a small improvement with $2 < q < 4$. Azzollini and Pomponio [2] obtained the existence of a ground state solution for (1.1) under one of the conditions

(i) $4 \le q < 6$ and $m > \omega$;

(ii) $2 < q < 4$ and $m\sqrt{q-2} > \omega\sqrt{6-q}$.

Soon afterwards, it is improved by Wang [25]. Motivated by the methods of Benci and Fortunato, Cassani [8] considered (1.1) for the critical case by adding a lower order perturbation:

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\mu > 0$ and $2^* = 6$. He showed that (1.2) has at least a radially symmetric solution under one of the following conditions:

(i) $4 < q < 6$, $|m| > |\omega|$ and $\mu > 0$;

(ii) $q = 4$, $|m| > |\omega|$ and μ is sufficiently large.

It is improved by the result in [9] provided one of the following conditions is satisfied:

(i) $4 < q < 6$, $|m| > |\omega| > 0$ and $\mu > 0$;

(ii) $q = 4$, $|m| > |\omega| > 0$ and μ is sufficiently large;

(iii) $2 < q < 4$, $|m|\sqrt{\frac{q-2}{2}} > |\omega| > 0$ and μ is sufficiently large.

Subsequently, Wang [24] generalized the result of [9]. Recently, the authors in [10] proved the existence of positive ground state solutions for the problem (1.2) with a periodic potential V, that is,

$$\begin{cases} -\Delta u + V(x)u + [m^2 - (\omega + \phi)^2]\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases}$$

In [20], Georgiev and Visciglia introduced a system like homogeneous (KGM) with potentials, however they considered a small external Coulomb potential in the corresponding Lagrangian density. Cunha [14] considered the existence of positive ground state solutions for (KGM) with periodic potential $V(x)$. Other related results about homogeneous Klein–Gordon–Maxwell system can be found in [15, 17–19, 23].

Next, we consider the nonhomogeneous case, that is $h \not\equiv 0$. In [12], Chen and Song proved that (KGM) had two nontrivial solutions if $f(x,t)$ satisfies the local (AR) condition:

(CS) There exist $\mu > 2$ and $r_0 > 0$ such that $\mathcal{F}(x,t) := \frac{1}{\mu} f(x,t)t - F(x,t) \geq 0$ for every $x \in \mathbb{R}^3$ and $|t| \geq r_0$, where $F(x,t) = \int_0^t f(x,s)ds$.

Xu and Chen [26] studied the existence and multiplicity of solutions for system (KGM) for the pure power of nonlinearity with $f(x,u) = |u|^{q-2}u$. They also assumed that $V(x) \equiv 1$ and $h(x)$ is radially symmetric. For more results on the nonhomogeneous case see [11] and the references therein.

Motivated by above works, in the present paper we consider system (KGM) with more general assumptions on f and without any radially symmetric assumptions on f and h. More precisely, we assume

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$. Moreover, for every $M > 0$, $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < +\infty$, where meas denotes the Lebesgue measures;

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $C_1 > 0$ and $p \in (2,6)$ such that

$$|f(x,t)| \leq C_1(|t| + |t|^{p-1});$$

(f_2) $f(x,t) = o(t)$ uniformly in x as $|t| \to 0$;

(f_3) There exist $\theta > 2$ and $D_1, D_2 > 0$ such that $F(x,t) \geq D_1|t|^\theta - D_2$, for a.e. $x \in \mathbb{R}^3$ and every t sufficiently large;

(f_4) There exist C_2, r_0 are two positive constants and $\mu > 2$ such that

$$\mathcal{F}(x,t) := \frac{1}{\mu} f(x,t)t - F(x,t) \geq -C_2|t|^2, \quad |t| \geq r_0;$$

(H) $h \in L^2(\mathbb{R}^3), h(x) \geq 0, h(x) \not\equiv 0.$

Before giving our main results, we give some notations. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar and norm

$$(u,v)_H = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx; \qquad \|u\|_H^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D^2 := \|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^3} |u|^s dx$.

Under condition (V), we define a new Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm $\|u\| = \langle u, u \rangle^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2, 2^*]$. Consequently, for each $s \in [2, 6]$, there exists a constant $d_s > 0$ such that

$$|u|_s \leq d_s \|u\|, \quad \forall u \in E. \tag{1.3}$$

Furthermore, it follows from the condition (V) that the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6)$ (see [4]).

System (KGM) has a variational structure. In fact, we consider the functional $\mathcal{J} : E \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{J}(u, \phi) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \phi|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega + \phi)\phi u^2 dx$$
$$- \int_{\mathbb{R}^3} F(x, u)dx - \int_{\mathbb{R}^3} h(x)u dx.$$

The solutions $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ of system (KGM) are the critical points of $\mathcal{J}$. As it is pointed in [12], the functional $\mathcal{J}$ is strongly indefinite and is difficult to investigate. By using the reduction method described in [7], we are led to the study of a new functional $I(u)$ ($I(u)$ is defined in (2.1)) which does not present such strongly indefinite nature.

Now we can state our main result.

Theorem 1.1. *Suppose (V), (f_1)–(f_4) and (H) hold. Then there exists a positive constant m_0 such that system (KGM) admits at least two different solutions $u_0, \tilde{u}_0$ in E satisfying $I(u_0) < 0$ and $I(\tilde{u}_0) > 0$ if $|h|_2 < m_0$.*

Remark 1.2. It is well known that, the (AR) condition is employed not only to prove that the Euler–Lagrange function associated has a mountain pass geometry, but also to guarantee that the Palais–Smale sequences, or Cerami sequences are bounded.

Compared with the local (AR) condition (CS), in our paper $\mathcal{F}(x, t)$ may have negative values.

Another widely used condition is the following condition introduced by Jeanjean [22].

(Je) There exists $\theta \geq 1$ such that $\theta \mathcal{F}_1(x, t) \geq \mathcal{F}_1(x, st)$ for all $s \in [0, 1]$ and $t \in \mathbb{R}$, where $\mathcal{F}_1(x, t) := \frac{1}{4}f(x, t)t - F(x, t)$.

We can observe that when $s = 0$, then $\mathcal{F}_1(x, t) \geq 0$, but for our condition (f_4), $\mathcal{F}(x, t)$ may assume negative values.

In [1,13], the authors studied the Schrödinger–Poisson equation by assuming the following global condition to replace the (AR) condition:

(ASS) There exists $0 \leq \beta < \alpha$ such that $tf(t) - 4F(t) \geq -\beta t^2$, for all $t \in \mathbb{R}$, where α is a positive constant such that $\alpha \leq V(x)$.

Notice that we only need the local condition (f_4) in order to get nontrivial solutions.

In [23], Li and Tang used the following condition to get infinitely many solutions for homogeneous system (KGM):

(LT) There exist two positive constants D_3 and r_0 such that $\frac{1}{4}f(x, t)t - F(x, t) \geq -D_3|t|^2$, if $|t| \geq r_0$.

Obviously, our condition (f_4) is weaker than (LT). Therefore, it is interesting to consider the nonhomogeneous system (KGM) under the conditions (f_3) and (f_4).

Remark 1.3. As it is pointed in [14], many technical difficulties arise to the presence of a non-local term ϕ, which is not homogeneous as it is in the Schrödinger–Poisson systems. Hence, a more careful analysis of the interaction between the couple (u, ϕ) is required.

Throughout this paper, letters $C_i, d_i, L_i, M_i, i = 1, 2, 3\ldots$ will be used to denote various positive constants which may vary from line to line and are not essential to the problem. We denote the weak convergence by "$\rightharpoonup$" and the strong convergence by "$\rightarrow$". Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again by $\{u_n\}$.

The paper is organized as follows. In Section 2, we will introduce the variational setting for the problem and give some related preliminaries. We give the proof of our main result in Section 3.

2 Variational setting and compactness condition

By [3], we know that the signs of ω is not relevant for the existence of solutions, so we can assume that $\omega > 0$.

Evidently, the properties of ϕ_u plays an important role in the study of $\mathcal{J}$. So we need the following technical results.

Proposition 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi = (\phi + \omega)u^2 \quad \text{in } \mathbb{R}^3.$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \le \phi_u \le 0$ on the set $\{x \in \mathbb{R}^3 | u(x) \neq 0\}$;

(ii) $\|\phi_u\|_D^2 \le C\|u\|^2$ and $\int_{\mathbb{R}^3} \phi_u u^2 dx \le C|u|_{12/5}^4 \le C\|u\|^4$.

The proof is similar to Proposition 2.1 in [21] by using the fact $E \hookrightarrow L^s(\mathbb{R}^3)$, for any $s \in [2, 6]$ is continuous.

Multiplying $-\triangle\phi_u + \phi_u u^2 = -\omega u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} (|\nabla\phi_u|^2 + \phi_u^2 u^2) dx = -\int_{\mathbb{R}^3} \omega\phi_u u^2 dx.$$

By the above equality and the definition of $\mathcal{J}$, we obtain a C^1 functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx - \int_{\mathbb{R}^3} h(x) u dx \tag{2.1}$$

and its Gateaux derivative is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) dx - \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uv dx - \int_{\mathbb{R}^3} f(x, u) v dx - \int_{\mathbb{R}^3} h(x) v dx$$

for all $v \in E$. Here we use the fact that $(\triangle - u^2)^{-1}[\omega u^2] = \phi_u$.

Now we will prove the function I has the mountain pass geometry.

Lemma 2.2. *Let $h \in L^2(\mathbb{R}^3)$. Suppose (V), (f_1) and (f_2) hold. Then there exist some positive constants ρ, α, m_0 such that $I(u) \ge \alpha$ for all $u \in E$ satisfying $\|u\| = \rho$ and h satisfying $|h|_2 < m_0$.*

Proof. By (f_2), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,t)| \le \varepsilon|t|$ for all $x \in \mathbb{R}^3$ and $|t| \le \delta$. By (f_1), we obtain

$$\begin{aligned}|f(x,t)| &\le C_1(|t| + |t|^{p-1}) \le C_1\left(|t||\frac{t}{\delta}|^{p-2} + |t|^{p-1}\right)\\ &= C_1\left(\frac{1}{\delta^{p-2}} + 1\right)|t|^{p-1}, \quad \text{for } |t| \ge \delta, \text{ a.e. } x \in \mathbb{R}^3.\end{aligned}$$

Then for all $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^3$ we have

$$|f(x,t)| \le \varepsilon|t| + C_1\left(\frac{1}{\delta^{p-2}} + 1\right)|t|^{p-1} =: \varepsilon|t| + C_\varepsilon|t|^{p-1}$$

and

$$|F(x,t)| \le \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{p}|t|^p. \tag{2.2}$$

Therefore, due to (2.2), Proposition 2.1 and the Hölder inequality, we obtain

$$\begin{aligned}I(u) &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u|^2dx - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^3}|u|^pdx - |h|_2|u|_2\\ &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}d_2^2\|u\|^2 - \frac{C_\varepsilon}{p}d_p^p\|u\|^p - d_2|h|_2\|u\|\\ &= \|u\|\left\{\left(\frac{1}{2} - \frac{\varepsilon}{2}d_2^2\right)\|u\| - \frac{C_\varepsilon}{p}d_p^p\|u\|^{p-1} - d_2|h|_2\right\}.\end{aligned}$$

Let $\varepsilon = \frac{1}{2d_2^2}$ and $g(t) = \frac{t}{4} - \frac{C_\varepsilon}{p}d_p^p t^{p-1}$ for $t \ge 0$. Because $2 < p < 6$, we can see that there exists a positive constant ρ such that $\tilde{m}_0 := g(\rho) = \max_{t\ge 0} g(t) > 0$. Taking $m_0 := \frac{1}{2d_2^2}\tilde{m}_0$, then it follows that there exists a positive constant α such that $I(u)|_{\|u\|=\rho} \ge \alpha$ for all h satisfying $|h|_2 < m_0$. The proof is complete. □

Lemma 2.3. *Assume that (V), (f_1)–(f_4) are satisfied, then there exists a function $u_0 \in E$ with $\|u_0\| > \rho$ such that $I(u_0) < 0$, where ρ is given in Lemma 2.2.*

Proof. By (f_3), there exist $L_1 > 0$ large enough and $M_1 > 0$, such that

$$F(x,t) \ge M_1|t|^\theta, \quad \text{for } |t| \ge L_1. \tag{2.3}$$

By (2.2), we get that

$$|F(x,t)| \le C_3(1 + |t|^{p-2})|t|^2, \quad \text{where } C_3 = \max\left\{\frac{\varepsilon}{2}, \frac{C_\varepsilon}{p}\right\}, \tag{2.4}$$

and then

$$|F(x,t)| \le C_3(1 + L_1^{p-2})|t|^2, \quad \text{when } |t| \le L_1. \tag{2.5}$$

By (2.3) and (2.5), we have

$$F(x,t) \ge M_1|t|^\theta - M_2|t|^2, \quad \text{for all } t \in \mathbb{R}, \tag{2.6}$$

where $M_2 = M_1L_1^{\theta-2} + C_3(1 + L_1^{p-2})$.

Thus, by Proposition 2.1, taking $u \in E, u \neq 0$ and $t > 0$ we have

$$\begin{aligned} I(tu) &= \frac{t^2}{2}\|u\|^2 - \frac{t^2}{2}\int_{\mathbb{R}^3} \omega\phi_{tu}u^2dx - \int_{\mathbb{R}^3} F(x,tu)dx - t\int_{\mathbb{R}^3} h(x)udx \\ &\leq \frac{t^2}{2}\|u\|^2 + \frac{t^2}{2}\int_{\mathbb{R}^3} \omega^2u^2dx - M_1t^\theta \int_{\mathbb{R}^3} |u|^\theta dx + M_2t^2\int_{\mathbb{R}^3} u^2dx - t\int_{\mathbb{R}^3} h(x)udx, \end{aligned}$$

thus $I(tu) \to -\infty$ as $t \to +\infty$ and $\theta > 2$. The lemma is proved by taking $u_0 = t_0u$ with $t_0 > 0$ large enough and $u \neq 0$. □

Lemma 2.4. *Under assumptions (V), (f_1)–(f_4) and (H), any sequence $\{u_n\} \subset E$ satisfying*

$$I(u_n) \to c > 0, \qquad \langle I'(u_n), u_n\rangle \to 0$$

is bounded in E. Moreover, $\{u_n\}$ has a strongly convergent subsequence in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that, up to subsequences, we have $\|u_n\| \to +\infty$ as $n \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded. Going if necessary to a subsequence, for some $v \in E$, we obtain that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{in } E, \\ v_n &\to v \quad \text{in } L^s,\ 2 \leq s < 6, \\ v_n(x) &\to v(x) \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Let $\Lambda = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Suppose that meas$(\Lambda) > 0$, then $|u_n(x)| \to +\infty$ as $n \to \infty$ for a.e. $x \in \Lambda$. By (1.3) and (2.6), we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^\theta}dx &\geq M_1\int_{\mathbb{R}^3} |v_n|^\theta dx - M_2\frac{|u_n|^2}{\|u_n\|^\theta} \\ &\geq M_1\int_{\mathbb{R}^3} |v_n|^\theta dx - \frac{M_2d_2^2}{\|u_n\|^{\theta-2}} \to M_1\int_\Lambda |v|^\theta dx > 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.7}$$

By Proposition 2.1, as from (2.4) and (2.6) it follows that $2 < \theta \leq p < 6$, so we can obtain that

$$\left|\int_{\mathbb{R}^3} \frac{\omega\phi_{u_n}u_n^2}{\|u_n\|^\theta}dx\right| \leq \frac{\omega^2|u_n|_2^2}{\|u_n\|^\theta} \leq \frac{\omega^2d_2^2}{\|u_n\|^{\theta-2}} \to 0 \quad \text{as } n \to \infty.$$

Since $h \in L^2(\mathbb{R}^3)$, we can obtain that

$$\left|\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^\theta}dx\right| \leq \frac{|h|_2|u_n|_2}{\|u_n\|^\theta} \leq \frac{|h|_2d_2}{\|u_n\|^{\theta-1}} \to 0 \quad \text{as } n \to \infty.$$

By the definition of I, we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \frac{I(u_n)}{\|u_n\|^\theta} \\ &= \lim_{n\to+\infty}\left[\frac{1}{2\|u_n\|^{\theta-2}} - \int_{\mathbb{R}^3} \frac{\omega\phi_{u_n}u_n^2}{2\|u_n\|^\theta}dx - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^\theta}dx - \int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^\theta}dx\right] \\ &< 0, \end{aligned}$$

which is a contradiction. Therefore, $\operatorname{meas}(\Lambda) = 0$, which implies $v(x) = 0$ for almost every $x \in \mathbb{R}^3$. By (f_1) and (2.4), we have for all $x \in \mathbb{R}^3$ and $|t| \le r_0$,

$$\begin{aligned}|f(x,t)t - \mu F(x,t)| &\le |f(x,t)t| + \mu|F(x,t)| \\ &\le C_1(|t|^2 + |t|^p) + \mu C_3(1+|t|^{p-2})t^2 \le C_6(1+|t|^{p-2})t^2 \\ &\le C_6(1+r_0^{p-2})t^2,\end{aligned}$$

where $C_6 := 2\max\{C_1, \mu C_3\}$. Together with (f_4), we obtain

$$f(x,t)t - \mu F(x,t) \ge -C_7 t^2, \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.8}$$

By $h \in L^2(\mathbb{R}^3)$, we can also obtain the following

$$\left|\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^2} dx\right| \le \frac{|h|_2 |u_n|_2}{\|u_n\|^2} \le \frac{|h|_2 d_2}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

Case i. $2 < \mu < 4$. By (2.8), (2.9), Proposition 2.1 and $2 < \mu < 4$, we have

$$\begin{aligned}&\frac{\mu I(u_n) - \langle I'(u_n), u_n\rangle}{\|u_n\|^2} \\ &= \left(\frac{\mu}{2} - 1\right) + \int_{\mathbb{R}^3} \frac{f(x,u_n)u_n - \mu F(x,u_n)}{\|u_n\|^2} dx + \left(2 - \frac{\mu}{2}\right)\int_{\mathbb{R}^3} \frac{\omega\phi_{u_n}u_n^2}{\|u_n\|^2}dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\phi_{u_n}^2 u_n^2}{\|u_n\|^2}dx + (1-\mu)\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^2}dx \\ &\ge \left(\frac{\mu}{2} - 1\right) - C_7|v_n|_2^2 + \left(2 - \frac{\mu}{2}\right)\int_{\mathbb{R}^3} \frac{\omega\phi_{u_n}u_n^2}{\|u_n\|^2}dx + (1-\mu)\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^2}dx \\ &\ge \left(\frac{\mu}{2} - 1\right) - C_7|v_n|_2^2 - \left(2 - \frac{\mu}{2}\right)\omega^2|v_n|_2^2 + (1-\mu)\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^2}dx \\ &\to \frac{\mu}{2} - 1 \textit{ as } n \to \infty.\end{aligned}$$

Then we get $0 \ge \frac{1}{2} - \frac{1}{\mu}$, which contradicts with $\mu > 2$.

Case ii. $\mu \ge 4$. By (2.8), (2.9), Proposition 2.1 and $\mu \ge 4$, we have

$$\begin{aligned}\frac{\mu I(u_n) - \langle I'(u_n), u_n\rangle}{\|u_n\|^2} &\ge \left(\frac{\mu}{2} - 1\right) - C_7|v_n|_2^2 + \left(2 - \frac{\mu}{2}\right)\int_{\mathbb{R}^3} \frac{\omega\phi_{u_n}u_n^2}{\|u_n\|^2}dx + (1-\mu)\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^2}dx \\ &\ge \left(\frac{\mu}{2} - 1\right) - C_7|v_n|_2^2 + (1-\mu)\int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|^2}dx \\ &\to \frac{\mu}{2} - 1 \quad \text{as } n \to \infty.\end{aligned}$$

Then we have $0 \ge \frac{1}{2} - \frac{1}{\mu}$, which contradicts with $\mu \ge 4$. Therefore $\{u_n\}$ is a bounded in E.

Now we shall prove $\{u_n\}$ contains a convergent subsequence. Without loss of generality, passing to a subsequence if necessary, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. By using the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ are compact for any $s \in [2,6)$, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$. So by (2.2) and the Hölder inequality, we have

$$\int_{\mathbb{R}^3} (f(x,u_n) - f(x,u))(u_n - u)dx \to 0 \quad \text{as } n \to +\infty.$$

By an easy computing, we can get that

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} &[(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u](u_n - u)dx \\ &= 2\omega \int_{\mathbb{R}^3} [(\phi_{u_n} u_n - \phi_u u)(u_n - u)dx + \int_{\mathbb{R}^3} [(\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u)dx \to 0\end{aligned}$$

as $n \to +\infty$. Indeed, by the Hölder inequality, the Sobolev inequality and Proposition 2.1, we can get

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)(u_n - u)u_n dx\right| &\le |(\phi_{u_n} - \phi_u)(u_n - u)|_2 |u_n|_2 \\ &\le |\phi_{u_n} - \phi_u|_6 |u_n - u|_3 |u_n|_2 \\ &\le C\|\phi_{u_n} - \phi_u\|_D |u_n - u|_3 |u_n|_2,\end{aligned}$$

where C is a positive constant. Since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, we get

$$\left|\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)(u_n - u)u_n dx\right| \to 0 \quad \text{as } n \to +\infty,$$

and

$$\left|\int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u)dx\right| \le |\phi_u|_6 |u_n - u|_3 |u_n - u|_2 \to 0 \quad \text{as } n \to +\infty.$$

Thus we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} &[(\phi_{u_n} u_n - \phi_u u)(u_n - u)dx \\ &= \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)(u_n - u)u_n dx + \int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u)dx \to 0\end{aligned}$$

as $n \to +\infty$.

In view of that the sequence $\{\phi_{u_n}^2 u_n\}$ is bounded in $L^{3/2}(\mathbb{R}^3)$, since

$$|\phi_{u_n}^2 u_n|_{3/2} \le |\phi_{u_n}|_6^2 |u_n|_3,$$

so

$$\begin{aligned}\left|\int_{\mathbb{R}^3} [(\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u)dx\right| &\le |\phi_{u_n}^2 u_n - \phi_u^2 u|_{3/2} |u_n - u|_3 \\ &\le (|\phi_{u_n}^2 u_n|_{3/2} + |\phi_u^2 u|_{3/2}) |u_n - u|_3 \to 0,\end{aligned}$$

as $n \to +\infty$. Thus, we get

$$\begin{aligned}\|u_n - u\|^2 = \langle I'(u_n) - I'(u), u_n - u\rangle &- \int_{\mathbb{R}^3} [(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u](u_n - u)dx \\ &+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx \to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

Therefore we get $\|u_n - u\| \to 0$ in E as $n \to \infty$. The proof is complete. □

3 Proof of main result

Now, we are ready to prove our main result.

Proof of Theorem 1.1. Firstly, we prove that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) < 0$.

Since $h \in L^2(\mathbb{R}^3)$, $h \geq 0$ and $h \not\equiv 0$, we can choose a function $\varphi \in E$ such that

$$\int_{\mathbb{R}^3} h(x)\varphi(x)dx > 0.$$

Hence, by Proposition 2.1, $\theta > 2$ and (2.6), we obtain that

$$I(t\varphi) \leq \frac{t^2}{2}\|\varphi\|^2 + \frac{t^2}{2}\int_{\mathbb{R}^3} \omega^2\varphi^2 dx - M_1 t^\theta |\varphi|_\theta^\theta + M_2 t^2 |\varphi|_2^2 - t\int_{\mathbb{R}^3} h(x)\varphi dx < 0.$$

for $t > 0$ small enough. Thus, we obtain

$$c_0 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 2.1, $B_\rho = \{u \in E : \|u\| < \rho\}$. By the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq I(u_n) < c_0 + \frac{1}{n},$$

and

$$I(v) \geq I(u_n) - \frac{1}{n}\|v - u_n\|$$

for all $v \in \overline{B}_\rho$. Then by a standard procedure, we can prove that $\{u_n\}$ is a bounded (PS) sequence of I. Hence, by Lemma 2.4 we know that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Secondly, we prove that there exists a function $\widetilde{u}_0 \in E$ such that $I'(\widetilde{u}_0) = 0$ and $I(\widetilde{u}_0) > 0$.

By Lemma 2.2, Lemma 2.3 and the Mountain Pass Theorem, there is a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to \widetilde{c}_0 > 0 \quad \text{and} \quad I'(u_n) \to 0.$$

In view of Lemma 2.4, we know that $\{u_n\}$ has a strongly convergent subsequence (still denoted by $\{u_n\}$) in E. So there exists a function $\widetilde{u}_0 \in E$ such that $\{u_n\} \to \widetilde{u}_0$ as $n \to \infty$ and $I'(\widetilde{u}_0) = 0$ and $I(\widetilde{u}_0) > 0$. The proof is complete. □

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