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Uniform exponential trisplitting – a new criterion for discrete skew-product semiflows

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Abstract. In the present paper the concept of uniform exponential trisplitting for skew-product flows in Banach space is considered. This concept is a generalisation of the well-known concept of uniform exponential trichotomy. Connections between these concepts are presented and some illustrating examples prove that these are distinct. Also, we present necessary and sufficient conditions for the uniform exponential trisplitting concept with invariant and strongly invariant projectors. Finally, a characterisation in terms of Lyapunov sequences is given.

Keywords: linear skew-product semiflows, discrete skew-product, uniform exponential trisplitting, uniform exponential trichotomy.

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1 Introduction

The notion of linear discrete/continuous skew-product semiflow associated to a linear non-autonomous difference/differential equation plays a central role in a large part of the theory of dynamical systems, especially in the infinite-dimensional case. Starting with the famous work [22], a significant number of papers were published regarding this issue. One of the important problems of asymptotic behaviour for skew-product semiflow is exponential dichotomy. The interested readers should refer to [6,7,11,14,16] and the references therein. See also [1,12,13] for the case of difference equations.

A considerable progression has been achieved in the direction of non-invertible systems since 1970's starting with paper [8]. Regarding this, in [8] this phenomenon is described as follows:

"...our main concern is, therefore, to develop our analysis without any assumption on the invertibility of the 'transition operators'..."

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This idea that the system defined on the positive integers should not be invertible is natural and has been widely used recently. Later, instead of dichotomy concept for non-invertible systems motivated by the fact that there are equations whose backward solutions are not guaranteed to exist has been considered in [2] the splitting concept for difference equations. We refer to [18] for results done in the same direction. For further reading and applications for the case of skew-product semiflows one may consult the monograph [21] and references therein.

In connection with the theory of exponential dichotomies the notion of exponential trichotomy can be shown to yield as a natural generalization. Roughly speaking, a linear dynamics admits an exponential trichotomy if and only if two appropriate linear shifts admit exponential dichotomies. We should mention [10, 19, 20] where trichotomy theory for difference equations is presented. Using above mentioned concept, in [15] is used to relatively compare the solutions of two difference equations, one linear and other semi linear. Also, in [3, 17] is presented the concept for skew-product semiflows. In this line, the results dealing with non-invertible systems are almost inexistens and that motivates us to proceed in this direction. Inspired by the above, we consider two main problems in this paper:

(a) how to define the notion of exponential trichotomy in a more general setting;

(b) how to construct Lyapunov sequences and how to provide characterizations of Datko for both invertible/non-invertible dynamical syatems.

The first issue regarding the notion of exponential trichotomy in a more general setting is pointed out by a motivating example which shows that the relation between the growth rates from the classical definition of exponential trichotomy for skew-product semiflow considered in [17] is too restrictive. Thus, we will consider two notions of exponential trisplitting for discrete skew-product semiflows. Those concepts use two ideas of projections sequences: invariant (without any assumptions regarding the invertibility of the dynamical system) and strongly invariant for the respective dynamical system. In addition, we present an example of a dynamical system with invariant family of projections, but not strongly invariant, hence the study of trisplitting for noninvertible dynamical systems is of interest.

Second, based on Lyapunov norms one reveals the construction of Lyapunov sequences for both invertible/non-invertible dynamical systems. It is important to notice here that Lyapunov sequences can be defined without having any regularity condition verified for the dynamical systems. As a particular case we recover these characterizations for the classical notion of exponential trichotomy. Also, we give characterizations of Datko type, these ones can be partly seen as a development of somewhat related approaches from [9]. This paper is a companion of our earlier work [5] where some preliminary results have been presented.

Outline. The remainder of the paper is organized as follows. In the next section the notion of uniform exponential trisplitting with invariant projectors is considered and a counterexample is presented showing that the inequality between the growth rates from the classical definition of uniform exponential trichotomy is not always true. In Section 3, the results from Section 2 are extended, considering strongly invariant projectors. Finally, some concluding remarks are presented in Section 4.

2 Uniform exponential trisplitting with invariant projectors

Let X be a metric space, V a Banach space and denote by $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators on V. The norm on V and on $\mathcal{B}(V)$ will be denoted by $\|\cdot\|$. The identity operator on V is denoted I. Also, we consider $Y = X \times V$.

Definition 2.1. A mapping $S : \mathbb{N} \times X \to X$ is called a *discrete semiflow* on X, if

$$S(0,x) = x, \quad \text{for every } x \in X;$$

$$S(m, S(n,x)) = S(m+n,x), \quad \text{for all } (m,n,x) \in \mathbb{N}^2 \times X.$$

Definition 2.2. We say that $C : \mathbb{N} \times X \to \mathcal{B}(V)$ is *a discrete cocycle over the discrete semiflow* $S : \mathbb{N} \times X \to X$ on the space Y if

$$C(0,x) = I, \quad \text{for every } x \in X;$$

$$C(m, S(n,x))C(n,x) = C(m+n,x), \ \text{ for all } (m,n,x) \in \mathbb{N}^2 \times X.$$

The discrete linear skew-product semiflow associated with the above cocycle is the dynamical system $\pi = (S, C)$ on $Y = X$ defined by

$$\pi : \mathbb{N} \times Y \to Y, \quad \pi(n,x,v) = (S(n,x), C(n,x)v).$$

Definition 2.3. A mapping $P\colon X \to \mathcal{B}(V)$ is called a *family of projectors* on the Banach space V if

$$P^2(x) = P(x), \quad \text{for every } x \in X.$$

Definition 2.4. A family of projectors $P\colon X \to \mathcal{B}(V)$ is said to be *invariant for the discrete cocycle* C if

$$C(n,x)P(x) = P(S(n,x))C(n,x), \quad \text{for all } (n,x) \in \mathbb{N} \times X.$$

Definition 2.5. If $P_1, P_2, P_3 : X \to \mathcal{B}(V)$ are three families of projectors on V, then we say that $\mathcal{P} = \{P_1, P_2, P_3\}$ is

(i) *orthogonal* if

$$P_1(x) + P_2(x) + P_3(x) = I, \quad \text{for every } x \in X;$$

$$P_k(x)P_j(x) = 0, \quad \text{for all } x \in X,\ k,j \in \{1,2,3\},\ k \neq j.$$

(ii) *invariant for* C if P_j is invariant for C, for all $j \in \{1,2,3\}$.

We further consider $C : \mathbb{N} \times X \to \mathcal{B}(V)$, a discrete cocycle over the discrete semiflow $S : \mathbb{N} \times X \to X$ on Y and $\mathcal{P} = \{P_1, P_2, P_3\}$ orthogonal and invariant for C.

Definition 2.6. We say that the pair $(C, \mathcal{P})$ has *uniform exponential trisplitting* if there exist some constants $N \geq 1$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, with $\alpha < \beta$ and $\gamma < \delta$ such that

$$\|C(n,x)P_1(x)v\| \leq N e^{\alpha n} \|P_1(x)v\|, \tag{2.1}$$
$$e^{\beta n}\|P_2(x)v\| \leq N\|C(n,x)P_2(x)v\|, \tag{2.2}$$
$$e^{\gamma n}\|C(n,x)P_3(x)v\| \leq N\|P_3(x)v\|, \tag{2.3}$$
$$\|P_3(x)v\| \leq N e^{\delta n}\|C(n,x)P_3(x)v\|, \tag{2.4}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Remark 2.7.

(i) If we consider in the previous definition $\alpha < 0 < \beta$ and $\gamma < 0 < \delta$, then we obtain for the pair $(C, \mathcal{P})$ the classical definition of *uniform exponential trichotomy* (see [17]). Also, if $P_3(x) = 0$ we recover the notion of *uniform exponential dichotomy*;

(ii) If $P_3(x) = 0$ for all $x \in X$, then for the pair $(C, \mathcal{P})$ we recall the definition of *uniform exponential splitting*. For a deeper discussion regarding the splitting concept we refer the reader to [18] and the reference therein;

Remark 2.8. The pair $(C, \mathcal{P})$ is uniformly exponentially trichotomic if and only if there exist $N \geq 1$, $\nu_1, \nu_2 > 0$ such that

$$\begin{aligned} \|C(n,x)P_1(x)v\| &\leq Ne^{-\nu_1 n}\|P_1(x)v\|, \\ e^{\nu_1 n}\|P_2(x)v\| &\leq N\|C(n,x)P_2(x)v\|, \\ \|C(n,x)P_3(x)v\| &\leq Ne^{\nu_2 n}\|P_3(x)v\|, \\ e^{-\nu_2 n}\|P_3(x)v\| &\leq N\|C(n,x)P_3(x)v\|, \end{aligned}$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

The following example shows a connection between the concepts considered above, i.e., if the pair $(C, \mathcal{P})$ admits a uniform exponential trichotomy then also admits a uniform exponential trisplitting. The converse is not generally true. For the particular case $P_3(x) = 0$, we recover the notion of uniform exponential trispitting. In this way it is also pointed out that also the notions the uniform exponential dichotomy and uniform exponential splitting are not equivalent. For a deeper analysis of this, the reader is referred to [4].

Example 2.9. Let $X = \mathbb{R}_+$ and $V = l^\infty(\mathbb{N}, \mathbb{R})$ with the norm

$$\|v\| = \sup_{n \in \mathbb{N}} |v_n|.$$

For $0 < a < b$, $c < 0$ and $S : \mathbb{N} \times X \to X$ we consider the discrete semiflow defined by $S(n, x) = n + x$ for all $(n, x) \in \mathbb{N} \times X$.

Also, we consider $C : \mathbb{N} \times X \to \mathcal{B}(V)$ given by

$$C(n,x)v = (v_0 e^{na}, v_1 e^{nb}, v_2 e^{-nc}, v_3 e^{na}, v_4 e^{nb}, v_5 e^{-nc}, \dots).$$

The family of projectors $\mathcal{P} = \{P_1, P_2, P_3\}$ is given by

$$\begin{aligned} P_1(x)v &= (v_0, 0, 0, v_3, 0, 0, \dots), \\ P_2(x)v &= (0, v_1, 0, 0, v_4, 0, 0, \dots), \\ P_3(x)v &= (0, 0, v_2, 0, 0, v_5, 0, 0, \dots). \end{aligned}$$

Thus, for all $(n, x, v) \in \mathbb{N} \times Y$, the following relations hold

$$\begin{aligned} \|C(n,x)P_1(x)v\| &\leq e^{na}\|P_1(x)v\|; \\ \|C(n,x)P_2(x)v\| &\geq e^{nb}\|P_2(x)v\|; \\ e^{2nc}\|C(n,x)P_3(x)v\| &= e^{nc}\|P_3(x)v\| \leq \|P_3(x)v\|; \\ e^{nc}\|C(n,x)P_3(x)v\| &\geq \|P_3(x)v\|. \end{aligned}$$

Hence, $(C, \mathcal{P})$ admits an uniform exponential trisplitting.

Assume that there exist some constants $N \geq 1$ and $\alpha < 0$ such that

$$\|C(n,x)P_1(x)v\| \leq Ne^{\alpha n}\|P_1(x)v\|,$$

for all $(n, x, v) \in \mathbb{N} \times Y$. This leads to $e^{na} \leq Ne^{\alpha n}$, for all $n \in \mathbb{N}$, which is a contradiction. Thus, we can conclude that the pair $(C, \mathcal{P})$ is not uniformly exponentially trichotomic.

Proposition 2.10. *The pair $(C, \mathcal{P})$ admits uniform exponential trisplitting if and only if there exist some constants $N \geq 1$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha < \beta$, $\gamma < \delta$ such that*

$$\|C(m+n,x)P_1(x)v\| \leq Ne^{\alpha m}\|C(n,x)P_1(x)v\|, \tag{2.5}$$

$$e^{\beta m}\|C(n,x)P_2(x)v\| \leq N\|C(m+n,x)P_2(x)v\|, \tag{2.6}$$

$$e^{\gamma m}\|C(m+n,x)P_3(x)v\| \leq N\|C(n,x)P_3(x)v\|, \tag{2.7}$$

$$\|C(n,x)P_3(x)v\| \leq Ne^{\delta m}\|C(m+n,x)P_3(x)v\|, \tag{2.8}$$

for all $(m, n, x, v) \in \mathbb{N}^2 \times Y$.

Proof. The proof is straightforward considering $x \to S(n, x)$ and $v \to C(n, x)v$ in the relations (2.1)-(2.4) and $n = 0$ for the sufficiency part. □

Definition 2.11. The pair $(C, \mathcal{P})$ admits *uniform exponential trisplitting of Datko type* if there exist some constants $D \geq 1$ and $\mu, \nu, \omega, \eta \in \mathbb{R}$ with $\mu < \nu$, $\omega < \eta$ such that

$$\sum_{k=n}^{+\infty} e^{\mu(n-k)}\|C(k,x)P_1(x)v\| \leq D\|C(n,x)P_1(x)v\|, \tag{2.9}$$

$$\sum_{k=0}^{n} e^{\nu(n-k)}\|C(k,x)P_2(x)v\| \leq D\|C(n,x)P_2(x)v\|, \tag{2.10}$$

$$\sum_{k=n}^{+\infty} e^{\omega(k-n)}\|C(k,x)P_3(x)v\| \leq D\|C(n,x)P_3(x)v\|, \tag{2.11}$$

$$\sum_{k=0}^{n} e^{\eta(k-n)}\|C(k,x)P_3(x)v\| \leq D\|C(n,x)P_3(x)v\|, \tag{2.12}$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

In particular, if $\mu < 0 < \nu$ and $\omega < 0 < \eta$, then we have that the pair $(C, \mathcal{P})$ admits *uniform exponential trichotomy of Datko type*.

Remark 2.12. The pair $(C, \mathcal{P})$ admits uniform exponential trichotomy of Datko type if and only if there are some constants $D \geq 1$ and $d_1, d_2 > 0$ such that

$$\sum_{k=n}^{+\infty} e^{d_1(k-n)}\|C(k,x)P_1(x)v\| \leq D\|C(n,x)P_1(x)v\|, \tag{2.13}$$

$$\sum_{k=0}^{n} e^{d_1(n-k)}\|C(k,x)P_2(x)v\| \leq D\|C(n,x)P_2(x)v\|, \tag{2.14}$$

$$\sum_{k=n}^{+\infty} e^{d_2(n-k)}\|C(k,x)P_3(x)v\| \leq D\|C(n,x)P_3(x)v\|, \tag{2.15}$$

$$\sum_{k=0}^{n} e^{d_2(k-n)}\|C(k,x)P_3(x)v\| \leq D\|C(n,x)P_3(x)v\|, \tag{2.16}$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

Theorem 2.13. *The pair $(C, \mathcal{P})$ has uniform exponential trisplitting if and only if $(C, \mathcal{P})$ has uniform exponential trisplitting of Datko type.*

Proof. Necessity. From Proposition 2.10 we have that there are $N \geq 1$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha < \beta$, $\gamma < \delta$ such that $(C, \mathcal{P})$ has uniform exponential trisplitting.

We consider $\mu, \nu, \omega, \eta \in \mathbb{R}$, $\alpha < \mu < \nu < \beta$, $\omega < \gamma < \delta < \eta$,

$$D = 1 + N\left(\frac{e^{\mu}}{e^{\mu} - e^{\alpha}} + \frac{e^{\beta}}{e^{\beta} - e^{\nu}} + \frac{e^{\gamma}}{e^{\gamma} - e^{\omega}} + \frac{e^{\eta}}{e^{\eta} - e^{\delta}}\right).$$

Further, in order to prove the implication between (2.5)–(2.8) and (2.9)–(2.12) we have the following situations. First, for all $(n, x, v) \in \mathbb{N} \times Y$ we obtain

$$\begin{aligned}
\sum_{k=n}^{+\infty} e^{\mu(n-k)} \|C(k,x)P_1(x)v\| &\leq N \sum_{k=n}^{+\infty} e^{\mu(n-k)} e^{\alpha(k-n)} \|C(n,x)P_1(x)v\| \\
&= N e^{n(\mu-\alpha)} \frac{e^{(\alpha-\mu)n}}{1 - e^{\alpha-\mu}} \|C(n,x)P_1(x)v\| \\
&= \frac{N e^{\mu}}{e^{\mu} - e^{\alpha}} \|C(n,x)P_1(x)v\| \\
&\leq D \|C(n,x)P_1(x)v\|.
\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}
\sum_{k=0}^{n} e^{\nu(n-k)} \|C(k,x)P_2(x)v\| &\leq N \sum_{k=0}^{n} e^{\nu(n-k)} e^{-\beta(n-k)} \|C(n,x)P_2(x)v\| \\
&\leq N e^{n(\nu-\beta)} \frac{e^{(\beta-\nu)(n+1)}}{e^{\beta-\nu} - 1} \|C(n,x)P_2(x)v\| \\
&= \frac{N e^{\beta}}{e^{\beta} - e^{\nu}} \|C(n,x)P_2(x)v\| \\
&\leq D \|C(n,x)P_2(x)v\|,
\end{aligned}$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

Next, we deduce that

$$\begin{aligned}
\sum_{k=n}^{+\infty} e^{\omega(k-n)} \|C(k,x)P_3(x)v\| &\leq N \sum_{k=n}^{+\infty} e^{\omega(k-n)} e^{-\gamma(k-n)} \|C(n,x)P_3(x)v\| \\
&= N e^{n(\gamma-\omega)} \frac{e^{(\omega-\gamma)n}}{1 - e^{\omega-\gamma}} \|C(n,x)P_3(x)v\| \\
&= \frac{N e^{\gamma}}{e^{\gamma} - e^{\omega}} \|C(n,x)P_3(x)v\| \\
&\leq D \|C(n,x)P_3(x)v\|.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\sum_{k=0}^{n} e^{\eta(k-n)} \|C(k,x)P_3(x)v\| &\leq N \sum_{k=0}^{n} e^{\eta(k-n)} e^{\delta(n-k)} \|C(n,x)P_3(x)v\| \\
&\leq N e^{n(\delta-\eta)} \frac{e^{(\eta-\delta)(n+1)}}{e^{\eta-\delta} - 1} \|C(n,x)P_3(x)v\| \\
&= \frac{N e^{\eta}}{e^{\eta} - e^{\delta}} \|C(n,x)P_3(x)v\| \\
&\leq D \|C(n,x)P_3(x)v\|.
\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Sufficiency. Let $(n,x,v) \in \mathbb{N} \times Y$. We observe that

$$e^{-\mu n}\|C(n,x)P_1(x)v\| \le \sum_{k=0}^{+\infty} e^{-\mu k}\|C(k,x)P_1(x)v\| \le D\|P_1(x)v\|;$$

and

$$e^{\nu n}\|P_2(x)v\| \le \sum_{k=0}^{n} e^{\nu(n-k)}\|C(k,x)P_2(x)v\| \ \le D\|C(n,x)P_2(x)v\|.$$

Using a similar technique, we have that

$$e^{\omega n}\|C(n,x)P_3(x)v\| \le \sum_{k=0}^{+\infty} e^{\omega k}\|C(k,x)P_3(x)v\| \le D\|P_3(x)v\|,$$

and

$$e^{-\eta n}\|P_3(x)v\| \le \sum_{k=0}^{n} e^{\eta(k-n)}\|C(k,x)P_3(x)v\| \ \le D\|C(n,x)P_3(x)v\|,$$

for all $(n,x,v) \in \mathbb{N} \times Y$. It follows that the pair $(C,\mathcal{P})$ admits uniform exponential trisplitting, which ends the proof. □

Corollary 2.14. *The pair $(C,\mathcal{P})$ is uniformly exponentially trichotomic if and only if $(C,\mathcal{P})$ has uniform exponential trichotomy of Datko type.*

Proof. This is seen from Theorem 2.13 and Remark 2.12. □

Definition 2.15. We say that $L = (L_1, L_2, L_3) : \mathbb{N} \times Y \to \mathbb{R}^3_+$ is *Lyapunov function* for the pair $(C,\mathcal{P})$ if there exist some constants $a, b, c, d \in \mathbb{R}$, with $a < b, c < d$ such that

$$L_1(n,x,P_1(x)v) + \sum_{k=0}^{n-1} e^{-ak}\|C(k,x)P_1(x)v\| \le L_1(0,x,P_1(x)v), \tag{2.17}$$

$$L_1(0,x,P_2(x)v) + \sum_{k=0}^{n-1} e^{b(n-k)}\|C(k,x)P_2(x)v\| \le L_1(n,x,P_2(x)v), \tag{2.18}$$

$$L_2(n,x,P_3(x)v) + \sum_{k=0}^{n-1} e^{ck}\|C(k,x)P_3(x)v\| \le L_2(0,x,P_3(x)v), \tag{2.19}$$

$$L_3(0,x,P_3(x)v) + \sum_{k=0}^{n-1} e^{d(k-n)}\|C(k,x)P_3(x)v\| \le L_3(n,x,P_3(x)v), \tag{2.20}$$

for all $(n,x,v) \in \mathbb{N}^* \times Y$.

Theorem 2.16. *The pair $(C,\mathcal{P})$ admits uniform exponential trisplitting if and only if there exist a Lyapunov function $L = (L_1, L_2, L_3) : \mathbb{N} \times Y \to \mathbb{R}^3_+$ for $(C,\mathcal{P})$ and a constant $M \ge 1$ such that the following relations are satisfied*

$$L_1(0,x,P_1(x)v) \le M\|P_1(x)v\|, \tag{2.21}$$

$$L_1(n,x,P_2(x)v) \le M\|C(n,x)P_2(x)v\|, \tag{2.22}$$

$$L_2(0,x,P_3(x)v) \le M\|P_3(x)v\|, \tag{2.23}$$

$$L_3(n,x,P_3(x)v) \le M\|C(n,x)P_3(x)v\|, \tag{2.24}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Proof. Necessity. We consider $L = (L_1, L_2, L_3) : \mathbb{N} \times Y \to \mathbb{R}_+$ given by

$$\begin{aligned}
L_1(n,x,v) &= \sum_{k=n}^{+\infty} e^{-\mu k}\|C(k,x)P_1(x)v\| + \sum_{k=0}^{n} e^{\nu(n-k)}\|C(k,x)P_2(x)v\|,\\
L_2(n,x,v) &= \sum_{k=n}^{+\infty} e^{\omega k}\|C(k,x)P_3(x)v\|,\\
L_3(n,x,v) &= \sum_{k=0}^{n} e^{\eta(k-n)}\|C(k,x)P_3(x)v\|,
\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$, where μ, ν, ω, η are given by Theorem 2.13. Further, for all $(n,x,v) \in \mathbb{N}^* \times Y$ we obtain

$$\begin{aligned}
&L_1(n,x,P_1(x)v) + \sum_{k=0}^{n-1} e^{-\mu k}\|C(k,x)P_1(x)v\|\\
&= \sum_{k=n}^{+\infty} e^{-\mu k}\|C(k,x)P_1(x)v\| + \sum_{k=0}^{n-1} e^{-\mu k}\|C(k,x)P_1(x)v\|\\
&= \sum_{k=0}^{+\infty} e^{-\mu k}\|C(k,x)P_1(x)v\|\\
&= L_1(0,x,P_1(x)v),\\
&L_1(0,x,P_2(x)v) + \sum_{k=0}^{n-1} e^{\nu(n-k)}\|C(k,x)P_2(x)v\|\\
&= \sum_{k=0}^{n-1} e^{\nu(n-k)}\|C(k,x)P_2(x)v\|\\
&\le \sum_{k=0}^{n} e^{\nu(n-k)}\|C(k,x)P_2(x)v\|\\
&= L_1(n,x,P_2(x)v),\\
&L_2(n,x,P_3(x)v) + \sum_{k=0}^{n-1} e^{\omega k}\|C(k,x)P_3(x)v\|\\
&= \sum_{k=n}^{+\infty} e^{\omega k}\|C(k,x)P_3(x)v\| + \sum_{k=0}^{n-1} e^{\omega k}\|C(k,x)P_3(x)v\|\\
&= \sum_{k=0}^{+\infty} e^{\omega k}\|C(k,x)P_3(x)v\|\\
&= L_2(0,x,P_3(x)v),\\
&L_3(0,x,P_3(x)v) + \sum_{k=0}^{n-1} e^{\eta(k-n)}\|C(k,x)P_3(x)v\|\\
&= \sum_{k=0}^{n-1} e^{\eta(k-n)}\|C(k,x)P_3(x)v\|\\
&\le \sum_{k=0}^{n} e^{\eta(k-n)}\|C(k,x)P_3(x)v\|\\
&= L_3(n,x,P_3(x)v).
\end{aligned}$$

Thus, we have that L is a Lyapunov function for the pair $(C, \mathcal{P})$. Finally, using Theorem 2.13 we deduce that the conditions (2.21)–(2.24) hold.

Sufficiency. Using condition (2.17) from Definition 2.15 and (2.21), one sees that

$$\sum_{k=0}^{n-1} e^{-ak}\|C(k,x)P_1(x)v\| \le L_1(0,x,P_1(x)v) \le M\|P_1(x)v\|$$

which gives

$$\sum_{k=0}^{+\infty} e^{-ak}\|C(k,x)P_1(x)v\| \le (M+1)\|P_1(x)v\|,$$

for all $(x,v) \in Y$. In a similar manner, (2.18) from Definition 2.15 and (2.22) provides

$$\begin{aligned}\sum_{k=0}^{n} e^{b(n-k)}\|C(k,x)P_2(x)v\| &= \sum_{k=0}^{n-1} e^{b(n-k)}\|C(k,x)P_2(x)v\| + \|C(n,x)P_2(x)v\| \\ &\le L_1(n,x,P_2(x)v) + \|C(n,x)P_2(x)v\| \\ &\le (M+1)\|C(n,x)P_2(x)v\|.\end{aligned}$$

Hence

$$\sum_{k=0}^{n} e^{b(n-k)}\|C(k,x)P_2(x)v\| \le (M+1)\|C(n,x)P_2(x)v\|,$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Also, the relations (2.19) from Definition 2.15 and (2.23) implies

$$\sum_{k=0}^{n-1} e^{ck}\|C(k,x)P_3(x)v\| \le L_2(0,x,P_3(x)v) \le M\|P_3(x)v\|$$

and then

$$\sum_{k=0}^{+\infty} e^{ck}\|C(k,x)P_3(x)v\| \le (M+1)\|P_3(x)v\|,$$

for all $(x,v) \in Y$. Similarly, from (2.20) from Definition 2.15 and (2.24) it follows that

$$\begin{aligned}\sum_{k=0}^{n} e^{d(k-n)}\|C(k,x)P_3(x)v\| &= \sum_{k=0}^{n-1} e^{d(k-n)}\|C(k,x)P_3(x)v\| + \|C(n,x)P_3(x)v\| \\ &\le L_3(n,x,P_3(x)v) + \|C(n,x)P_3(x)v\| \\ &\le (M+1)\|C(n,x)P_3(x)v\|\end{aligned}$$

which yields

$$\sum_{k=0}^{n} e^{d(k-n)}\|C(k,x)P_3(x)v\| \le (M+1)\|C(n,x)P_3(x)v\|,$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Applying Theorem 2.13, we conclude that the pair $(C, \mathcal{P})$ admits uniform exponential trisplitting, completing the proof. □

In particular, we obtain

Corollary 2.17. *The pair $(C,\mathcal{P})$ admits uniform exponential trichotomy if and only if there exist a Lyapunov function $L = (L_1, L_2, L_3) : \mathbb{N} \times Y \to \mathbb{R}^3_+$ for $(C,\mathcal{P})$ and a constant $M \geq 1$ such that*

$$\begin{aligned}
L_1(0, x, P_1(x)v) &\leq M\|P_1(x)v\|, \\
L_1(n, x, P_2(x)v) &\leq M\|C(n, x)P_2(x)v\|, \\
L_2(0, x, P_3(x)v) &\leq M\|P_3(x)v\|, \\
L_3(n, x, P_3(x)v) &\leq M\|C(n, x)P_3(x)v\|,
\end{aligned}$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

3 Uniform exponential trisplitting with strongly invariant projectors

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be an orthogonal and invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ over the discrete semiflow $S : \mathbb{N} \times X \to X$ on Y.

Definition 3.1. We say that $\mathcal{P} = \{P_1, P_2, P_3\}$ is *strongly invariant* for the discrete cocycle C if the map $C(n, x)$ is an isomorphism from *Range* $P_j(x)$ to *Range* $P_j(S(n, x))$, $j = 2, 3$, for all $(n, x) \in \mathbb{N} \times X$.

Previous definition provides a so-called "regularity condition". For a more detailed discussion about some properties regarding this see ([21]). Through the following example we will show for a pair $(C,\mathcal{P})$ that admits a uniform exponential trisplitting with invariant projections but fails to be strongly invariant.

Example 3.2. Let $X = \mathbb{R}_+$ and $S : \mathbb{N} \times X \to \mathcal{B}(V)$ defined by $S(n, x) = n + x$. It is obvious that the semiflow properties are satisfied, i.e. $S(0, x) = x$ and $S(m, S(n, x)) = S(m + n, x)$.

Let $V = \mathbb{R}^4$ with the norm $\|v\| = \|(v_1, v_2, v_3, v_4)\| = |v_1| + |v_2| + |v_3| + |v_4|$. For all $(n, x) \in \mathbb{N} \times X$ we consider $C(n, x) : V \to V$ given by

$$C(n, x)v = \begin{cases} (v_1 e^{-n}, v_2 e^n, v_3 e^n, v_4 e^{2n}), & n \geq 0,\ x > 0 \\ (v_1 e^{-n}, 0, v_3 e^n, v_4 e^{2n}), & n > 0,\ x = 0 \\ (v_1, v_2, v_3, v_4), & n = 0,\ x = 0 \end{cases}$$

Let $m, n \in \mathbb{N}$, $x \in X$ and $v \in V$. Clearly, $C(0, x)v = v$. Further, we verify the cocycle properties. If $n = 0$ the relation

$$C(m, S(n, x))C(n, x)v = C(m + n, x)v \iff C(m, n + x)C(n, x)v = C(m + n, x)v$$

is automatically satisfied. Further, we suppose that $n > 0$. If $x = 0$ then we have

$$\begin{aligned}
&C(m, n)C(n, 0)v = C(m + n, 0)v, \\
&\qquad \iff C(m, n)(v_1 e^{-n}, 0, v_3 e^n, v_4 e^{2n}) = (v_1 e^{-(m+n)}, 0, v_3 e^{m+n}, v_4 e^{2(m+n)}), \\
&\qquad \iff (v_1 e^{-n} e^{-m}, 0, v_3 e^n e^m, v_4 e^{2n} e^{2m}) = (v_1 e^{-(m+n)}, 0, v_3 e^{m+n}, v_4 e^{2(m+n)}).
\end{aligned}$$

If $x > 0$ then we have

$$\begin{aligned}
&C(m, n + x)C(n, x)v = C(m + n, x)v \\
&\quad \iff C(m, n + x)(v_1 e^{-n}, v_2 e^n, v_3 e^n, v_4 e^{2n}) = (v_1 e^{-(m+n)}, v_2 e^{m+n}, v_3 e^{m+n}, v_4 e^{2(n+m)}) \\
&\quad \iff (v_1 e^{-n} e^{-m}, v_2 e^n e^m, v_3 e^n e^m, v_4 e^{2n} e^{2m}) = (v_1 e^{-(m+n)}, v_2 e^{n+m}, v_3 e^{n+m}, v_4 e^{2(n+m)})
\end{aligned}$$

Hence $C : \mathbb{N} \times X \to \mathcal{B}(V)$ is a discrete cocycle over the semiflow S.

Now, for all $x \in X$ we consider the family of projections $\mathcal{P} = \{P_j(x)\}$, $P_j(x) : V \to V$, $j \in \{1,2,3\}$ defined by

$$P_1(x)v = \begin{cases} (0, v_2, v_3, 0), & x = 0 \\ (-v_2 e^{-2x}, v_2, v_3, 0), & x > 0 \end{cases}$$
$$P_2(x)v = \begin{cases} (0, 0, 0, v_4), & x \geq 0 \end{cases}$$
$$P_3(x)v = \begin{cases} (v_1, 0, 0, 0), & x = 0 \\ (v_1 + v_2 e^{-2x}, 0, 0, 0), & x > 0. \end{cases}$$

One can easily see that $(P_1(x) + P_2(x) + P_3(x))v = v$ and

$$P_k(x)P_j(x)v = \begin{cases} P_k(x)v, & k = j \\ 0, & k \neq j, \end{cases}$$

for all $x \in X$, $v \in V$ and $k, j \in \{1,2,3\}$. Further, we will show that the pair $(C, \mathcal{P})$ considered above admits an exponential trisplitting. Let $n \in \mathbb{N}$, $x \in X$, $v \in V$. First, we have that

$$\begin{aligned} &P_1(S(n,x))C(n,x)v = C(n,x)P_1(x)v \\ &\quad\iff P_1(n+x)C(n,x)v = C(n,x)P_1(x)v. \end{aligned}$$

For $n = 0$ the equality is obvious. For $n > 0$ and $x = 0$ we have that

$$\begin{aligned} &P_1(n)C(n,0)v = C(n,0)P_1(0)v \\ &\quad\iff P_1(n)(v_1 e^{-n}, 0, v_3 e^n, v_4 e^{2n}) = C(n,0)(0, v_2, v_3, 0) \\ &\quad\iff (0, 0, v_3 e^n, 0) = (0, 0, v_3 e^n, 0). \end{aligned}$$

For the case $n > 0$ and $x > 0$ we obtain

$$\begin{aligned} &P_1(n+x)C(n,x)v = C(n,x)P_1(x)v \\ &\quad\iff P_1(n+x)(v_1 e^{-n}, v_2 e^n, v_3 e^n, v_4 e^{2n}) = C(n,x)(-v_2 e^{-2x}, v_2, v_3, 0) \\ &\quad\iff (-v_2 e^{-n} e^{-2x}, v_2 e^n, v_3 e^n, 0) = (-v_2 e^{-2x} e^{-n}, v_2 e^n, v_3 e^n, 0). \end{aligned}$$

Hence

$$\|C(0,x)P_1(x)v\| = \|P_1(x)v\|.$$

Further, $n > 0$ and $x = 0$ leads us to

$$\|C(n,0)P_1(0)v\| = e^n|v_3| \leq e^n(|v_2| + |v_3|) = e^n\|P_1(0)v\|.$$

Similarly, the case $n > 0$ and $x > 0$ provides us

$$\begin{aligned} \|C(n,x)P_1(x)v\| &= |v_2|e^{-2x}e^{-n} + |v_2|e^n + |v_3|e^n \\ &\leq e^n(|v_2|e^{-2x} + |v_2| + |v_3|) \\ &= e^n\|P_1(x)v\|. \end{aligned}$$

We conclude that

$$\|C(n,x)P_1(x)v\| \leq e^n\|P_1(x)v\|. \tag{3.1}$$

Then, for the second projector it is clear that

$$\begin{aligned} &P_2(S(n,x))C(n,x)v = C(n,x)P_2(x)v \\ &\qquad \iff P_2(n+x)C(n,x)v = C(n,x)P_2(x)v. \end{aligned}$$

The equality is also true for $n = 0$. For $n > 0$ and $x = 0$ we have that

$$\begin{aligned} &P_2(n)C(n,0)v = C(n,0)P_2(0)v \\ &\qquad \iff P_2(n)(v_1e^{-n}, 0, v_3e^n, v_4e^{2n}) = C(n,0)(0,0,0,v_4) \\ &\qquad \iff (0,0,0,v_4e^{2n}) = (0,0,0,v_4e^{2n}). \end{aligned}$$

For the case $n > 0$ and $x > 0$ we obtain

$$\begin{aligned} &P_2(n+x)C(n,x)v = C(n,x)P_2(x)v \\ &\qquad \iff P_2(n+x)(v_1e^{-n}, v_2e^n, v_3e^n, v_4e^{2n}) = C(n,x)(0,0,0,v_4) \\ &\qquad \iff (0,0,0,v_4e^{2n}) = (0,0,0,v_4e^{2n}). \end{aligned}$$

Hence

$$\|C(n,x)P_2(x)v\| = e^{2n}\|P_2(x)v\|. \tag{3.2}$$

Finally, for the third projector we have

$$\begin{aligned} &P_3(S(n,x))C(n,x)v = C(n,x)P_3(x)v \\ &\qquad \iff P_3(n+x)C(n,x)v = C(n,x)P_3(x)v. \end{aligned}$$

The equality is also true for $n = 0$. For $n > 0$ and $x = 0$ we have that

$$\begin{aligned} &P_3(n)C(n,0)v = C(n,0)P_3(0)v \\ &\qquad \iff P_3(n)(v_1e^{-n}, 0, v_3e^n, v_4e^{2n}) = C(n,0)(v_1,0,0,0) \\ &\qquad \iff (v_1e^{-n},0,0,0) = (v_1e^{-n},0,0,0). \end{aligned}$$

For the case $n > 0$ and $x > 0$ we obtain

$$\begin{aligned} &P_3(n+x)C(n,x)v = C(n,x)P_3(v) \\ &\qquad \iff P_3(n+x)(v_1e^{-n}, v_2e^n, v_3e^n, v_4e^{2n}) = C(n,x)(v_1 + v_2e^{-2x}, 0,0,0) \\ &\qquad \iff e^{-n}(v_1 + v_2e^{-2x}, 0,0,0) = e^{-n}(v_1 + v_2e^{-2x}, 0,0,0). \end{aligned}$$

Hence

$$\|C(0,x)P_3(x)v\| = \|P_3(x)v\|.$$

Further, $n > 0$ and $x > 0$ leads us to

$$\|C(n,x)P_3(x)v\| = e^{-n}\|P_3(x)v\|.$$

Similarly, if $n > 0$ we have

$$\|C(n,0)P_3(0)v\| = e^{-n}|v_1| = e^{-n}\|P_3(0)v\|.$$

It follows that

$$\|C(n,x)P_3(x)v\| \leq \|P_3(x)v\|, \tag{3.3}$$

respectively

$$\|P_3(x)v\| = e^n\|C(n,x)P_3(x)v\|. \tag{3.4}$$

Using (3.1)–(3.4), we conclude that relations (2.1)-(2.4) fron Definition 2.6 holds with $N = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = 0$ and $\delta = 1$.

Finally, we will prove that $C(1,0) : P_1(0)V \to P_1(S(1,0))V = P_1(1)V$ is not surjective. Clearly

$$(-e^{-2},1,1,0) = P_1(1)(1,1,1,1) \in P_1(1)V.$$

Let $v \in V$. Then

$$\begin{aligned} C(1,0)P_1(0)v &= C(1,0)(0,v_2,v_3,0) = (0,0,v_3e,0) \\ &\neq (-e^{-2},1,1,0) = P_1(1)(1,1,1,1). \end{aligned}$$

Therefore $C(1,0)$ is not an isomorphism and we get the desired conclusion.

Further, we will provide an example where the isomorphism property is verified. In fact this example provides a general class of pairs $(C,\mathcal{P})$ which has uniform exponential trisplitting with strongly invariant projections.

Example 3.3. Let $X = \mathbb{R}_+$ and $S : \mathbb{N} \times X \to X$ defined by $S(n,x) = n + x$. Let $V = l^\infty(\mathbb{N},\mathbb{R})$ endowed with the norm $\|v\| = \|(v_j)_j\| = \sup_{j\in\mathbb{N}} |v_j|$. For all $x \in X$ we consider the family of projections $\mathcal{P} = \{P_j(x)\}$, $P_j(x) : V \to V$, $j \in \{1,2,3\}$, defined by

$$\begin{aligned} P_1(x)v &= (v_0,0,0,v_3,0,0,v_6,0,0,\dots), \\ P_2(x)v &= (0,v_1,0,0,v_4,0,0,v_7,0,\dots), \\ P_3(x)v &= (0,0,v_2,0,0,v_5,0,0,v_8,\dots). \end{aligned}$$

Let $a_1,a_2,a_3,a_4 \in \mathbb{R}$ such that $a_1 < a_2$ and $a_3 < a_4$. For all $(n,x) \in \mathbb{N} \times X$ we consider $C(n,x) : V \to V$ given by

$$C(n,x)v = e^{a_1n}P_1(x)v + e^{a_2n}P_2(x)v + e^{-a_3n}P_3(x)v.$$

For all $m,n \in \mathbb{N}$, $x \in X$ and $v \in V$ we have that

$$S(0,x) = x, \qquad S(m,S(n,x)) = S(m+n,x),$$

respectively

$$C(0,x)v = v, \qquad C(m,S(n,x))C(n,x)v = C(m+n,x)v.$$

Further, for all $j \in \{1,2,3\}$ we have that

$$P_j(S(n,x)) = P_j(n+x) = P_j(x)$$

respectively

$$\begin{aligned} &P_j(S(n,x))C(n,x)v = C(n,x)P_j(x)v \\ &\Longleftrightarrow P_j(x)C(n,x)v = C(n,x)P_j(x)v = \begin{cases} e^{a_jn}P_j(x)v, & j = 1,2 \\ e^{-a_jn}P_j(x)v, & j = 3. \end{cases} \end{aligned}$$

It follows that relations (2.1)-(2.4) fron Definition 2.6 holds with $N = 1$, $\alpha = a_1$, $\beta = a_2$, $\gamma = a_3$ and $\delta = a_4$. Let $j \in \{1,2,3\}$. It remains to show that $C(n,x) : P_j(x)V \to P_j(S(n,x))V = P_j(x)V$ is an isomorphism. This follows from the definition of the projections $P_j(x) : V \to V$ and the fact that

$$C(n,x)P_j(x)v = \begin{cases} P_j(x)(e^{a_jn}v), & j = 1,2 \\ P_j(x)(e^{-a_jn}v), & j = 3. \end{cases}$$

Remark 3.4. If the family of projectors $\mathcal{P} = \{P_1, P_2, P_3\}$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ over the discrete semiflow $S : \mathbb{N} \times X \to X$, then there exists $D_j : \mathbb{N} \times X \to \mathcal{B}(V)$ such that for all $(n, x) \in \mathbb{N} \times X$, $D_j(n, x)$ is an isomorphism from Range $P_j(S(n, x))$ to Range $P_j(x)$, $j = 2, 3$ and

$$C(n,x)D_j(n,x)P_j(S(n,x)) = P_j(S(n,x)), \tag{3.5}$$

$$D_j(n,x)C(n,x)P_j(x) = P_j(x) \Longleftrightarrow D_j(n,x)P_j(S(n,x))C(n,x)P_j(x) = P_j(x), \tag{3.6}$$

$$P_j(x)D_j(n,x)P_j(S(n,x)) = D_j(n,x)P_j(S(n,x)), \tag{3.7}$$

for all $(n, x) \in \mathbb{N} \times X$.

Proposition 3.5. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$, then $D_j : \mathbb{N} \times X \to \mathcal{B}(V)$, $j = 2, 3$ satisfies*

$$D_j(n+m,x)P_j(S(n+m,x)) = D_j(m,x)D_j(n,S(m,x))P_j(S(n+m,x)),$$

for all $(n, m, x) \in \mathbb{N}^2 \times X$.

Proof. We have that

$$\begin{aligned} D_j(n+m,x)P_j(S(n+m,x)) &= P_j(x)D_j(n+m,x)P_j(S(n+m,x)) \\ &= D_j(m,x)C(m,x)P_j(x)D_j(n+m,x)P_j(S(n+m,x)) \\ &= D_j(m,x)P_j(S(m,x))C(m,x)D_j(n+m,x)P_j(S(n+m,x)) \\ &= D_j(m,x)D_j(n,S(m,x))C(n,S(m,x))P_j(S(m,x))C(m,x)D_j(n+m,x) \end{aligned}$$

and

$$\begin{aligned} &P_j(S(n+m,x)) \\ &\quad = D_j(m,x)D_j(n,S(m,x))C(n,S(m,x))C(m,x)P_j(x)D(n+m,x)P_j(S(n+m,x)) \\ &\quad = D_j(m,x)D_j(n,S(m,x))C(n+m,x)P_j(x)D_j(n+m,x)P_j(S(n+m,x)) \\ &\quad = D_j(m,x)D_j(n,S(m,x))C(n+m,x)D_j(n+m,x)P_j(S(n+m,x)) \\ &\quad = D_j(m,x)D_j(n,S(m,x))P_j(S(n+m,x)), \end{aligned}$$

for all $(n, m, x) \in \mathbb{N}^2 \times X$. □

Proposition 3.6. *Let $n, k \in \mathbb{N}$ such that $0 \le k \le n$. If $\mathcal{P} = \{P_1, P_2, P_3\}$ is a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$, then $D_j : \mathbb{N} \times X \to \mathcal{B}(V)$, $j = 2, 3$ verifies the following properties*

$$\begin{aligned} D_j(n-k,S(k,x))P_j(S(n,x)) &= C(k,x)P_j(x)D_j(n,x)P_j(S(n,x)), \\ C(k,x)P_j(x) &= D_j(n-k,S(k,x))C(n,x)P_j(x), \end{aligned}$$

for all $x \in X$.

Proof. It is easily seen that $S(n-k, S(k, x)) = S(n, x)$. Further, replacing $n \to n-k$ and $x \to S(k, x)$ in (3.7) from Remark 3.4 leads to

$$\begin{aligned} &P_j(S(k,x)D_j(n-k,S(k,x))P_j(S(n-k,S(k,x))) \\ &\quad = D_j(n-k,S(k,x)P_j(S(n-k,S(k,x))) \\ &\quad = D_j(n-k,S(k,x))P_j(S(n,x)). \end{aligned}$$

Hence

$$\begin{aligned}
&D_j(n-k,S(k,x))P_j(S(n,x))\\
&\quad = P_j(S(k,x))D_j(n-k,S(k,x))P_j(S(n,x))\\
&\quad = C(k,x)D_j(k,x)P_j(S(k,x))D_j(n-k,S(k,x))P_j(S(n,x))\\
&\quad = C(k,x)P_j(x)D_j(k,x)P_j(S(k,x))D_j(n-k,S(k,x))P_j(S(n,x))\\
&\quad = C(k,x)P_j(x)D_j(k,x)D_j(n-k,S(k,x))P_j(S(n,x))\\
&\quad = C(k,x)P_j(x)D_j(n,x)P_j(S(n,x)).
\end{aligned}$$

Using similar arguments in (3.6) from Remark 3.4 for $n \to n-k$ and $x \to S(k,x)$ one can check that

$$D_j(n-k,S(k,x))P_j(S(n-k,S(k,x)))C(n-k,S(k,x))P_j(S(k,x)) = P_j(S(k,x)),$$

and so

$$D_j(n-k,S(k,x))P_j(S(n,x))C(n-k,S(k,x))P_j(S(k,x)) = P_j(S(k,x)).$$

Finally, we have that

$$\begin{aligned}
C(k,x)P_j(x) &= P_j(S(k,x))C(k,x)P_j(x)\\
&= D_j(n-k,S(k,x))P_j(S(n,x))C(n-k,S(k,x))P_j(S(n,x))\\
&\quad \cdot C(n-k,S(k,x))C(k,x)P_j(x)\\
&= D_j(n-k,S(k,x))P_j(S(n,x))C(n-k,S(k,x))C(k,x)P_j(x)\\
&= D_j(n-k,S(k,x))P_j(S(n,x))C(n,x)P_j(x).
\end{aligned}$$

This completes the proof. □

Proposition 3.7. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Then the pair $(C, \mathcal{P})$ admits uniform exponential trisplitting if and only if there exist $N \geq 1$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha < \beta$, $\gamma < \delta$ such that*

$$\|C(n,x)P_1(x)v\| \leq Ne^{\alpha n}\|P_1(x)v\|, \tag{3.8}$$

$$e^{\beta n}\|D_2(n,x)P_2(S(n,x))v\| \leq N\|P_2(S(n,x))v\|, \tag{3.9}$$

$$e^{\gamma n}\|C(n,x)P_3(x)v\| \leq N\|P_3(x)v\|, \tag{3.10}$$

$$\|D_3(n,x)P_3(S(n,x))v\| \leq Ne^{\delta n}\|P_3(S(n,x))v\|, \tag{3.11}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Proof. We only show the equivalence between (2.2)–(3.9) and (2.4)–(3.11). Assume that (2.2) is satisfied, thus

$$\begin{aligned}
e^{\beta n}\|D_2(n,x)P_2(S(n,x))v\| &= e^{\beta n}\|P_2(x)D_2(n,x)P_2(S(n,x))v\|\\
&\leq N\|C(n,x)P_2(x)D_2(n,x)P_2(S(n,x))v\|\\
&= N\|P_2(S(n,x))C(n,x)D_2(n,x)P_2(S(n,x))v\|\\
&= N\|P_2(S(n,x))v\|,
\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Also, from (2.4) is obtained that

$$\begin{aligned}\|D_3(n,x)P_3(S(n,x))v\| &= \|P_3(x)D_3(n,x)P_3(S(n,x))v\|\\ &\leq Ne^{\delta n}\|P_3(S(n,x))C(n,x)D_3(n,x)P_3(S(n,x))v\\ &= Ne^{\delta n}\|P_3(S(n,x))v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Conversely, if (3.9) is satisfied, then

$$\begin{aligned}e^{\beta n}\|P_2(x)v\| &= e^{\beta n}\|D_2(n,x)C(n,x)P_2(x)v\|\\ &= e^{\beta n}\|D_2(n,x)P_2(S(n,x))C(n,x)P_2(x)v\|\\ &\leq N\|P_2(S(n,x))C(n,x)P_2(x)v\|\\ &= N\|C(n,x)P_2(x)v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Similarly, by (3.11) one has

$$\begin{aligned}\|P_3(x)v\| &= \|D_3(n,x)C(n,x)P_3(x)v\|\\ &= \|D_3(n,x)P_3(S(n,x))C(n,x)P_3(x)v\|\\ &\leq Ne^{\delta n}\|P_3(S(n,x))C(n,x)P_3(x)v\|\\ &= Ne^{\delta n}\|C(n,x)P_3(x)v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$. □

Proposition 3.8. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Then the pair $(C, \mathcal{P})$ admits uniform exponential trisplitting of Datko type if and only if there exist some constants $D \geq 1$ and $\mu, \nu, \omega, \eta \in \mathbb{R}$ with $\mu < \nu$, $\omega < \eta$ such that*

$$\sum_{k=n}^{+\infty} e^{\mu(n-k)}\|C(k,x)P_1(x)v\| \leq D\|C(n,x)P_1(x)v\|, \tag{3.12}$$

$$\sum_{k=0}^{n} e^{\nu(n-k)}\|D_2(n-k,S(k,x))P_2(S(n,x))v\| \leq D\|P_2(S(n,x))v\|, \tag{3.13}$$

$$\sum_{k=n}^{+\infty} e^{\omega(k-n)}\|C(k,x)P_3(x)v\| \leq D\|C(n,x)P_3(x)v\|, \tag{3.14}$$

$$\sum_{k=0}^{n} e^{\eta(k-n)}\|D_3(n-k,S(k,x))P_3(S(n,x))v\| \leq D\|P_3(S(n,x))v\|, \tag{3.15}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Proof. Again, as in the proof of Proposition 3.7 it is enough to prove the equivalences (2.10)–(3.13), and (2.12)–(3.15). The "if" part follows from Definition 2.11. Using (2.10) we have that

$$\begin{aligned}&\sum_{k=0}^{n} e^{\nu(n-k)}\|D_2(n-k,S(k,x))P_2(S(n,x))v\|\\ &\quad= \sum_{k=0}^{n} e^{\nu(n-k)}\|C(k,x)P_2(x)D_2(k,x)P_2(S(k,x))D_2(n-k,S(k,x))P_2(S(n,x))v\|\\ &\quad\leq D\|C(n,x)P_2(x)D_2(n,x)P_2(S(n,x))v\|\\ &\quad= D\|P_2(S(n,x))v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$, hence (3.13). Using a similar technique, we obtain that condition (2.12) from Definition 2.11 implies (3.15).

Let us now show the "only if" part. From (3.13) it yields that

$$\begin{aligned}
&\sum_{k=0}^{n} e^{\nu(n-k)} \|C(k,x)P_2(x)v\| \\
&\quad = \sum_{k=0}^{n} e^{\nu(n-k)} \|D_2(n-k,S(k,x))P_2(S(n,x))C(n-k,S(k,x))P_2(S(k,x))C(k,x)P_2(x)v\| \\
&\quad \leq D\|P_2(S(n,x))C(n,x)P_2(x)v\| \\
&\quad = D\|C(n,x)P_2(x)v\|,
\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$, so (2.10) is satisfied. Using a similar calculation we obtain (2.12) and hence we have the assertion. □

Theorem 3.9. *We consider $\mathcal{P} = \{P_1, P_2, P_3\}$ a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Then $(C,\mathcal{P})$ admits uniform exponential trisplitting if and only if $(C,\mathcal{P})$ admits uniform exponential trisplitting of Datko type.*

Proof. Necessity. Let $\mu, \nu, \omega, \eta \in \mathbb{R}$, such that $\alpha < \mu < \nu < \beta$, $\omega < \gamma < \delta < \eta$, and

$$D = 1 + N\left(\frac{e^{\mu}}{e^{\mu} - e^{\alpha}} + \frac{e^{\beta}}{e^{\beta} - e^{\nu}} + \frac{e^{\gamma}}{e^{\gamma} - e^{\omega}} + \frac{e^{\eta}}{e^{\eta} - e^{\delta}}\right).$$

Using (3.9) from Proposition 3.7 it follows that

$$\begin{aligned}
\sum_{k=0}^{n} e^{\nu(n-k)} \|D_2(n-k,S(k,x))P_2(S(n,x))v\| &\leq N \sum_{k=0}^{n} e^{(\nu-\beta)(n-k)} \|P_2(S(n,x))v\| \\
&\leq \frac{Ne^{\beta}}{e^{\beta} - e^{\nu}} \|P_2(S(n,x))v\| \\
&\leq D\|P_2(S(n,x))v\|,
\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$, hence (3.13). Similarly, the relation (3.11) from Proposition 3.7 implies (3.15), i.e.,

$$\begin{aligned}
\sum_{k=0}^{n} e^{\eta(k-n)} \|D_3(n-k,S(k,x))P_3(S(n,x))v\| &\leq N \sum_{k=0}^{n} e^{(\delta-\eta)(n-k)} \|P_3(S(n,x))v\| \\
&\leq \frac{Ne^{\eta}}{e^{\eta} - e^{\delta}} \|P_3(S(n,x))v\| \\
&\leq D\|P_3(S(n,x))v\|,
\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Sufficiency. We shall prove that the conditions in Proposition 3.7 hold. Relations (3.8) and (3.10) holds as in Theorem 2.13. Further, making use of (3.13) and (3.15) we deduce

$$\begin{aligned}
e^{\nu n} \|D_2(n,x)P_2(S(n,x))v\| &\leq \sum_{k=0}^{n} e^{\nu(n-k)} \|D_2(n-k,S(k,x))P_2(S(n,x))v\| \\
&\leq D\|P_2(S(n,x))v\|,
\end{aligned}$$

respectively

$$e^{-\eta n}\|D_3(n,x)P_3(S(n,x))v\| \le \sum_{k=0}^{n} e^{\eta(k-n)}\|D_3(n-k,S(k,x))P_3(S(n,x))v\| \le D\|P_3(S(n,x))v\|,$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Thus, we obtain that the pair $(C,\mathcal{P})$ admits uniform exponential trisplitting. □

Corollary 3.10. *We consider $\mathcal{P} = \{P_1, P_2, P_3\}$ a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. The pair $(C,\mathcal{P})$ has uniform exponential trichotomy if and only if $(C,\mathcal{P})$ has uniform exponential trichotomy of Datko type.*

Proof. The proofs here are straightforward from Theorem 3.9. □

Proposition 3.11. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Then $L = (L_1, L_2, L_3) : \mathbb{N} \times Y \to \mathbb{R}^3_+$ is a Lyapunov function for the pair $(C,\mathcal{P})$ if and only if there exist some constants $a,b,c,d \in \mathbb{R}$, with $a < b$ and $c < d$ such that*

$$L_1(n,x,P_1(x)v) + \sum_{k=0}^{n-1} e^{-ak}\|C(k,x)P_1(x)v\| \le L_1(0,x,P_1(x)v), \tag{3.16}$$

$$L_1(0,x,D_2(n,x)P_2(S(n,x))v) + \sum_{k=0}^{n-1} e^{b(n-k)}\|D_2(n-k,S(k,x))P_2(S(n,x))v\| \le L_1(n,x,D_2(n,x)P_2(S(n,x))v), \tag{3.17}$$

$$L_2(n,x,P_3(x)v) + \sum_{k=0}^{n-1} e^{ck}\|C(k,x)P_3(x)v\| \le L_2(0,x,P_3(x)v), \tag{3.18}$$

$$L_3(0,x,D_3(n,x)P_3(S(n,x))v) + \sum_{k=0}^{n-1} e^{d(k-n)}\|D_3(n-k,S(k,x))P_3(S(n,x))v\| \le L_3(n,x,D_3(n,x)P_3(S(n,x))v), \tag{3.19}$$

for all $(n,x,v) \in \mathbb{N}^* \times Y$.

Proof. It is easily seen that (2.17)–(3.16) and (2.19)–(3.18), respectively, are equivalent. Further, we prove the equivalence between (2.18) and (3.21). Assume that (2.18) is satisfied, then there exists a unique $w \in V$ such that

$$D_2(n,x)P_2(S(n,x))v = P_2(x)w,$$

or, in equivalent form

$$C(n,x)P_2(x)w = P_2(S(n,x))v.$$

It follows that

$$\begin{aligned}
&\sum_{k=0}^{n-1} e^{b(n-k)}\|D_2(n-k,S(k,x))P_2(S(n,x))v\| \\
&\quad= \sum_{k=0}^{n-1} e^{b(n-k)}\|C(k,x)P_2(x)D_2(n,x)P_2(S(n,x))v\| \\
&\quad= \sum_{k=0}^{n-1} e^{b(n-k)}\|C(k,x)P_2(x)w\| \le L_1(n,x,P_2(w)) - L_1(0,x,P_2(x)w) \\
&\quad= L_1(n,x,D_2(n,x)P_2(S(n,x))v) - L_1(0,x,D_2(n,x)P_2(S(n,x))v).
\end{aligned}$$

Conversely, there exists a unique $t \in V$ such that

$$C(n,x)P_2(x)v = P_2(S(n,x))t,$$

or, in equivalent form

$$D_2(n,x)P_2(S(n,x))t = P_2(x)v.$$

Hence,

$$\begin{aligned}
&\sum_{k=0}^{n-1} e^{b(n-k)} \|C(k,x)P_2(x)v\| \\
&\quad = \sum_{k=0}^{n-1} e^{b(n-k)} \|D_2(n-k, S(k,x))P_2(S(n,x))C(n,x)P_2(x)v\| \\
&\quad = \sum_{k=0}^{n-1} e^{b(n-k)} \|D_2(n-k, S(k,x))P_2(S(n,x))t\| \\
&\quad \leq L_1(n,x,D_2(n,x)P_2(S(n,x))t) - L_1(0,x,D_2(n,x)P_2(S(n,x))t) \\
&\quad = L_1(n,x,P_2(x)v) - L_1(0,x,P_2(x)v).
\end{aligned}$$

In a similar manner it can be proved the equivalence between (2.20) and (3.19), which ends the proof. □

Theorem 3.12. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. The pair $(C, \mathcal{P})$ has uniform exponential trisplitting if and only if there exists a Lyapunov function $L = (L_1, L_2, L_3) : \mathbb{N} \times Y \to \mathbb{R}_+^3$ for $(C, \mathcal{P})$ and a constant $M \geq 1$ such that the following relations are satisfied*

$$L_1(0,x,P_1(x)v) \leq M\|P_1(x)v\|, \tag{3.20}$$
$$L_1(n,x,D_2(n,x)P_2(S(n,x))v) \leq M\|P_2(S(n,x))v\|, \tag{3.21}$$
$$L_2(0,x,P_3(x)v) \leq M\|P_3(x)v\|, \tag{3.22}$$
$$L_3(n,x,D_3(n,x)P_3(S(n,x))v) \leq M\|P_3(S(n,x))v\|, \tag{3.23}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.

Proof. It is enough to show the equivalence between (2.22) and (3.21). Assume that (2.22) is satisfied. Then, there exists a unique $w \in V$ such that

$$D_2(n,x)P_2(S(n,x))v = P_2(x)w,$$

or, in equivalent form

$$C(n,x)P_2(n,x)w = P_2(S(n,x))v.$$

Hence, we have

$$\begin{aligned}
L_1(n,x,D_2(n,x)P_2(S(n,x))v) &= L_1(n,x,P_2(x)w) \\
&\leq M\|C(n,x)P_2(x)w\| = M\|P_2(S(n,x))v\|.
\end{aligned}$$

Conversely suppose (3.21) holds. There exists a unique $t \in V$ such that,

$$C(n,x)P_2(x)v = P_2(S(n,x))t$$

which is equal to

$$D_2(n,x)P_2(S(n,x))t = P_2(x)v.$$

In this way we may conclude that

$$\begin{aligned} L_1(n,x,P_2(x)v) &= L_1(n,x,D_2(n,x)P_2(S(n,x))t) \\ &\leq M\|P_2(S(n,x)t\| = M\|C(n,x)P_2(x)v\|. \end{aligned}$$

□

4 Conclusion

The notion of uniform exponential trichotomy, which has been proved so useful in characterizing dynamical systems, has been extended here to the so called concept of uniform exponential trisplitting. This new concept provides a better view regarding the growth rate constants from the classical definition of uniform exponential trichotomy. Furthermore, we have derived new Datko-type and Lyapunov-type results for both characterisations of the dynamical system in terms of invariant and strongly invariant projections.

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