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EXISTENCE RESULTS FOR SECOND ORDER CONVEX SWEEPING PROCESSES IN p-UNIFORMLY SMOOTH AND q-UNIFORMLY CONVEX BANACH SPACES

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Abstract. In a previous work the authors proved under a complex assumption on the set-valued mapping, the existence of Lipschitz solutions for second order convex sweeping processes in p-uniformly smooth and q-uniformly convex Banach spaces. In the present work we prove the same result, under a condition on the distance function to the images of the set-valued mapping. Our assumption is much simpler than the one used in the former paper.

1. Introduction

In [5], the authors studied the following extensions of convex sweeping processes from Hilbert spaces H to reflexive smooth Banach spaces X:

(SSP) Find $T > 0$, $x^* : [0,T] \to J(cl(\nu_0))$ and $u^* : [0,T] \to X^*$ such that

$$\begin{cases} u^*(0) = J(u_0), J^*(u^*(t)) \in K(J^*(x^*(t))), \text{ for all } t \in [0,T]; \\ x^*(t) = J(x_0) + \displaystyle\int_0^t u^*(s)ds, \text{ for all } t \in [0,T]; \\ (u^*)'(t) \in -N(K(J^*(x^*(t))); J^*(u^*(t))) \text{ a.e. on } [0,T], \end{cases}$$

where $x_0 \in X$, $u_0 \in K(x_0)$, $\nu_0 := J^*(\nu_0^*)$, ν_0^* be an open neighborhood of $J(x_0)$ in X^*, $K : cl(\nu_0) \rightrightarrows X$ be a set-valued mapping taking nonempty closed convex values in X, and $J : X \to X^*$ is the duality mapping defined from X into X^* (see Section 2 for the definitions). The mapping x^* is called a solution of (SSP)
Clearly, (SSP) coincides with the well known second order convex sweeping process studied in many works (see for instance [4, 7, 10] and the reference therein) in the Hilbert space setting in which J is the identity mapping. The authors in [5] proved the following theorem.

Theorem 1.1. *Let $p, q > 1$, X be a separable Banach space which is p-uniformly convex and q-uniformly smooth, and let $K : cl(\nu_0) \rightrightarrows X$ be a set-valued mapping taking nonempty closed convex values in X and satisfying: for any $x, x' \in cl(\nu_0)$ and any $\varphi, \varphi' \in X^*$,*

$$(1.1) \qquad |(d^V_{K(x')})^{\frac{q-1}{q}}(\varphi') - (d^V_{K(x)})^{\frac{q-1}{q}}(\varphi)| \le \lambda \|J(x') - J(x)\| + \gamma\|\varphi' - \varphi\|.$$

Assume that $J(K(x)) \subset \mathcal{L}$, for some convex compact set $\mathcal{L} \in X^*$. Then (SSP) has at least one Lipschitz solution.

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They proved the existence of solutions under (1.1) the Lipschitz behavior of the function $(x,\psi) \mapsto (d^V_{K(J^*(x))})^{\frac{q-1}{q}}(\psi)$ defined on $X^* \times X^*$, where $d^V_S(\psi) := \inf_{x\in S} V(\psi, x)$ and $V(\varphi, x) = \|\varphi\|^2 - 2\langle\varphi, x\rangle + \|x\|^2$. In this paper we prove the previous theorem under the Lipschitz continuity of the function $x \mapsto d^{\frac{q}{p}}_{K(x)}(u), \forall u \in X$, (with constant depending on u see Theorem 3.1), which is defined on X and is easier to handle with, than the function $(y,\psi) \mapsto (d^V_{K(J^*(y))})^{\frac{q-1}{q}}(\psi)$ used in (1.1). Also, in the case of Banach spaces (not necessarily Hilbert) the Lipschitz assumption (3.1) is much easier to be checked than (1.1).
Before proving our main result in Theorem 3.1, we recall from [5] some needed concepts and results and for more details we refer the reader to [5] and the references therein.

2. Preliminaries.

Let X be a Banach space with topological dual space X^*. We denote by d_S the usual distance function to S, i.e., $d_S(x) := \inf_{u\in S} \|x-u\|$. Let S be a nonempty closed convex set of X and $\bar{x}$ be a point in S. The convex normal cone of S at $\bar{x}$ is defined by (see for instance [11])

$$N(S;\bar{x}) = \{\varphi \in X^* : \langle\varphi, x-\bar{x}\rangle \le 0 \text{ for all } x \in S\}. \tag{2.1}$$

The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) = \{j(x) \in X^* : \langle j(x), x\rangle = \|x\|^2 = \|j(x)\|^2\}.$$

Many properties of the normalized duality mapping J have been studied. For the details, one may see the books [1, 14, 15]. Let $V : X^* \times X \to \mathbb{R}$ be defined by

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle\varphi, x\rangle + \|x\|^2, \text{ for any } \varphi \in X^* \text{ and } x \in X.$$

Based on the functional V, a set $\pi_S(\varphi)$ of generalized projections of $\varphi \in X^*$ onto S is defined as follows (see [2]).

Definition 2.1. Let S be a nonempty subset of X and $\varphi \in X^*$. If there exists a point $\bar{x} \in S$ satisfying

$$V(\varphi, \bar{x}) = \inf_{x\in S} V(\varphi, x),$$

then $\bar{x}$ is called a generalized projection of φ onto S. The set of all such points is denoted by $\pi_S(\varphi)$. When the space X is not reflexive $\pi_S(\varphi)$ may be empty for some elements $\varphi \in X^*$ even when S is closed and convex (see Example 1.4. in [12]).

The following proposition is needed in the proof of the main theorem. For its proof we refer the reader to [13].

Proposition 2.2. *For a nonempty closed convex subset S of a reflexive smooth Banach space X and $u \in S$, the following assertions are equivalent:*

i) *$\bar{x} \in S$ is a projection of u onto S, that is $\bar{x} \in P_S(u)$;*
ii) *$\langle J(u-\bar{x}), x-\bar{x}\rangle \le 0$ for all $x \in S$;*
iii) *$J(u-\bar{x}) \in N(S;\bar{x})$.*

Assume now that X is p-uniformly convex and q-uniformly smooth Banach space (for their definitions we refer the reader to the reference [5] and the references therein) and let S be closed nonempty set in X. Recall the definition of the function $d_S^V : X^* \to [0, \infty[$, given by $d_S^V(\varphi) = \inf_{x \in S} V(\varphi, x)$. Clearly, in Hilbert spaces d_S^V coincide with d_S^2. We need the two following lemma proved in [5] respectively.

Lemma 2.3. *Let $p, q > 1$, X be a p-uniformly convex and q-uniformly smooth Banach space, and let S be a bounded set. Then there exist two constants $\alpha > 0$ and $\beta > 0$ so that $\alpha\|x - y\|^p \leq V(J(x), y) \leq \beta\|x - y\|^q$, for all $x, y \in S$.*

Proposition 2.4. *If S is a bounded set in X, then $d_S^V(\varphi) \leq \beta(d_S(J^*(\varphi)))^q$, where β depends on the bound of S and on φ. As a consequence, for sets S_1 and S_2 in X and X^* bounded by l_1 and l_2 respectively, we have $d_S^V(\varphi) \leq \beta(d_S(J^*(\varphi)))^q$, for all $\varphi \in S_2$, where β depends on l_1 and l_2.*

The following lemma is taken from [1].

Proposition 2.5. *Let $p \geq 2, q > 1$ and let X be a p-uniformly convex and q-uniformly smooth Banach space. The duality mapping $J : X \to X^*$ is Lipschitz on bounded sets, that is,*

$$\|J(x) - J(y)\| \leq C(R)\|x - y\|, \text{ for all } \|x\| \leq R, \|y\| \leq R.$$

Here $C(R) := 32Lc_2^2(q-1)^{-1}$ and $c_2 = \max\{1, R\}$ and $1 < L < 1.7$.

Let us mention that the Lipschitz continuity on bounded sets of the duality mapping J_* on X^*, is not ensured in general by Proposition 2.5 because X^* is p'-uniformly convex and q'-uniformly smooth Banach space with $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$ and by the fact that $p' \in [1, 2]$ whenever $p \geq 2$. However, J^* is uniformly continuous on bounded sets.
The following proposition summarizes two important results proved respectively in [12, 6]

Proposition 2.6. *Let X be a reflexive Banach space with dual space X^* and S be a nonempty, closed and convex subset of X. The following properties hold:*

- (π_1) *$\pi_S(\varphi) \neq \emptyset$, for any $\varphi \in X^*$;*
- (π_2) *If X is also smooth, then $\varphi \in N(S, \bar{x})$, if and only if, $\exists \alpha > 0$ so that $\bar{x} \in \pi_S(J(\bar{x}) + \alpha\varphi)$.*

We end this section with the following lemma needed in our proofs (for the proof we refer the reader for instance to [9]).

Lemma 2.7. *Let X be a reflexive Banach space and let $C : I \to X$ be a set-valued mapping with nonempty closed convex values. Then the functional $I : v \mapsto I(v) := \int_0^T \delta^*_{C(t)}(v(t))dt$ from X^* to $\mathbb{R}$ is weakly lower semi-continuous in the following sense: for any (v_n) a sequence of mappings $v_n : I \to X^*$ such that $v_n \to v_*$ in the weak star topology of $L^\infty(I, X^*)$, we have*

$$\int_0^T \delta^*_{C(t)}(v_*(t))dt \leq \liminf_n \int_0^T \delta^*_{C(t)}(v_n(t))dt.$$

3. Main result.

Now, we are ready to prove the main result in the following theorem.

Theorem 3.1. *Instead of (1.1) in Theorem 1.1, assume that*

$$|d^{\frac{q}{p}}_{K(x')}(u) - d^{\frac{q}{p}}_{K(x)}(u)| \leq \lambda(u)\|x' - x\|, \text{ for all } u \in X, \tag{3.1}$$

with $\lambda : X \to [0, \infty)$ is bounded on bounded sets. Then (SSP) has at least one Lipschitz solution.

Proof. We give the proof in four steps.
Step 1. Construction of approximants. Let $\mu > 0$ such that $J(x_0) + \mu\mathbf{B}_* \subset \nu_0^*$ and let $l > 0$ such that $L \subset l\mathbf{B}_*$. Let $T \in (0, \frac{\mu}{l})$ and put $I := [0, T]$. For each $n \in N$, we consider the partition of I given by $I_{n,i} := [t_{n,i}, t_{n,i+1})$, for all $i = 0, \dots, n-1$, with $t_{n,i} = i\mu_n$, $\mu_n := \frac{T}{n}$, and $I_{n,n} := \{T\}$.
For every $n \in \mathbb{N}$ we define the following approximating mappings on each interval $I_{n,i}$ as follows

$$\begin{cases} u_n^*(t) := J(u_{n,i}), \quad u_n(t) = J^*(u_n^*(t)) = u_{n,i}, \\ x_n^*(t) = J(x_0) + \displaystyle\int_0^t u_n^*(s)ds, \quad x_n(t) = J^*(x_n^*(t)), \end{cases} \tag{3.2}$$

where $u_{n,0} = u_0$ and for all $i = 0, \dots, n-1$ the point $u_{n,i+1}$ is given by

$$u_{n,i+1} = \pi_{K(x_n(t_{n,i+1}))}(J(u_{n,i})). \tag{3.3}$$

As

$$x_n^*(t_{n,1}) = J(x_0) + \int_0^{t_{n,1}} u_n^*(s)ds \subset J(x_0) + lt_{n,1}\mathbb{B}_* \subset J(x_0) + \mu\mathbb{B}_* \subset \nu_0^*,$$

so

$$x_n(t_{n,1}) = J^*(x_n^*(t_{n,1})) \subset J^*(\nu_0^*) = \nu_0,$$

and as K has nonempty closed convex values, by Proposition 2.6 one can choose a point $u_{n,1} \in \pi_{K(x_n(t_{n,1}))}(J(u_{n,0}))$. Similarly, we can define, by induction, all the points $(u_{n,i})_i$

Let us define $\theta_n(t) := t_{n,i}$, and $\rho_n(t) := t_{n,i+1}$ if $t \in I_{n,i}$. Then, the definition of $x_n(\cdot)$ and $u_n(\cdot)$ yield for all $t \in I$,

$$u_n(t) \in K(x_n(\theta_n(t))) \subset l\mathbb{B}. \tag{3.4}$$

So, the mappings $x_n^*(\cdot)$ are Lipschitz with ratio l and they are also equibounded, with $\|x_n^*\|_\infty \leq \|x_0\| + lT$. Hence the mappings $x_n(\cdot)$ are continuous.

Observe also that for all $n \in \mathbb{N}$ and $t \in I$ one has

$$x_n(t) \in \nu_0. \tag{3.5}$$

Indeed, the definition of $x_n(\cdot)$ and $u_n(\cdot)$ ensure that, for all $t \in I$,

$$x_n^*(t) = J(x_0) + \int_0^t u_n^*(s)ds \subset J(x_0) + lt\mathbb{B}_* \subset J(x_0) + \mu\mathbb{B}_* \subset \nu_0^*,$$

and so

$$x_n(t) = J^*(x_n^*(t)) \subset J^*(\nu_0^*) = \nu_0,$$

and hence $K(x_n(t))$ is well defined for all $t \in I$.

Now we define the piecewise affine approximants from I to X^* as follows

$$v_n^*(t) := J(u_{n,i}) + \mu_n^{-1}(t - t_{n,i})(J(u_{n,i+1}) - J(u_{n,i})), \text{ if } t \in I_{n,i}. \tag{3.6}$$

Define the mapping v from I to X by

$$v_n(t) = J^*(v_n^*(t)), \text{ for all } t \in I. \tag{3.7}$$

Observe that $v_n^*(\theta_n(t)) = J(u_{n,i})$ and $v_n(\theta_n(t)) = u_{n,i}$, for all $i = 0, \ldots, n$ and so by (3.3),(3.5), and (3.6) one has

$$v_n(\theta_n(t)) \in K(x_n(t_{n,i})) = K(x_n(\theta_n(t))) \subset l\mathbb{B}.$$

Now, we check that the mappings v_n^* are equi-Lipschitz. Let us first find an upper bound estimate for the expression $\|J(u_{n,i+1}) - J(u_{n,i})\|$. Clearly, the sequence (u_i^n) is bounded by l. Consequently, $\lambda(u_i^n)$ is bounded for any i, n. Let $\bar{\lambda}$ be its bound, that is, $\lambda(u_i^n) \leq \bar{\lambda}$, for any i, n. Now, since X is q-uniformly smooth and p-uniformly convex and the sequence (u_i^n) is bounded by l, there exists some constants α and β depending on l such that

$$\alpha\|u_{n,i+1} - u_{n,i}\|^p \leq V(J(u_{n,i}), u_{n,i+1}) \leq \beta\|u_{n,i+1} - u_{n,i}\|^q,$$

and so by the construction of the sequence u_i^n and Proposition 2.4 we get

$$\alpha\|u_{n,i+1} - u_{n,i}\|^p \leq d^V_{K(x_n(t_{n,i+1}))}(J(u_{n,i})) \leq \beta d^q_{K(x_n(t_{n,i+1}))}(u_{n,i})$$

and so by the Lipschitz property in (3.1) we obtain

$$\begin{aligned}(\frac{\alpha}{\beta})^{\frac{1}{p}}\|u_{n,i+1} - u_{n,i}\| &\leq d^{\frac{q}{p}}_{K(x_n(t_{n,i+1}))}(u_{n,i}) - d^{\frac{q}{p}}_{K(x_n(t_{n,i}))}(u_{n,i}) \\ &\leq \lambda(u_{n,i})|x_n(t_{n,i+1}) - x_n(t_{n,i})| \leq \bar{\lambda}l|t_{n,i+1} - t_{n,i}| \leq l\bar{\lambda}\mu_n,\end{aligned}$$

and so

$$\|u_{n,i+1} - u_{n,i}\| \quad \leq \quad \hat{\lambda}\mu_n,$$

where $\hat{\lambda} = l(\frac{\beta}{\alpha})^{\frac{1}{p}}\bar{\lambda}$. Using now the Lipschitz property of the duality mapping J in Proposition 2.5, we can write

$$\|J(u_{n,i+1}) - J(u_{n,i})\| \leq C(l)\|u_{n,i+1} - u_{n,i}\| \leq C(l)\hat{\lambda}\mu_n. \tag{3.8}$$

So, for any $t, t' \in I_{n,i}$ one has

$$\|v_n^*(t') - v_n^*(t)\| = \mu_n^{-1}|t' - t|\|J(u_{n,i+1}) - J(u_{n,i})\| \leq C(l)\hat{\lambda}|t' - t|.$$

This inequality, with the continuity of v_n^* on $(t_{n,i})_i$, shows that the mappings v_n^* are equi-Lipschitz on all I with ratio $\delta := C(l)\hat{\lambda}$ and hence the mappings $v_n = J^*(v_n^*)$ are uniformly continuous on I because v_n^* is bounded and J^* is uniformly continuous on bounded sets. By the definition of $u_n^*(\cdot)$ and $v_n^*(\cdot)$ one has

$$\|v_n^*(t) - u_n^*(t)\| \quad \leq \quad \mu_n^{-1}|t - t_{n,i}|\|J(u_{n,i+1}) - J(u_{n,i})\| \leq \delta\mu_n,$$

and hence

$$\|v_n^* - u_n^*\|_\infty \to 0. \tag{3.9}$$

The definition of $v_n(\cdot)$ given by (3.7) and the relation (3.3) yield

$$v_n(\theta_n(t)) \in K(x_n(\theta_n(t))), \text{ for all } t \in I_{n,i}, (i = 0, \ldots, n-1), \tag{3.10}$$

and by the definition of $v_n^*(\cdot)$, one has for a.e. $t \in I_{n,i}$

$$(v_n^*)'(t) = \mu_n^{-1}(J(u_{n,i+1}) - J(u_{n,i})). \tag{3.11}$$

So, by the characterization of the convex normal cones stated in Proposition 2.6, we get for a.e. $t \in I$

$$(v_n^*)'(t) \in -N(K(x_n(\rho_n(t))); v_n(\rho_n(t))). \tag{3.12}$$

Indeed, by construction

$$\begin{aligned} u_{n,i+1} &\in \pi_{K(x_n(t_{n,i+1}))}(J(u_{n,i})) \\ &= \pi_{K(x_n(t_{n,i+1}))}(J(u_{n,i+1}) - [J(u_{n,i+1}) - J(u_{n,i})]) \\ \Leftrightarrow \quad & J(u_{n,i+1}) - J(u_{n,i}) \in -N(K(x_n(t_{n,i+1})); u_{n,i+1}) \\ \Leftrightarrow \quad & \mu_n^{-1}(J(u_{n,i+1}) - J(u_{n,i})) \in -N(K(x_n(\rho_n(t))); v_n(\rho_n(t))). \end{aligned}$$

and hence (3.12) holds.

Step 2. Uniform convergence of the sequences $x_n(\cdot)$ and $v_n(\cdot)$. Since $\mu_n^{-1}(t - t_{n,i}) \le 1$, for all $t \in I_{n,i}$ and $J(u_{n,i+1}), J(u_{n,i}) \in L$, and L is a convex set in X^* one gets for all $t \in I$,

$$\begin{aligned} v_n^*(t) &= J(u_{n,i}) + \mu_n^{-1}(t - t_{n,i})[J(u_{n,i+1}) - J(u_{n,i})] \\ &= \left(1 - \frac{t - t_{n,i}}{\mu_n}\right) J(u_{n,i}) + \frac{t - t_{n,i}}{\mu_n} J(u_{n,i+1}) \in L. \end{aligned}$$

Thus for every $t \in I$, the set $\{v_n^*(t) : n \in \mathbb{N}\}$ is relatively compact in X^*. On the other hand, it is clear by (3.8) and (3.11) that

$$\|(v_n^*)'(t)\| \le \delta. \tag{3.13}$$

Therefore, this estimate and Theorem 0.3.4 in [3] ensure the existence of a Lipschitz mapping $u^* : I \to X^*$ such that:

- (v_n^*) converges uniformly to u^* on I.

Clearly, we have the weak star convergence of $((v_n^*)')$ to some limit ω in $L^\infty(I, X^*)$ and easily, we can check that $\omega = (u^*)'$ a.e. on I. Indeed, the weak star convergence of $((v_n^*)')$ to ω in $L^\infty(I, X^*)$ ensures for any $t \in I$ and any $y \in L^1(I, X)$

$$\lim_n \langle (v_n^*)' - \omega, y \rangle_{L^\infty(I,X^*), L^1(I,X)} = 0,$$

that is,

$$\lim_n \int_0^T \langle (v_n^*)'(s) - \omega(s), y(s) \rangle_{X^*,X} ds = 0.$$

Here $\langle \cdot, \cdot \rangle_{L^\infty(I,X^*), L^1(I,X)}$ denotes the dual pairing between the spaces $L^1(I, X)$ and $L^\infty(I, X^*)$, and $\langle \cdot, \cdot \rangle_{X^*,X}$ denotes the dual pairing between the spaces X and X^*. Fix now any $t \in [0, T]$ and define $y_k : I \to X$ by $y_k \equiv \psi_{[0,t]}(\cdot) \cdot e_k$, where $(e_k) \subset X$

is a sequence separating points in X^* (such sequences exist in reflexive separable Banach spaces). Then for any $k \in \mathbb{N}$ we have

$$\langle \lim_n \int_0^t (v_n^*)'(s)ds, e_k\rangle_{X^*,X} = \langle \int_0^t \omega(s)ds, e_k\rangle_{X^*,X}.$$

This ensures

$$\lim_n \int_0^t (v_n^*)'(s)ds = \int_0^t \omega(s)ds.$$

Consequently,

$$u^*(t) = \lim_n v_n^*(t) = \lim_n [J(u_0) + \int_0^t (v_n^*)'(s)ds] = J(u_0) + \int_0^T w(s)ds,$$

and since u^* is absolutely continuous, we deduce that $\omega = (u^*)'$ a.e. on I.

We define now the continuous mapping $u : I \to X$ by

$$u(t) = J^*(u^*(t)), \text{ for all } t \in I. \tag{3.14}$$

Then, it is clear that (v_n) converges uniformly to u, because J^* is uniformly continuous on bounded sets.

Now, we define the Lipschitz mapping $x^* : I \to X$ by

$$x^*(t) = J(x_0) + \int_0^t u^*(s)ds, \text{ for all } t \in I, \tag{3.15}$$

and the continuous mapping $x : I \to X$ is given by

$$x(t) = J^*(x^*(t)), \text{ for all } t \in I. \tag{3.16}$$

Then by the definition of x_n^* one obtains for all $t \in I$,

$$\|x_n^*(t) - x^*(t)\| = \| \int_0^t (u_n^*(s) - u^*(s))ds\| \le T\|u_n^* - u^*\|_\infty,$$

and by (3.9) we get

$$\|x_n^* - x^*\|\infty \le T\|u_n^* - v_n^*\|_\infty + T\|v_n^* - u^*\|_\infty \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Hence (x_n^*) converges uniformly to x^* on I and so $(x_n) = (J^*(x_n^*))$ converges uniformly to $J^*(x^*) = x$ on I because J^* is uniformly continuous on bounded sets. This completes the second step.

Step 3. Existence of a solution. First observe that $(x_n^* \circ \theta_n)$, $(x_n^* \circ \rho_n)$ and $(v_n^* \circ \theta_n)$, $(v_n^* \circ \rho_n)$ converge uniformly on I to x^* and u^* respectively. Recall now that $v_n(\rho_n(t)) \in K(x_n(\rho_n(t)))$, for all $t \in I$ and $n \in \mathbb{N}$. It follows then by our assumptions that

$$\begin{aligned}
d_{K(x(t))}^{\frac{p}{q}}(v_n(\rho_n(t))) &= d_{K(x(t))}^{\frac{p}{q}}(v_n(\rho_n(t))) - d_{K(x_n(\rho_n(t)))}^{\frac{p}{q}}(v_n(\rho_n(t))) \\
&\le \bar{\lambda}\|x(t) - x_n(\rho_n(t))\| \\
&\le \bar{\lambda}\|x(t) - x_n(t)\| + \bar{\lambda}\|x_n(t) - x_n(\rho_n(t))\| \\
&\le \bar{\lambda}\|J^*(x^*(t)) - j^*(x_n^*(t))\| + \bar{\lambda}\|j^*(x_n^*(t)) - j^*(x_n^*(\rho_n(t)))\|.
\end{aligned}$$

Hence, by the fact that $\|x_n^* - x^*\|_\infty \to 0$ and $\|x_n^*(t) - x^*(\rho_n(t))\| \leq l\mu_n \to 0$, and the unifrom continuity of J^*, we obtain

$$d_{K(x(t))}(v_n(\rho_n(t))) \to 0,$$

and so

$$\begin{aligned} d_{K(x(t))}(u(t)) &\leq d_{K(x(t))}(v_n(\rho_n(t))) + \|v_n(\rho_n(t)) - u(t)\| \\ &\leq d_{K(x(t))}(v_n(\rho_n(t))) + \|v_n(\rho_n(t)) - v_n(t)\| + \|v_n(t) - u(t)\| \to 0, \end{aligned}$$

which ensures by the closedness of the values of K, that $u(t) \in K(x(t))$, for all $t \in I$.

Now, let us prove that the mapping x^* is a solution of our problem (SSP). Using the weak star convergence of $((v_n^*)')$ to $(u^*)'$ in $L^\infty(I, X^*)$ and Lemma 2.7 we obtain

$$\liminf_n \int_0^T \delta^*_{K(x(t))}(-(v_n^*)'(t))dt \geq \int_0^T \delta^*_{K(x(t))}(-(u^*)'(t))dt.$$

Again, we use the weak star convergence of $(v_n^*)'$ to $(u^*)'$ in $L^\infty(I, X^*)$ with the uniform convergence of $v_n \circ \rho_n$ to u to get

$$\lim_n \int_0^T \langle (v_n^*)'(t), v_n(\rho_n(t))\rangle\, dt = \int_0^T \langle (u^*)'(t), u(t)\rangle\, dt.$$

Therefore,

$$\begin{aligned} &\int_0^T \left[\delta^*_{K(x(t))}(-(u^*)'(t)) + \langle (u^*)'(t), u(t)\rangle\right] dt \leq \\ &\liminf_n \int_0^T \left[\delta^*_{K(x(t))}(-(v_n^*)'(t)) + \langle (v_n^*)'(t), v_n(\rho_n(t))\rangle\right] dt. \end{aligned} \tag{3.18}$$

By (3.12) and the definition of the normal cone we have

$$\langle -(v_n^*)'(t); y - v_n(\rho_n(t))\rangle \leq 0, \quad \forall y \in K(x_n(\rho_n(t))), \text{ a.e. } t \in I. \tag{3.19}$$

Fix any t in I for which (3.19) holds and let any $v \in K(x(t))$. By (3.1) we have

$$\begin{aligned} d^{\frac{p}{q}}_{K(x_n(\rho_n(t)))}(v) &= |d^{\frac{p}{q}}_{K(x_n(\rho_n(t)))}(v) - d^{\frac{p}{q}}_{K(x(t))}(v)| \\ &\leq \bar\lambda \|x_n(\rho_n(t)) - x(t)\| \\ &\leq \bar\lambda\left[\|x_n(\rho_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\|\right] \\ &\leq \bar\lambda\left[\|x_n \circ \rho_n - x_n\|_\infty + \|x_n - x\|_\infty\right]. \end{aligned} \tag{3.20}$$

Put $\lambda_n := \left(\bar\lambda\left[\|x_n \circ \rho_n - x_n\|_\infty + \|x_n - x\|_\infty\right]\right)^{\frac{q}{p}}$. Clearly $\lambda_n \to 0$ as $n \to \infty$ by the uniform convergence of the sequence (x_n) to x. Thus,

$$d_{K(x_n(\rho_n(t)))}(v) \leq \lambda_n, \tag{3.21}$$

which ensures the existence of $w_n \in K(x_n(\rho_n(t)))$ with $\|v - w_n\| \leq \lambda_n$. Hence by (3.19) we have

$$\langle -(v_n^*)'(t); w_n - v_n(\rho_n(t))\rangle \leq 0 \tag{3.22}$$

and so by using (3.13) we obtain

$$\begin{aligned}\langle -(v_n^*)'(t); v - v_n(\rho_n(t))\rangle &= \langle -(v_n^*)'(t); w_n - v_n(\rho_n(t))\rangle + \langle -(v_n^*)'(t); v - w_n\rangle \\ &\leq \|(v_n^*)'(t)\| \|v - w_n\| \leq \delta\lambda_n.\end{aligned}$$

Thus

$$\langle -(v_n^*)'(t); v\rangle + \langle (v_n^*)'(t); v_n(\rho_n(t))\rangle \leq \delta\lambda_n, \forall v \in K(x(t)), \text{ a.e. } t \in I.$$

Taking the supremum on v over $K(x(t))$ and integrating over I we get

$$\int_0^T \left[\delta^*_{K(x(t))}(-(v_n^*)'(t)) + \langle (v_n^*)'(t); v_n(\rho_n(t))\rangle\right] dt \leq \delta T\lambda_n.$$

Hence

$$\liminf_n \int_0^T \left[\delta^*_{K(x(t))}(-(v_n^*)'(t)) + \langle (v_n^*)'(t); v_n(\rho_n(t))\rangle\right] dt \leq 0,$$

and so, combining with (3.18) we obtain

$$\int_0^T \left[\delta^*_{K(x(t))}(-(u^*)'(t)) + \langle (u^*)'(t), u(t)\rangle\right] dt \leq 0,$$

that is,

$$\int_0^T \delta^*_{K(x(t))}(-(u^*)'(t))dt \leq \int_0^T \langle -(u^*)'(t), u(t)\rangle dt.$$

Since $u(t) \in K(x(t))$, the last inequality becomes equality and we write

$$\int_0^T \delta^*_{K(x(t))}(-(u^*)'(t))dt = \int_0^T \langle -(u^*)'(t), u(t)\rangle dt,$$

and hence for a.e. $t \in I$ we have

$$\delta^*_{K(x(t))}(-(u^*)'(t)) = \langle -(u^*)'(t), u(t)\rangle,$$

that is,

$$\langle -(u^*)'(t), w\rangle \leq \langle -(u^*)'(t), u(t)\rangle, \forall w \in K(x(t)), \text{ a.e. } t \in I.$$

Thus,

$$(u^*)'(t) \in -N(K(J^*(x^*(t))); J^*(u^*(t))), \text{ a.e. on } I,$$

that is, x^* is a solution of (SSP) and so the proof of the theorem is complete. □

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