



Strongly formal Weierstrass non-integrability for polynomial differential systems in \mathbb{C}^2

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Abstract. Recently a criterion has been given for determining the weakly formal Weierstrass non-integrability of polynomial differential systems in \mathbb{C}^2 . Here we extend this criterion for determining the strongly formal Weierstrass non-integrability which includes the weakly formal Weierstrass non-integrability of polynomial differential systems in \mathbb{C}^2 . The criterion is based on the solutions of the form $y = f(x)$ with $f(x) \in \mathbb{C}[[x]]$ of the differential system whose integrability we are studying. The results are applied to a differential system that contains the famous force-free Duffing and the Duffing–Van der Pol oscillators.


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1 Introduction and statement of the main result

One of the main problems in the qualitative theory of differential systems is the integrability problem. For differential systems in \mathbb{C}^2 this problem consists in to determine if the system has or not an explicit first integral. When this first integral can be expressed as quadratures of elementary functions we have the so-called Liouville integrability, which is the most studied, see for instance [16, 30, 31] and references therein. The Liouville integrability is based on the cofactors of the invariant algebraic curves and the exponential factors (see definitions below). Some generalizations of the Liouville integrability theory defining the generalized cofactors have been obtained, see [7, 8, 10, 11, 19, 20, 30, 31].

Some differential systems have an explicit first integral that cannot be expressed as quadratures of elementary functions. Hence these systems are not Liouville integrable. Sometimes these first integrals can be expressed in terms of special functions, as for instance functions that are solutions of second order linear differential equations (in [11, 19, 29] several examples are given). To determine when a differential system is not Liouville integrable is an open problem, see [25]. A partial answer to this question has been recently given in [23].

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In this work we present a criterion to detect the strongly formal Weierstrass non-integrability which is a generalization of the criterion for detecting weakly formal Weierstrass non-integrability given in [23]. Finally we apply this new criterion to some differential systems. Puiseux Weierstrass integrability is a generalization of formal Weierstrass integrability which includes the Liouville integrability and is based on the Puiseux Weierstrass polynomials, see again [23] and below.

First we provide some preliminary definitions and results.

In this paper we consider polynomial differential systems in the plane \mathbb{C}^2 that are given by

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1.1)$$

where the functions P and Q are polynomials in the complex variables x and y . We define by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1.1) with $P(0, 0) = Q(0, 0) = 0$. Along the paper we also consider the associated differential equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \quad (1.2)$$

and the associated *vector field* $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$.

An *invariant algebraic curve* of system (1.1) is an invariant curve $f = 0$ with $f \in \mathbb{C}[x, y]$, such that the orbital derivative $\dot{f} = \mathcal{X}f = P\partial f/\partial x + Q\partial f/\partial y$ vanishes on $f = 0$. This condition implies that there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$ of degree less than or equal to $m - 1$ such that

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \quad (1.3)$$

This polynomial K is called the *cofactor* of the curve $f(x, y) = 0$.

A function of the form $e^{f/g}$ with f and g polynomials is called an *exponential factor* if there is a polynomial L of degree at most $m - 1$ such that

$$\mathcal{X}(e^{f/g}) = P\frac{\partial e^{f/g}}{\partial x} + Q\frac{\partial e^{f/g}}{\partial y} = L e^{f/g}.$$

The polynomial L is called the *cofactor* of the exponential factor $e^{f/g}$.

A non-locally constant function $H : U \subset \mathbb{C}^2 \rightarrow \mathbb{C}$ is a *first integral* of system (1.1) in the open set U if this function is constant on each solution $(x(t), y(t))$ of system (1.1) contained in U . In fact if $H \in C^1(U)$ is a first integral of system (1.1) on U if and only if $\mathcal{X}H = P\partial H/\partial x + Q\partial H/\partial y \equiv 0$ on U . A non-constant function $M : U \subset \mathbb{C}^2 \rightarrow \mathbb{C}$ is an *integrating factor* in U if

$$P\frac{\partial M}{\partial x} + Q\frac{\partial M}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)M = -\operatorname{div}(\mathcal{X})M. \quad (1.4)$$

This integrating factor M is associated to a first integral H when $MP = -\partial H/\partial y$ and $MQ = \partial H/\partial x$. Moreover $V = 1/M$ is an *inverse integrating factor* in $U \setminus \{M = 0\}$.

A polynomial differential system (1.1) has a *Liouville first integral* H if its associated integrating factor is of the form

$$M = \exp\left(\frac{D}{E}\right) \prod_i C_i^{\alpha_i}, \quad (1.5)$$

where D , E and the C_i are polynomials in $\mathbb{C}[x, y]$ and $\alpha_i \in \mathbb{C}$, see [3, 17, 30, 31]. The curves $C_i = 0$ and $E = 0$ are invariant algebraic curves of the differential system (1.1), and the

exponential $\exp(D/E)$ is a product of some exponential factors associated to the multiple invariant algebraic curves of system (1.1) or to the invariant straight line at infinity, see for instance [2, 4, 5, 15] or Chapter 8 of [10].

The Liouville integrability is based on the existence of algebraic cofactors for the invariant algebraic curves and for the exponential factors. The first generalization of this theory is to consider non-algebraic invariant curves but still with algebraic cofactors, see [11]. In [12] a method for detecting non-algebraic invariant curves for polynomial differential systems was given. However there exist non-algebraic invariant curves without an algebraic cofactor, see [20].

Now we recall the definition of Puiseux Weierstrass integrability introduced in [23].

Let $\mathbb{C}((x))$ be the set of series in fractionary powers in the variable x with coefficients in \mathbb{C} (these series are called *Puiseux series*), and $\mathbb{C}[y]$ the set of the polynomials in the variable y with coefficients in the ring \mathbb{C} . A function of the form

$$\sum_{i=0}^{\ell} a_i(x)y^i \in \mathbb{C}((x))[y] \quad (1.6)$$

is a *Puiseux Weierstrass polynomial* in y of degree ℓ , i.e. a polynomial in the variable y with coefficients in $\mathbb{C}((x))$. Here we have privileged the variable y but of course we can privileged the variable x instead of the y .

In the next result we provide the expression of the cofactor of an invariant curve $y - g(x) = 0$ with $g(x)$ being a Puiseux series, for a proof see [23], see also [13].

Proposition 1.1. *Let $g(x) \in \mathbb{C}((x))$. An invariant curve of the form $y - g(x) = 0$ of a polynomial differential system (1.1) of degree m has a Puiseux Weierstrass polynomial cofactor of the form*

$$K(x, y) = k_{m-1}(x)y^{m-1} + \cdots + k_1(x)y + k_0(x). \quad (1.7)$$

A planar autonomous differential system is *Puiseux Weierstrass integrable* if admits an integrating factor of the form (1.5) where D , E and the C_i 's are Puiseux Weierstrass polynomials. This definition is a generalization of the Weierstrass integrability given in [19] and studied in [21, 22, 24, 28]. We remark that by definition that all the Liouvillian integrable systems are particular cases of the Puiseux Weierstrass integrable systems.

Let $\mathbb{C}[[x, y]]$ be the set of all formal power series in the variables x and y with coefficients in \mathbb{C} .

Theorem 1.2. *If $f \in \mathbb{C}[[x, y]]$ then it has a unique decomposition of the form*

$$f = ux^s \prod_{j=1}^{\ell} (y - g_j(x)), \quad (1.8)$$

where $g_j(x)$ are Puiseux series and $s \in \mathbb{Z}$, $s \geq 0$ and $u \in \mathbb{C}[[x, y]]$ is invertible inside the ring $\mathbb{C}[[x, y]]$.

For a proof of Theorem 1.2 see Corollary 1.5.6 of [1].

We note that a Darboux integrating factor (1.5) is analytic function where it is defined consequently by Theorem 1.2 it can be written into the form (1.8).

The first aim of this work was to give a necessary condition for detecting the Puiseux Weierstrass integrability but when $g_j(x) \in \mathbb{C}[[x]]$ of a polynomial differential system (1.1).

However this has been impossible using only the formal solutions of the form $y = f(x)$ of the associated differential equation for the reasons that we will see later on.

We say that a polynomial differential system (1.1) is *strongly formal Weierstrass integrable* if it has an integrating factor of the form

$$M(x, y) = \alpha(x) \prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k}, \quad (1.9)$$

where the functions $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$ for $i = 1, \dots, k$. Note that the definition of strongly formal Weierstrass integrability is a generalization of the definition of weakly formal Weierstrass integrability given in [23], where that the functions $\alpha(x)$ is constant equal to one.

In this work we give a criterion for detecting when a polynomial differential system (1.1) is not strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$. This criterion is based on the following result which provides a necessary condition in order that a polynomial differential system (1.1) be strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$.

Our main result is the following one.

Theorem 1.3. *Assume that a polynomial differential system (1.1) is strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$, and let $H(x, y)$ be a first integral provided by the strongly formal Weierstrass integrability.*

- (a) *Let $h(x) \in \mathbb{C}[[x]]$ and $y = h(x)$ be an invariant curve of the system such that $H(x, y)$ is defined on the curve $y = h(x)$. Then there exists an integrating factor $M(x, y)$ of the form (1.9) such that $M(x, h(x)) = 0$.*
- (b) *Assume that the origin of system (1.1) is a singular point, and the first integral $H(x, y)$ and $M(x, y)$ of statement (a) are well-defined at the origin. Then a linear combination of the formal Weierstrass polynomial cofactors up to order r of the solutions of the form $y = f(x)$ satisfying $E_q := \dot{x}dy/dx - \dot{y} = 0$ must be equal to minus the divergence of system (1.1) up to order r .*

Theorem 1.3 is proved in Section 2.

Now we apply Theorem 1.3 to a differential system that contains the force-free Duffing and Duffing–Van der Pol oscillators. Hence we consider the differential system

$$\dot{x} = y, \quad \dot{y} = -(\zeta x^2 + \alpha)y - (\varepsilon x^3 + \sigma x). \quad (1.10)$$

This system contains the famous force-free Duffing ($\zeta = 0, \varepsilon \neq 0$) and the Duffing–Van der Pol ($\zeta \neq 0, \varepsilon \neq 0$) oscillators that appear in several fields of mathematics, physics, biology, see [18] and references therein. The Liouville integrability of system (1.10) was studied in [9] where the following results were established.

Theorem 1.4. *System (1.10) with $\zeta = 0$ and $\varepsilon \neq 0$ is Liouvillian integrable if and only if either $\alpha = 0$, or $\sigma = 2\alpha^2/9$.*

In the case $\zeta \neq 0$ by a suitable rescaling of the variables for the Duffing–Van der Pol system we can take $\zeta = 3$ without loss of generality.

Theorem 1.5. *System (1.10) with $\zeta = 3$ and $\varepsilon \neq 0$ is Liouvillian integrable if and only if $\alpha = 4\varepsilon/3$ and $\sigma = \varepsilon^2/3$.*

Applying Theorem 1.3 to system (1.10) we obtain the following result.

Theorem 1.6. *System (1.10) can be strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$ if, and only if, one of the following cases holds:*

- (a) $\sigma = 2\alpha^2/9$,
- (b) $\sigma \neq 2\alpha^2/9, \sigma \neq 0$ and $3\alpha\varepsilon - 4\zeta\sigma = 0$,
- (c) $\sigma \neq 2\alpha^2/9, \sigma \neq 0$ and $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$,
- (d) $\sigma \neq 2\alpha^2/9, \sigma = 0$ and $-6\varepsilon(7\varepsilon - 2\alpha\zeta) = 0$.

We can see that all the Liouvillian integrable cases given in Theorems 1.4 and 1.5 are included in Theorem 1.6. In particular the case $\zeta = 3$ with $\alpha = 4\varepsilon/3$ and $\sigma = \varepsilon^2/3$ vanish the condition $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$.

Theorem 1.6 is proved in Section 3.

The following proposition shows that if a polynomial differential system has a Puiseux Weierstrass first integral of the form (1.5) then it has an integrating factor of the same form.

Proposition 1.7. *If system (1.1) has a Puiseux Weierstrass first integral of the form (1.5), then it has a Puiseux Weierstrass integrating factor of the form (1.5).*

The proof is straightforward because $M = (\partial H/\partial y)/P(x, y)$ which has the form (1.5). This proposition was generalized in [17] for non-Liouville integrable systems.

2 Proof of Theorem 1.3

Proof of statement (a) of Theorem 1.3. By assumptions the first integral $H(x, y)$ is defined on the invariant curve $y = h(x)$. So $H(x, h(x)) = c \in \mathbb{C}$, and the first integral $\bar{H}(x, y) = H(x, y) - c$ satisfies $\bar{H}(x, h(x)) = 0$. Now we consider the integrating factor $M(x, y)$ associated to the first integral \bar{H} . Perhaps this inverse integrating factor does not vanish at $y = h(x)$, but we consider the function $\bar{M} = MF(\bar{H})$ being F an arbitrary function of \bar{H} such that $F(0) = 0$. This function \bar{M} is also an inverse integrating factor of system (1.1) because

$$\begin{aligned} \mathcal{X}(\bar{M}) &= \mathcal{X}(MF(\bar{H})) = \mathcal{X}(M)F(\bar{H}) + M\mathcal{X}(F(\bar{H})) = F(\bar{H})\mathcal{X}(M) \\ &= -F(\bar{H})\operatorname{div}(\mathcal{X})M = -\operatorname{div}(\mathcal{X})MF(\bar{H}) = -\operatorname{div}(\mathcal{X})\bar{M}. \end{aligned}$$

Hence we obtain that $\bar{M}(x, h(x)) = 0$ because $F(0) = 0$. □

We can repeat this process to obtain an integrating factor that vanish in a finite number of the solutions of the form $y = h(x)$ such $H(x, h(x)) = c \in \mathbb{C}$.

In the proof of statement (b) of Theorem 1.3 we shall need the following result, for a proof see for instance Proposition 8.4 of [10].

Proposition 2.1. *Assume that $f \in \mathbb{C}[x, y]$ and let $f = f_1^{n_1} \dots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x, y]$. Then for a polynomial system (1.1), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \dots, r$ with cofactor K_{f_i} . Moreover $K_f = n_1K_{f_1} + \dots + n_rK_{f_r}$.*

Proof of statement (b) of Theorem 1.3. We assume that the system is strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$ this means by definition that the system has an integrating factor of the form (1.9). Hence we know that a first integral H and an integrating factor

M of the form given in statement (a) can be found. We compute the solutions $y = f_i(x)$ where $f_i(x) = \sum_{j=0}^{\infty} a_j x^j$ with a_i arbitrary coefficients that must satisfy the equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ up to certain order r . Note that these solutions satisfy that

$$\text{either } M(x, f_i(x)) = \mathcal{O}(x^r), \quad \text{or } M(x, f_i(x)) = c_2 + \mathcal{O}(x^r),$$

with $c_2 \neq 0$, this case appears when the integrating factor (1.8) has $s = 0$. The first ones correspond to the $f_i(x)$ that approximate the invariant curves $y = g_k(x)$ that appear in the integrating factor (1.9). For such $f_i(x)$ we compute the cofactor K_i up to certain order r though the equation

$$\mathcal{X}(y - f_i(x)) = \bar{K}_i(y - f_i(x)) + \mathcal{O}(x^r). \quad (2.1)$$

Hence these cofactors \bar{K}_i of the solutions $y - f_i(x)$ are the approximations up to order r of the cofactors K_k of the invariant curves $y - g_k(x)$ of the integrating factor (1.9).

The second ones satisfy

$$M(x, f_i(x)) = \alpha(x) \prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} = c_2 + \mathcal{O}(x^r). \quad (2.2)$$

Hence, since $c_2 \neq 0$, $M(x, f_i(x)) = c_2 + \mathcal{O}(x^r)$, and from (1.9) we have that $\alpha(0) \neq 0$. Then up to order r we have

$$\prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} = \left[\frac{c_2}{\alpha(x)} \right]_r + \mathcal{O}(x^r), \quad (2.3)$$

where here $[\cdot]_r$ means up to order r . Consequently $y = f_i(x)$ is an approximation up to order r of the equation

$$\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} = \frac{c_2}{\alpha(x)}. \quad (2.4)$$

We apply the vector field operator to (2.4) and we obtain

$$\mathcal{X} \left(\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} \right) = \mathcal{X} \left(\frac{c_2}{\alpha(x)} \right) = -\frac{c_2 \alpha'(x)}{\alpha(x)^2} P = -K_{\alpha} \frac{c_2}{\alpha(x)}, \quad (2.5)$$

because $\mathcal{X}(\alpha(x)) = K_{\alpha}(x, y)\alpha(x)$ where K_{α} is a formal Weierstrass polynomial cofactor. This happens because $\alpha(x) = 0$ is an invariant algebraic curve of the vector field \mathcal{X} . Indeed, $\alpha(x)$ is a factor of the integrating factor $M(x, y)$ given in (1.9), and $M(x, y) = 0$ is an invariant curve because it satisfies (1.4), and the factors of an invariant curve are also invariant curves. Moreover we have taken into account that $\mathcal{X}(\alpha(x)) = \alpha'(x)\dot{x} = \alpha'(x)P(x, y)$ and then $K_{\alpha} = \alpha'(x)P(x, y)/\alpha(x)$.

In summary from equations (2.4) and (2.5) we have

$$\mathcal{X} \left(\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} \right) = -K_{\alpha} \prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k}. \quad (2.6)$$

Now we apply the vector field operator to (2.3) and we obtain

$$\mathcal{X} \left(\prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} \right) = \mathcal{X} \left(\left[\frac{c_2}{\alpha(x)} \right]_r \right) + \mathcal{O}(x^r), \quad (2.7)$$

where $\mathcal{X}(\mathcal{O}(x^r)) = \mathcal{O}(x^{r-1})P(x, f_i(x)) = \mathcal{O}(x^r)$. Taking into account equation (2.5) we define the new cofactor \tilde{K}_α through the equation

$$\mathcal{X}\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right) = -\tilde{K}_\alpha\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right) \quad (2.8)$$

which is equation (2.5) taking the lower terms up to r and where \tilde{K}_α is an approximation up to r of the cofactor K_α . Therefore from (2.3), (2.7) and (2.8) we obtain an approximation of the cofactor of $\alpha(x)$ up to order r computing

$$\frac{\mathcal{X}\left(\prod_{k=1}^{\ell}(f_i(x) - g_k(x))^{\alpha_k}\right)}{\prod_{k=1}^{\ell}(f_i(x) - g_k(x))^{\alpha_k}} = -\tilde{K}_\alpha + \mathcal{O}(x^r). \quad (2.9)$$

By the definition of integrating factor (1.9) and from the extension of the Darboux theory to Weierstrass functions, see for instance Theorem 3 of [23], we have that

$$\mathcal{X}(M) = -\operatorname{div}(\mathcal{X})M. \quad (2.10)$$

In short the cofactors \tilde{K}_i of the solutions $y - f_i(x)$ passing through the origin are the approximations up to order r of the cofactors K_i of the solution $y = g_i(x)$. By Proposition 2.1 the other solutions $y - f_i(x)$ not passing through the origin with cofactor \tilde{K}_i give by equation (2.9) an approximation up to order r of the cofactor \tilde{K}_α of $\alpha(x)$, i.e.

$$\sum_{i=1}^s \mu_i \tilde{K}_i = -\tilde{K}_\alpha. \quad (2.11)$$

Therefore, from (2.2), (2.10) and (2.11) we obtain that

$$\sum_{i=1}^{\ell} \lambda_i \tilde{K}_i + \sum_{i=1}^s \mu_i \tilde{K}_i = -\operatorname{div}_r(\mathcal{X}) + \mathcal{O}(x^{r+1}). \quad (2.12)$$

This proves statement (b) of the theorem. \square

In summary, if condition (2.12) is not satisfied then system (1.1) does not admit an integrating factor of the form (1.9) and consequently is not strongly formal Weierstrass integrable. Hence we have a necessary condition to have strongly formal Weierstrass integrability. Note that if we have that $\sum_{i=1}^{\ell} \lambda_i \tilde{K}_i + \sum_{i=1}^s \mu_i \tilde{K}_i = \mathcal{O}(x^{r+1})$ system (1.1) satisfies a necessary condition to have a first integral of the form (1.9), see for more details statement (i) of Theorem 8.7 of [10].

3 Proof of Theorem 1.6

We apply the criterion provided by statement (b) of Theorem 1.3 to detect if system (1.10) can be strongly formal Weierstrass integrable, that is, if it can have an inverse integrating factor of the form (1.9). We propose a solution curve of the form

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

Substituting this solution in the first ordinary differential equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ we get an infinite system of equations. First we have studied the case when $a_0 \neq 0$, and in this

case it is easy to see that we find two solutions not passing through the origin but we do not find any possible integrable case. So we consider the case $a_0 = 0$. In order to determine the first coefficients we fix up to certain order the developments of $f(x)$ and Eg in power series of the variable x . If we compute the solutions up to order 6 we obtain the following finite system of equations

$$\begin{aligned} a_2(3a_1 + \alpha) &= 0, & a_1^2 + a_1\alpha + \sigma &= 0, \\ 2a_2^2 + 4a_1a_3 + a_3\alpha + \varepsilon + a_1\zeta &= 0, & 7a_3a_4 + 7a_2a_5 + a_4\zeta &= 0, \\ 5a_2a_3 + 5a_1a_4 + a_4\alpha + a_2\zeta &= 0, & 3a_3^2 + 6a_2a_4 + 6a_1a_5 + a_5\alpha + a_3\zeta &= 0. \end{aligned}$$

From the first equation we have two possibilities $a_2 = 0$ or $a_1 = -\alpha/3$. First we take $a_2 = 0$. The obtained system is compatible and we get two solutions. We denote them y_1 and y_2 but we do not write them here due to their long extensions. Now we study the case $a_1 = -\alpha/3$ with $a_2 \neq 0$. In this case the equation $a_1^2 + a_1\alpha + \sigma = 0$ takes the form $\sigma - 2\alpha^2/9 = 0$. Hence we must impose $\sigma = 2\alpha^2/9$ in order that the finite system of equations be compatible. Under this condition we find four more solutions that we denote by y_3, y_4, y_5 and y_6 , but again we do not write them here due to their big extensions. We recall that all these solutions pass through the origin, i.e., $y_i(0) = 0$ for $i = 1, \dots, 6$. Now we compute their cofactors using equation (2.1), that we denote by \bar{K}_i . Finally we verify if the equation

$$\sum_{i=1}^6 \lambda_i \bar{K}_i = -\operatorname{div}_6 \mathcal{X} + \mathcal{O}(x^7),$$

has any solution, and since it has a solution statement (a) of the theorem follows.

Now we consider the case $\sigma \neq 2\alpha^2/9$. In this case the solutions y_i for $i = 3, \dots, 6$ do not exist and we only have the solutions y_1 and y_2 . We compute their cofactors from equation (2.1), that we denote by \bar{K}_1 and \bar{K}_2 and we verify if the equation

$$\lambda_1 \bar{K}_1 + \lambda_2 \bar{K}_2 = -\operatorname{div}_6 \mathcal{X} + \mathcal{O}(x^7),$$

is satisfied. This equation gives a system of three equations. The first one is

$$\alpha(2 + \lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)\sqrt{\alpha^2 - 4\sigma} = 0. \quad (3.1)$$

From this condition we can isolate λ_1 if $\sigma \neq 0$ (we will consider $\sigma = 0$ below) and we have

$$\lambda_1 = \frac{\alpha(2 + \lambda_2) + \lambda_2\sqrt{\alpha^2 - 4\sigma}}{-\alpha + \sqrt{\alpha^2 - 4\sigma}}.$$

From the second equation we obtain

$$\left(-\alpha + 2\sqrt{\alpha^2 - 4\sigma} + \lambda_2\sqrt{\alpha^2 - 4\sigma}\right)(3\alpha\varepsilon - 4\zeta\sigma) = 0.$$

Hence we have two possibilities: If $3\alpha\varepsilon - 4\zeta\sigma = 0$ the third equation can vanish choosing the value of λ_2 and this proves statement (b) of the theorem. If $-\alpha + 2\sqrt{\alpha^2 - 4\sigma} + \lambda_2\sqrt{\alpha^2 - 4\sigma} = 0$ we isolate the value of λ_2 , i.e.

$$\lambda_2 = \frac{\alpha - 2\sqrt{\alpha^2 - 4\sigma}}{\sqrt{\alpha^2 - 4\sigma}},$$

and the third equation provides the condition $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$, which shows statement (c) of the theorem.

Now we study the case $\sigma = 0$. In this case condition (3.1) becomes $\alpha(1 + \lambda_2) = 0$. Taking into account that we are in the case $\sigma \neq 2\alpha^2/9$, we must take $\lambda_2 = -1$. The second condition is $\varepsilon(3 + \lambda_1) = 0$. The case $\varepsilon = 0$ gives a trivial integrable case. Hence we must consider $\lambda_1 = -3$. In this case the third condition gives $-6\varepsilon(7\varepsilon - 2\alpha\zeta) = 0$ which proves statement (d) of the theorem. Hence this completes the proof of theorem.

4 Examples

Example 4.1. Consider the differential system

$$\dot{x} = y + xy + x^2, \quad \dot{y} = 2y(y + x). \quad (4.1)$$

This system was studied in [14] where an algorithmic method to determine integrability was given. Using the method developed in [14] it was shown that system (4.1) has an integrating factor of the form $M(x, y) = e^{x^2/(2y)}y^{-5/2}$ and the a Liouville first integral

$$H(x, y) = \frac{e^{\frac{x^2}{2y}}}{\sqrt{y}} + \sqrt{2} \int_0^{x/\sqrt{2y}} e^{t^2} dt.$$

Now we are going to apply the criterion provided by statement (b) of Theorem 1.3 for detecting if system (4.1) can have an inverse integrating factor of the form (1.9). We propose a solution curve of the form

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

Substituting this solution in the first ordinary differential equation $Eq := xdy/dx - y = 0$ we get an infinite system of equations. In order to determine the first coefficients we fix up to certain order the developments of $f(x)$ and Eq in power series in variable x . If we do that up to order 6 and we solve the finite system of equations we obtain the following solutions.

- 1) $a_6 = a_5 = a_4 = a_3 = a_2 = a_1 = 0$,
- 2) $a_6 = \frac{-1368989 - 4007\sqrt{150829}}{607500}$, $a_5 = \frac{781 + 3\sqrt{150829}}{750}$, $a_4 = \frac{173 - \sqrt{150829}}{900}$,
 $a_3 = -\frac{2}{3}$, $a_2 = \frac{497 - \sqrt{150829}}{70}$, $a_1 = \frac{427 - \sqrt{150829}}{35}$, $a_0 = \frac{427 - \sqrt{150829}}{70}$,
- 3) $a_6 = \frac{-1368989 + 4007\sqrt{150829}}{607500}$, $a_5 = \frac{781 - 3\sqrt{150829}}{750}$, $a_4 = \frac{173 + \sqrt{150829}}{900}$,
 $a_3 = -\frac{2}{3}$, $a_2 = \frac{497 + \sqrt{150829}}{70}$, $a_1 = \frac{427 + \sqrt{150829}}{35}$, $a_0 = \frac{427 + \sqrt{150829}}{70}$.

The solutions correspond to the solution curves

- 1) $y_1 = 0 + \mathcal{O}(x^7)$,
- 2) $y_2 = f_2(x) = \frac{427 - \sqrt{150829}}{70} + \frac{427 - \sqrt{150829}}{35}x + \frac{497 - \sqrt{150829}}{70}x^2 - \frac{3}{2}x^3$
 $+ \frac{173 - \sqrt{150829}}{900}x^4 + \frac{781 + 3\sqrt{150829}}{750}x^5 - \frac{1368989 + 4007\sqrt{150829}}{607500}x^6 + \mathcal{O}(x^7)$,
- 3) $y_3 = f_3(x) = \frac{427 + \sqrt{150829}}{70} + \frac{427 + \sqrt{150829}}{35}x + \frac{497 + \sqrt{150829}}{70}x^2 - \frac{3}{2}x^3$
 $+ \frac{173 + \sqrt{150829}}{900}x^4 + \frac{781 - 3\sqrt{150829}}{750}x^5 - \frac{1368989 - 4007\sqrt{150829}}{607500}x^6 + \mathcal{O}(x^7)$,

respectively. The first one corresponds to the invariant algebraic curve $y = 0$ whose cofactor is $2x + 2y$. However, in general, we can have an approximation of a solution of the form $y = g_k(x)$ and an approximation of its cofactor. To compute the approximation of the Weierstrass polynomial cofactor of the solution curve $y = 0$, since the system is of degree 2, it must be of the form $\bar{K}_1 = k_0(x) + k_1(x)y$. Hence we have the equation

$$\frac{\partial(y - y_1)}{\partial x} \dot{x} + \frac{\partial(y - y_1)}{\partial y} \dot{y} = (k_0(x) + k_1(x)y)(y - y_1) + \mathcal{O}(x^7),$$

From here we obtain $k_0 = 2x$ and $k_1 = 2$.

For determining the cofactors of the other two solutions $y = f_i(x)$ for $i = 2, 3$ we use equation (2.7) that in this case are

$$\mathcal{X}(y_2(x) - y_1(x)) = \tilde{K}_2(x)(y_2(x) - y_1(x)) + \mathcal{O}(x^7),$$

$$\mathcal{X}(y_3(x) - y_1(x)) = \tilde{K}_3(x)(y_3(x) - y_1(x)) + \mathcal{O}(x^7),$$

We do not write here the expressions of \tilde{K}_2 and \tilde{K}_3 due to their extension but the reader can compute them straightforward. Now we study if the cofactors \bar{K}_1 , \bar{K}_2 and \bar{K}_3 satisfy (2.12), i.e.

$$\lambda_1 \bar{K}_1 + \mu_1 \bar{K}_2 + \mu_2 \bar{K}_3 = -\text{div}_6 \mathcal{X} + \mathcal{O}(x^7),$$

and this equation has not solution. Hence system (4.1) has not an integrating factor of the form (1.9), this implies that system (4.1) is not strongly formal Weierstrass integrable.

If we try to see if there is a linear combination that gives zero, then the system has the unique solution $\lambda_1 = \mu_2 = \mu_3 = 0$. Therefore the system has not a first integral of the form (1.9).

The conclusion is that system (4.1) is not strongly formal Weierstrass integrable in the original coordinates (x, y) . However we can ask if system (4.1) is strongly formal Weierstrass integrable after a change of variable. The answer to this question is positive as we will see below.

System (4.1) after doing the change of variables

$$z = \frac{x}{\sqrt{2y}}, \quad u = \sqrt{y},$$

takes the form

$$\dot{u} = \sqrt{2}u^2 + 2uz, \quad \dot{z} = 1.$$

First we rename the new variables of the form $u := x$ and $z := y$. So the equation associated to this differential system is the Bernoulli equation $dx/dy = \sqrt{2}x^2 + 2xy$, and then its integrability is straightforward. In fact an integrating factor is given by $M(x, y) = e^{-y^2} x^2$ and a first integral is

$$H(x, y) = \frac{e^{y^2}}{x} + \sqrt{2} \int_0^y e^{t^2} dt.$$

Anyway we are going to apply the necessary condition of strongly formal Weierstrass integrability to this system. Attending to the form of the integrating factor in this case the answer must be positive.

Hence consider the system of the form

$$\dot{x} = \sqrt{2}x^2 + 2xy, \quad \dot{y} = 1. \tag{4.2}$$

Now we study if system (4.2) is strongly formal Weierstrass integrable. We propose a solution curve of the form

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

Substituting this solution in the ordinary differential equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ we get an infinite system of equations without any solution. Therefore privileging the variable y the system has no solutions curves. Next we propose a solution curve of the form

$$x = f(y) = a_0 + a_1y + a_2y^2 + a_3y^3 + a_4y^4 + a_5y^5 + \dots$$

Substituting this solution in the first ordinary differential equation $Eq := \dot{x}dx/dy - \dot{x} = 0$ we get an infinite system of equations. We determine the first parameters fixing certain order in the developments of $f(y)$ and Eq in power series of the variable y . If we do that up to order 4 and we solve the finite system of equations we obtain the following solutions.

- 1) $x_1 = \mathcal{O}(x^5)$,
- 2) $x_2 = -\frac{i}{15}\sqrt{\frac{225-15\sqrt{15}}{2}} + \frac{-15+\sqrt{15}}{15\sqrt{2}}y - \frac{i}{15}\sqrt{\frac{15-\sqrt{15}}{2}}y^2 - \frac{\sqrt{2}}{45}(6 + \sqrt{15})y^3$
 $+ \frac{i}{450}\sqrt{\frac{15-\sqrt{15}}{2}}(10 + 3\sqrt{15})y^4 + \mathcal{O}(x^5)$,
- 3) $x_3 = \frac{i}{15}\sqrt{\frac{225-15\sqrt{15}}{2}} + \frac{-15+\sqrt{15}}{15\sqrt{2}}y + \frac{i}{15}\sqrt{\frac{15-\sqrt{15}}{2}}y^2 - \frac{\sqrt{2}}{45}(6 + \sqrt{15})y^3$
 $- \frac{i}{450}\sqrt{\frac{15-\sqrt{15}}{2}}(10 + 3\sqrt{15})y^4 + \mathcal{O}(x^5)$,
- 4) $x_4 = -i\sqrt{\frac{15+\sqrt{15}}{30}} - \frac{15+\sqrt{15}}{15\sqrt{2}}y + \frac{i}{15}\sqrt{\frac{15+\sqrt{15}}{2}}y^2 - \frac{\sqrt{2}}{45}(6 - \sqrt{15})y^3$
 $- \frac{i}{450}\sqrt{\frac{15+\sqrt{15}}{2}}(10 - 3\sqrt{15})y^4 + \mathcal{O}(x^5)$,
- 5) $x_5 = i\sqrt{\frac{15+\sqrt{15}}{30}} - \frac{15+\sqrt{15}}{15\sqrt{2}}y - \frac{i}{15}\sqrt{\frac{15+\sqrt{15}}{2}}y^2 - \frac{\sqrt{2}}{45}(6 - \sqrt{15})y^3$
 $+ \frac{i}{450}\sqrt{\frac{15+\sqrt{15}}{2}}(10 - 3\sqrt{15})y^4 + \mathcal{O}(x^5)$.

Next we compute their Weierstrass polynomial cofactor for the solution curve y_1 through the equation

$$\frac{\partial(x - x_1)}{\partial x}\dot{x} + \frac{\partial(x - x_1)}{\partial y}\dot{y} = (k_0(y) + k_1(y)x)(x - x_1) + \mathcal{O}(x^5),$$

which is $\bar{K}_1 = \sqrt{2}x + 2y$, and the cofactors of the other solutions through the equations

$$\mathcal{X}(x_i(x) - x_1(x)) = \bar{K}_i(x)(x_i(x) - x_1(x)) + \mathcal{O}(x^5),$$

for $i = 2, 3, 4, 5$. We do not write here the expressions of these cofactors due to their extension. Finally we try to see if there is a linear combination of these cofactors equals to minus the divergence, that is,

$$\lambda_1\bar{K}_1 + \mu_2\bar{K}_2 + \mu_3\bar{K}_3 + \mu_4\bar{K}_4 + \mu_5\bar{K}_5 = -\text{div}_4\mathcal{X} + \mathcal{O}(x^5),$$

and this system has the solution $\lambda_1 = -2$, $\mu_2 = 5/6 - \sqrt{5/3}$, $\mu_3 = 5/6 - \sqrt{5/3}$, $\mu_4 = 5/6 + \sqrt{5/3}$ and $\mu_5 = 5/6 + \sqrt{5/3}$. Hence system (4.2) satisfies the strongly formal Weierstrass integrability condition and it can have an integrating factor of the form (1.9) as indeed it

has. Moreover we can also study if the system has a first integral of the form (1.9) using the equation

$$\lambda_1 \bar{K}_1 + \mu_2 \bar{K}_2 + \mu_3 \bar{K}_3 + \mu_4 \bar{K}_4 + \mu_5 \bar{K}_5 = \mathcal{O}(x^5),$$

and this system has the only solution $\lambda_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$. Consequently system (4.2) has not a first integral of the form (1.9).

Example 4.2. In 1944 Kukles [27] studied the following system

$$\dot{x} = y, \quad \dot{y} = -x + Q(x, y), \quad (4.3)$$

where $Q(x, y) = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3$, giving the conditions in order that the origin of (4.3) be a center. However some decades later in [6, 26] was proved that the conditions were uncompleted showing that the origin of the following system has also a center. Consider the system

$$\dot{x} = y, \quad \dot{y} = -x + x^2 - \frac{x^3}{3} - \frac{x^2 y}{\sqrt{2}} - 2y^2 + \frac{y^3}{3\sqrt{2}}. \quad (4.4)$$

System (4.4) has an inverse integrating factor of the form

$$V(x, y) = e^{-x(1-\frac{x}{3})} (3\sqrt{2}(1-x) + x(\sqrt{2}x + y))^3,$$

and the following first integral

$$H(x, y) = \frac{y^2(x+1) + 2\sqrt{2}xy(x-2) + 6(3x-2) + 2x^3 - 10x^2}{(x(y + \sqrt{2}x) + 3\sqrt{2}(1-x))^2} e^{x(1-\frac{x}{3})} + \int e^{x(1-\frac{x}{3})} dx.$$

The analyticity of this first integral around the origin implies that the origin is a center.

Now we are going to apply the criterion to detect if system (4.1) can have a strongly formal Weierstrass first integral. We propose a solution curve of the form $y = f(x) = \sum_{i=0}^{\infty} a_i x^i$ and substitute this solution into the differential equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ and we get an infinite system of equations. If we develop up to order 3 and we solve the finite system of equations we obtain the solutions curves.

- 1) $y_1 = ix - ix^2 + \mathcal{O}(x^3),$
- 2) $y_2 = -ix + ix^2 + \mathcal{O}(x^3),$
- 3) $y_{3,4} = 3\sqrt{2} \pm \sqrt{3(4 - \sqrt{6})} - (\sqrt{2} + \sqrt{3})x + 1/12 \left(6\sqrt{2} - 6\sqrt{3} \pm \sqrt{3}(4 - \sqrt{6}) \right)^{3/2} \mp 10\sqrt{3(4 - \sqrt{6})} x^2 + \mathcal{O}(x^3),$
- 4) $y_{5,6} = 3\sqrt{2} \pm \sqrt{3(4 \mp \sqrt{6})} + (-\sqrt{2} + \sqrt{3})x + 1/12 \left(6\sqrt{2} + 6\sqrt{3} \pm \sqrt{3}(4 + \sqrt{6}) \right)^{3/2} \mp 10\sqrt{3(4 + \sqrt{6})} x^2 + \mathcal{O}(x^3).$

Now we compute the Weierstrass polynomial cofactor of the first two solutions curves. These cofactors, as the system is of degree 3 must be of the form $K = k_0(x) + k_1(x)y + k_2(x)y^2$. Applying equation (2.1) to the solution curves $y - f(x) = 0$ we obtain the Weierstrass polynomial cofactors up to order 3 in the variable x

- 1) $K_1 = -1/6(6i + 4(-3i + \sqrt{2})x^2) - 1/6(12 - \sqrt{2}ix + \sqrt{2}ix^2)y + 1/(3\sqrt{2})y^2 + \mathcal{O}(x^3),$

$$2) K_2 = 1/6(6i - 4(3i + \sqrt{2})x^2) - 1/6(12 + \sqrt{2}ix - \sqrt{2}ix^2))y + 1/(3\sqrt{2})y^2 + \mathcal{O}(x^3),$$

The cofactors of the other solutions must be computed through the equation (2.9) that in this case are

$$\mathcal{X}\left((y_i(x) - y_1(x))(y_i(x) - y_2(x))\right) = \tilde{K}_i(x)\left((y_i(x) - y_1(x))(y_i(x) - y_2(x))\right) + \mathcal{O}(x^3),$$

for $i = 3, 4, 5, 6$. We do not write here the expressions of these cofactors due to their extension.

From statement (b) of Theorem 1.3 we study if system (4.4) has a strongly formal Weierstrass first integral using the equation

$$\lambda_1 K_1 + \lambda_2 K_2 + \mu_3 K_3 + \mu_4 K_4 + \mu_5 K_5 + \mu_6 K_6 = \mathcal{O}(x^3),$$

and this system has the solution $\lambda_1 = \lambda_2 = 0$ and

$$\begin{aligned} \mu_6 &= \frac{\mu_5 \left(6\sqrt{10} + 5\sqrt{15} - 6\sqrt{24 - 6\sqrt{6}} - 15\sqrt{4 - \sqrt{6}} \right)}{6\sqrt{10} + 5\sqrt{15} + 6\sqrt{24 - 6\sqrt{6}} + 15\sqrt{4 - \sqrt{6}}}, \\ \mu_3 &= \frac{\mu_5}{D_1} \left[24\sqrt{10} + 20\sqrt{15} + 4\sqrt{60 - 15\sqrt{6}} + 5\sqrt{40 - 10\sqrt{6}} \right. \\ &\quad + 129\sqrt{24 - 6\sqrt{6}} + 316\sqrt{4 - \sqrt{6}} + 8\sqrt{4 + \sqrt{6}} \\ &\quad \left. + 3\sqrt{6(4 + \sqrt{6})} - 49\sqrt{10(4 + \sqrt{6})} - 40\sqrt{15(4 + \sqrt{6})} \right], \\ \mu_4 &= \frac{\mu_5}{D_2} \left[-6\sqrt{10} - 4\sqrt{15} + \sqrt{60 - 15\sqrt{6}} + \sqrt{40 - 10\sqrt{6}} \right. \\ &\quad - 29\sqrt{24 - 6\sqrt{6}} - 71\sqrt{4 - \sqrt{6}} + \sqrt{4 + \sqrt{6}} + \sqrt{6(4 + \sqrt{6})} \\ &\quad \left. + 11\sqrt{10(4 + \sqrt{6})} + 9\sqrt{15(4 + \sqrt{6})} \right], \end{aligned}$$

where we have $D_1 = 372 + 152\sqrt{6} + 80\sqrt{60 - 15\sqrt{6}} + 98\sqrt{40 - 10\sqrt{6}}$ and $D_2 = 84 + 34\sqrt{6} + 18\sqrt{60 - 15\sqrt{6}} + 22\sqrt{40 - 10\sqrt{6}}$. Consequently system (4.4) can have a strong formal Weierstrass first integral as indeed it has as we have seen before.

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Conflict of interest

The authors declare that they have no conflict of interest.

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